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MODULAR LATTICES AND DESARGUES' THEOREM

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Introduction. It has been proved (Frink [4]) that every complemented modular lattice is isomorphic to a sublattice of the lattice of all subspaces of a (possibly degenerate) projective space, but very little is known about the connection between the given lattice and the corresponding space. After summarizing in Section 1 some known results which will be used in our investigation, we devote the next section to this problem. The representation theorem shows that every complemented modular lattice B can be embedded in a complete and atomistic modular lattice A, satisfying certain additional conditions. Our main results consist in describing up to isomorphism the connection between B and A, and in showing that every identity which holds in B is also valid in A. Since it is known that Desargues' Theorem is equivalent to a lattice-theoretic identity, we are thus able to state under what conditions on B the corresponding space is Arguesian (for the definition, see p. 297). Inasmuch as every lattice of commuting equivalence relations satisfies the identity in question (cf. Jónsson [5]), we infer that, for a complemented modular lattice, the existence of a representation by commuting equivalence relations is equivalent to the existence of a representation by subspaces of an Arguesian projective space. For arbitrary modular lattices this is no longer true. In fact, in the third and last section we construct a five dimensional modular lattice which is isomorphic to a lattice of commuting equivalence relations, but not to a lattice of normal subgroups of a group.

1. Preliminaries. We use the symbol $\subseteq$ to denote set-inclusion, $\cup$ and $\cap$ for binary unions and intersections, $\bigcup$ and $\bigcap$ for unrestricted unions and intersections, $\in$ and $\notin$ for the relations of membership and non-membership, and Φ for the empty set. The symbols $\nsubseteq$, $\supseteq$ and $\nsupseteq$ are defined in the usual way in terms of $\subseteq$. By $\{f(x) \mid \varphi(x)\}$ we mean the set of all elements of the form $f(x)$, associated with elements x satisfying

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the condition $\varphi(x)$. If X is a set and n is a positive integer, then X^n is the set of all n-termed sequences whose terms belong to X. If x is an n-termed sequence, then $x_0, x_1, \ldots, x_{n-1}$ are the first, second, $\ldots$, n^{th} terms of x, and we write $x = \langle x_0, x_1, \ldots, x_{n-1}\rangle$. On the other hand $\{x_0, x_1, \ldots, x_{n-1}\}$ is the set whose elements are $x_0, x_1, \ldots, x_{n-1}$.

When referring to an arbitrary lattice, the symbols $+, \cdot, \Sigma, \Pi, \leq, \nleq, \geq, \ngeq, <, >$ will have their usual meaning. When applied to sequences (functions) whose terms (values) are elements of a given lattice A, these symbols should be interpreted in the sense of the cardinal power. Thus if $x, y \in A^n$, then $x \leq y$ means that $x_i \leq y_i$ for $i = 0, 1, \ldots, n-1$. When two lattices are being considered, one of which is a sublattice of the other, the symbols Σ and Π will always refer to the larger lattice. The zero and unit elements of a lattice (when they exist) will be denoted by 0 and 1. If A is a lattice, $a, b \in A$ and $a \leq b$, then we let

$$[a, b] = \{x \mid x \in A \text{ and } a \leq x \leq b\}.$$

As is well known, binary relations can be regarded as sets whose elements are ordered pairs, or two-termed sequences. Thus we can apply to them the usual set-theoretic operations. In addition, given two binary relations R and S, we define the *relative product* $R;S$ of R and S, the *converse* R^{-1} of R, and the *domain* $\operatorname{dmn} R$ of R by the formulas

$$\begin{aligned} R;S &= \{\langle x, y\rangle \mid \langle x, z\rangle \in R \text{ and } \langle z, y\rangle \in S \text{ for some } z\}, \\ R^{-1} &= \{\langle x, y\rangle \mid \langle y, x\rangle \in R\}, \\ \operatorname{dmn} R &= \{x \mid \langle x, y\rangle \in R \text{ for some } y\}. \end{aligned}$$

A binary relation R is called an *equivalence relation* if $R;R \subseteq R$ and $R^{-1} = R$. An equivalence relation whose domain is U is called an *equivalence relation over* U. If x is an element of the domain of an equivalence relation R, then the set $\{y \mid \langle x, y\rangle \in R\}$ is called an R *class*.

The family $\mathcal{A}$ of all equivalence relations over a set U is a lattice. Here lattice inclusion and multiplication coincide with set-theoretic inclusion and intersection while, for $R, S \in \mathcal{A}$, the lattice sum $R+S$ is the smallest equivalence relation over U which contains both R and S. In particular, if R and S commute, i.e., if $R;S = S;R$, then $R+S = R;S$. A sublattice $\mathcal{B}$ of $\mathcal{A}$, such that any two members of $\mathcal{B}$ commute, will be referred to as a *lattice of commuting equivalence relations.*

When speaking of a projective space S, we follow the lead of Frink [4] in not requiring the dimension to be finite, and in not excluding degenerate cases. Thus we assume only that any two distinct points of S determine a unique line which passes through both of them, and that any line of S which intersects two sides of a triangle, without passing through their point of intersection, shall intersect the third side. A projective space S is called *Arguesian* if it satisfies Desargues' Theorem, i.e., if any two triangles of S, which are centrally perspective, are also axially perspective.

We shall now summarize certain known results concerning the connection between projective geometry and lattice theory, and introduce some definitions related to these results. For general information on lattices we refer the reader to Birkhoff [2].

DEFINITION 1.1. *A lattice A is said to be projective if it is complete, atomistic, complemented and modular, and satisfies the following condition: If p is an atom of A, if I is any set, and if the elements $a_i \in A$ with $i \in I$ are such that*

$$p \leq \sum_{i \in I} a_i ,$$

then there exists a finite subset J of I such that

$$p \leq \sum_{i \in J} a_i .$$

THEOREM 1.2. *The set of all subspaces of a projective space is a projective lattice, where lattice inclusion and multiplication coincide with set-theoretic inclusion and intersection, while the lattice sum of two distinct, non-empty subspaces is the set-theoretic union of all lines passing through two distinct points, one from each subspace.*

THEOREM 1.3. *The set S of all atoms of a modular lattice A is a projective space, where by the line through two distinct atoms p and q we mean the set $\{r \mid p+q \geq r \in S\}$. Furthermore, if A is projective and if $\Phi(x)=\{r \mid x \geq r \in S\}$ for $x \in A$, then Φ maps A isomorphically onto the lattice of all subspaces of S.*

DEFINITION 1.4. *Given a modular lattice A, we shall refer to the projective space constructed in Theorem 1.3 as the projective space associated with A.*

THEOREM 1.5. *Every projective lattice is a direct product of indecomposable projective lattices. If A is an indecomposable projective lattice of dimension three or more, then one and only one of the following conditions holds:*

(i) *The projective space associated with A is a non-Arguesian plane.*

(ii) *A is isomorphic to the lattice of all vector subspaces of a vector space over a division ring.*

THEOREM 1.6. *If B is a complemented modular lattice, then the set $\mathcal{D}$ of all dual ideals of B is a complete, atomistic modular lattice, where lattice inclusion coincides with the relation $\supseteq$, lattice addition coincides with set-*

theoretic intersection, and the lattice product of two dual ideals is the dual ideal generated by their union. Furthermore, if $\mathcal{M}$ is the projective space associated with $\mathcal{D}$ (whose points are the maximal proper dual ideals of B), and if $\Phi(x) = \{M \mid x \in M \in \mathcal{M}\}$ for $x \in B$, then Φ maps B isomorphically onto a sublattice $\mathcal{B}$ of the lattice $\mathcal{A}$ of all subspaces of $\mathcal{M}$.

THEOREM 1.7. *If A is a projective lattice, then the following conditions are equivalent:*

(i) *A is isomorphic to a lattice of commuting equivalence relations.*
(ii) *A is isomorphic to a lattice of normal subgroups of a group.*
(iii) *A is isomorphic to a lattice of subgroups of an Abelian group.*
(iv) *A is isomorphic to the lattice of all subspaces of an Arguesian projective space.*
(v) *For any $a, b \in A^3$, if*
$$y = (a_0+a_1)\cdot(b_0+b_1)\cdot[(a_0+a_2)\cdot(b_0+b_2) + (a_1+a_2)\cdot(b_1+b_2)]\,,$$
then
$$(a_0+b_0)\cdot(a_1+b_1)\cdot(a_2+b_2) \leq a_0\cdot(y+a_1) + b_0\cdot(y+b_1)\,.$$

A result essentially equivalent to the finite dimensional case of Theorem 1.2 is contained in Menger [7], the present formulation can be found in Birkhoff [1] for the finite dimensional case, and in Frink [4] without any restriction on the dimension. Theorem 1.3 is stated in Mousinho [8], while the finite dimensional case of Theorem 1.5 was given in Birkhoff [1], and the extension to the infinite dimensional case in Frink [4]. Theorem 1.6 is the principal result of Frink [4]. As regards Theorem 1.7, it was first shown in Schützenberger [9] that Desargues' Theorem can be given the form of a lattice identity. The inequality (v) was first used in Jónsson [5], where it was shown that this condition is satisfied in every lattice of commuting equivalence relations. Combining this result with Theorem 1.5, one easily obtains Theorem 1.7.

The equivalence of the conditions (iv) and (v) in Theorem 1.7 suggests the following:

DEFINITION 1.8. *A lattice B is said to be Arguesian if it satisfies the following condition: For every $a, b \in B^3$, if*
$$y = (a_0+a_1)\cdot(b_0+b_1)\cdot[(a_0+a_2)\cdot(b_0+b_2) + (a_1+a_2)\cdot(b_1+b_2)]\,,$$
then
$$(a_0+b_0)\cdot(a_1+b_1)\cdot(a_2+b_2) \leq a_0\cdot(y+a_1) + b_0\cdot(y+b_1)\,.$$

THEOREM 1.9. *Every Arguesian lattice is modular.*

PROOF. Suppose B is an Arguesian lattice. Assuming that $u, v, w \in B$ and $u \leq w$, we apply the above definition with

$$a_0 = b_1 = v, \qquad a_1 = a_2 = b_0 = u, \qquad \text{and} \qquad b_2 = w\,,$$

and find that

$$(a_0+b_0)\cdot(a_1+b_1)\cdot(a_2+b_2) = (v+u)\cdot(u+v)\cdot(u+w) = (u+v)\cdot w\,,$$

$$y = (v+u)\cdot(u+v)\cdot[(v+u)\cdot(u+w) + (u+u)\cdot(v+w)] = (u+v)\cdot w\,,$$

$$a_0\cdot(y+a_1) + b_0\cdot(y+b_1) = v\cdot[(u+v)\cdot w+u] + u\cdot[(u+v)\cdot w+v]$$

$$= v\cdot w + u\,,$$

whence

$$(u+v)\cdot w \leq u+v\cdot w\,.$$

2. Perfect extensions of complemented modular lattices. We shall now study in some detail the connection between the two lattices $\mathcal{A}$ and $\mathcal{B}$ in Theorem 1.6.

DEFINITION 2.1. *We say that*

$$A \textit{ is a perfect extension of } B,$$

and that

$$B \textit{ is a regular sublattice of } A,$$

if the following conditions are satisfied:

(i) *A is a projective lattice and B is a complemented sublattice of A with the same zero and unit elements as A.*

(ii) *If I is any set, and if the elements $x_i \in B$ with $i \in I$ are such that*

$$\prod_{i \in I} x_i = 0\,,$$

then there exists a finite subset J of I such that

$$\prod_{i \in J} x_i = 0\,.$$

(iii) *If u is a finite dimensional element of A, if v is an atom of A, and if $u\cdot v = 0$, then there exist $x, y \in B$ such that $u \leq x$, $v \leq y$ and $x\cdot y = 0$.*

LEMMA 2.2. *If S is the projective space associated with a modular lattice A, n is a positive integer, and $p \in S^n$, then*

$$\{p_0\}+\{p_1\}+\ldots+\{p_{n-1}\} = \{q \mid p_0+p_1+\ldots+p_{n-1} \geq q \in S\}\,.$$

(It is essential here to distinguish between the point p_i of S and the corresponding one-element subspace $\{p_i\}$ of S. When no confusion is likely to arise, this distinction will sometimes not be made. The plus signs on the left of this equation refer to the addition in the lattice of all subspaces of S, while the plus signs on the right refer to the addition in A.)

Proof of Lemma 2.2. This is trivial in case $n = 1$ or $n = 2$. Assume that it holds for $n = m$, and consider the case in which $n = m+1$. Let

$$X = \{p_0\}+\{p_1\}+\ldots+\{p_{m-1}\} = \{q \mid p_0+p_1+\ldots+p_{m-1} \geq q \in S\},$$
$$Y = X+\{p_m\} \quad \text{and} \quad Z = \{q \mid p_0+p_1+\ldots+p_m \geq q \in S\}.$$

Then X, Y and Z are subspaces of S with $X \subseteq Y \subseteq Z$. For any $q \in$ we have

$$p_m \leq q+p_m \leq (p_0+p_1+\ldots+p_{m-1})+p_m,$$

and it follows by the modular law that

$$q+p_m = r+p_m, \quad \text{where} \quad r = (p_0+p_1+\ldots+p_{m-1})\cdot(q+p_m).$$

If $r = 0$, then $q = p_m \in Y$. If $r = q+p_m$, then $q \leq p_0+p_1+\ldots+p_{m-1}$, whence $q \in X \subseteq Y$. Finally, if $0 < r < q+p_m$, then r is an atom of A, and consequently $r \in X$. In this case

$$q \in \{r\}+\{p_m\} \subseteq X+\{p_m\} = Y.$$

Thus we have shown that $Z \subseteq Y$, and the proof is complete.

Theorem 2.3. *If B, $\mathcal{D}$, $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ and Φ are as in Theorem 1.6, then $\mathcal{B}$ is a regular sublattice of $\mathcal{A}$.*

Proof. Suppose I is a non-empty set, and let the elements $x_i \in B$ with $i \in I$ be such that

$$\bigcap_{i \in J} \Phi(x_i) \neq \Phi$$

whenever J is a finite subset of I. For every finite subset J of I we then have

$$\prod_{i \in J} x_i \neq 0.$$

It follows that the set $\{x_i \mid i \in I\}$ generates a proper dual ideal of B, and is therefore contained in a maximal proper dual ideal M of B. We infer that, for each $i \in I$,

$$x_i \in M, \quad \text{hence} \quad M \in \Phi(x_i).$$

Consequently

$$M \in \bigcap_{i \in I} \Phi(x_i).$$

Thus we have shown that the condition (ii) of Definition 2.1 holds with A and B replaced by $\mathcal{A}$ and $\mathcal{B}$.

Next suppose that U is a finite dimensional element of $\mathcal{A}$, and V is an atom of $\mathcal{A}$ with $U \cap V = \Phi$. Then

$$U = \{M_0\}+\{M_1\}+\ldots+\{M_n\} \quad \text{and} \quad V = \{N\},$$

with $M_0, M_1, \ldots, M_n, N \in \mathcal{M}$. By Lemma 2.2 we have

$$U = \{P \mid M_0 \cap M_1 \cap \ldots \cap M_n \subseteq P \in \mathcal{M}\}.$$

Hence the condition $U \cap V = \Phi$ means that

$$M_0 \cap M_1 \cap \ldots \cap M_n \nsubseteq N.$$

Choose $x \in M_0 \cap M_1 \cap \ldots \cap M_n$ with $x \notin N$. Then $x \cdot y = 0$ for some $y \in N$. Thus

$$U \subseteq \Phi(x) \in \mathcal{B}, \qquad V \subseteq \Phi(y) \in \mathcal{B} \qquad \text{and} \qquad \Phi(x) \cap \Phi(y) = \Phi(x \cdot y) = \Phi.$$

We have therefore verified the condition (iii) of Definition 2.1, and the proof is complete.

THEOREM 2.4. *Every complemented modular lattice is a regular sublattice of a projective lattice.*

PROOF. This is an immediate consequence of the fact that, according to Theorems 1.2, 1.6 and 2.3, every complemented modular lattice is isomorphic to a regular sublattice of a projective lattice.

LEMMA 2.5. *If A is a perfect extension of B, and if u is an atom of A, then*

$$u = \prod_{u \leq y \in B} y.$$

PROOF. Clearly

$$u \leq \prod_{u \leq y \in B} y.$$

On the other hand, if v is any atom of A distinct from u, then $u \cdot v = 0$, and it follows from the condition (iii) of Definition 2.1 that there exists an element $y \in B$ such that $u \leq y$ and $v \nleq y$. Consequently

$$v \cdot \prod_{u \leq y \in B} y = 0.$$

The conclusion now follows from the fact that A is complemented and atomistic.

THEOREM 2.6. *If A and A' are perfect extensions of B and B' respectively, and if φ maps B isomorphically onto B', then there exists a unique function ψ which maps A isomorphically onto A' in such a way that $\psi(x) = \varphi(x)$ for $x \in B$.*

PROOF. Let At and At' be the sets whose elements are the atoms of A and A' respectively. Using Lemma 2.5 we see that if ψ is a function which satisfies the required conditions, then

$$\psi(x) = \sum_{x \geq u \in At} \prod_{u \leq y \in B} \varphi(y) \qquad \text{for} \quad x \in A\,.$$

Hence there exists at most one such function.

Consider therefore the function ψ defined by this formula. Also let

$$\psi'(x') = \sum_{x' \geq u' \in At'} \prod_{u' \leq y' \in B} \varphi^{-1}(y') \qquad \text{for} \quad x' \in A'\,.$$

Then

$$\psi(u) = \prod_{u \leq y \in B} \varphi(y) \quad \text{for } u \in At, \qquad \psi'(u') = \prod_{u' \leq y' \in B'} \varphi^{-1}(y') \quad \text{for } u' \in At'\,,$$

$$\psi(x) = \sum_{x \geq u \in At} \psi(u) \quad \text{for } x \in A, \qquad \psi'(x') = \sum_{x' \geq u' \in At'} \psi'(u') \quad \text{for } x' \in A'\,.$$

We begin by showing that ψ establishes a one-to-one correspondence between At and At', and that $\psi'(\psi(u)) = u$ for $u \in At$.

If $u \in At$ and if L is any finite subset of the set

$$K = \{y \,|\, u \leq y \in B\}\,,$$

then

$$0 \neq \prod_{y \in L} y \in B\,,$$

so that

$$\prod_{y \in L} \varphi(y) = \varphi\Big(\prod_{y \in L} y\Big) \neq 0\,.$$

It follows by the condition (ii) of Definition 2.1 that

$$\psi(u) = \prod_{y \in K} \varphi(y) \neq 0\,.$$

Next consider two distinct elements u and v of At. Then $u \cdot v = 0$, and it follows from the condition (iii) of Definition 2.1 that there exist $y, z \in B$ such that

$$u \leq y, \qquad v \leq z \qquad \text{and} \qquad y \cdot z = 0\,.$$

Consequently

$$\psi(u) \cdot \psi(v) \leq \varphi(y) \cdot \varphi(z) = \varphi(y \cdot z) = \varphi(0) = 0, \qquad \psi(u) \cdot \psi(v) = 0\,.$$

We infer that ψ maps At univalently onto a family of pairwise disjoint non-zero elements of A'. Similarly ψ' maps At' univalently onto a family of pairwise disjoint non-zero elements of A.

Now consider again an element u of At, and suppose $\psi(u) \geq u' \in At'$. If L and L' are finite subsets of the sets

$$K = \{x \mid u \leq x \in B\} \qquad \text{and} \qquad K' = \{x' \mid u' \leq x' \in B'\}$$

respectively, then

$$u' = \psi(u) \cdot u' \leq \prod_{y \in L} \varphi(y) \cdot \prod_{y' \in L'} y'\,,$$

whence

$$\prod_{y \in L} \varphi(y) \cdot \prod_{y' \in L'} y' \neq 0 ,$$

$$\prod_{y \in L} y \cdot \prod_{y' \in L'} \varphi^{-1}(y') = \varphi^{-1}\Big(\prod_{y \in L} \varphi(y) \cdot \prod_{y' \in L'} y'\Big) \neq 0 .$$

It follows by the condition (ii) of Definition 2.1 that

$$\prod_{y \in K} y \cdot \prod_{y' \in K'} \varphi^{-1}(y') \neq 0 .$$

Observing that, by Lemma 2.5 and the definition of ψ',

$$\prod_{y \in K} y = u \qquad \text{and} \qquad \prod_{y' \in K'} \varphi^{-1}(y') = \psi'(u') ,$$

we infer that

$$u \leq \psi'(u') .$$

Similarly, if $\psi(u) \geq v' \in At'$ then $u \leq \psi'(v')$. Hence $\psi'(u') \cdot \psi'(v') \neq 0$, but we have already shown that this implies that $u' = v'$. Consequently $\psi(u) = u' \in At'$.

Thus ψ maps At univalently into At' and, similarly, ψ' maps At' univalently into At. The above argument also shows that, for any $u \in At$,

$$u \leq \psi'\big(\psi(u)\big) \in At, \qquad \text{hence} \qquad \psi'\big(\psi(u)\big) = u .$$

Analogously we have $\psi\big(\psi'(u')\big) = u'$ for $u' \in At'$, which shows that ψ actually maps At onto At'.

Next suppose $x \in A$, $u \in At$ and

$$\psi(u) \leq \psi(x) .$$

Then

$$\psi(u) \leq \sum_{x \geq v \in At} \psi(v) ,$$

and it follows from Definitions 1.1 and 2.1 that there exist a positive integer n and a sequence $w \in (At)^n$ such that $w_i \leq x$ for $i = 0, 1, \ldots, n-1$, and

$$\psi(u) \leq \sum_{i=0}^{n-1} \psi(w_i) .$$

We wish to show that

$$u \leq \sum_{i=0}^{n-1} w_i, \qquad \text{hence} \qquad u \leq x .$$

If this were not the case, then there would exist, by the condition (iii) of Definition 2.1, elements $y, z \in B$ such that

$$u \leq y, \qquad \sum_{i=0}^{n-1} w_i \leq z, \qquad \text{and} \qquad y \cdot z = 0 .$$

But this would imply that

$$\psi(u) \leq \varphi(y), \qquad \sum_{i=1}^{n-1} \psi(w_i) \leq \varphi(z), \qquad \text{and} \qquad \varphi(y) \cdot \varphi(z) = 0 ,$$

so that

$$\psi(u) \cdot \sum_{i=1}^{n-1} \psi(w_i) = 0 ,$$

which is a contradiction. We infer that, for any $x \in A$ and $u \in At$,

$$\psi(u) \leq \psi(x) \qquad \text{if and only if} \qquad u \leq x .$$

It follows that, for any $x \in A$,

$$\{\psi(u) \mid x \geq u \in At\} = \{u' \mid \psi(x) \geq u' \in At'\} ,$$

and hence

$$\psi'\big(\psi(x)\big) = \sum_{\psi(x) \geq u' \in At'} \psi'(u') = \sum_{x \geq u \in At} \psi'\big(\psi(u)\big) = \sum_{x \geq u \in At} u = x .$$

Similarly

$$\psi\big(\psi'(x')\big) = x' \qquad \text{for} \quad x' \in A' ,$$

and we infer that ψ maps A univalently onto A', and that $\psi' = \psi^{-1}$. Since ψ and ψ' are clearly monotonic, it follows that ψ maps A isomorphically onto A'.

Finally suppose $x \in B$. Then clearly $\psi(x) \leq \varphi(x)$. On the other hand, if $\varphi(x) \geq u' \in At'$, then

$$\psi'(u') \in At, \quad \psi'(u') \leq \varphi^{-1}\big(\varphi(x)\big) = x, \qquad \text{hence} \qquad u' = \psi\big(\psi'(u')\big) \leq \psi(x) .$$

Consequently $\psi(x) = \varphi(x)$ for $x \in B$, and the proof is complete.

COROLLARY 2.7. *If A and A' are perfect extensions of B, then there exists a unique function ψ which maps A isomorphically onto A' in such a way that $\psi(x) = x$ for $x \in B$.*

LEMMA 2.8. *Suppose A is a perfect extension of B. If $u, v \in A$ are finite dimensional with $u \cdot v = 0$, then there exist $x, y \in B$ such that $u \leq x$, $v \leq y$ and $x \cdot y = 0$.*

PROOF. This is trivial in case the dimension of v is 0. Suppose n is a positive integer, and assume the theorem to hold whenever the dimension of v is less than n. Consider the case when the dimension of v is n. Then there exist $p, w \in A$ such that p is an atom,

$$v = w + p \quad \text{and} \quad w \cdot p = 0 .$$

Then $u \cdot w = 0$ and $(u+w) \cdot p = 0$. Since the dimension of w is $n-1$ and the dimension of $u+w$ is finite, it follows from the inductive hypothesis and from the condition (iii) of Definition 2.1 that there exist $x_1, y_1, x_2, y_2 \in B$ such that

$$u \leq x_1, \quad w \leq y_1, \quad x_1 \cdot y_1 = 0, \quad u+w \leq x_2, \quad p \leq y_2, \quad \text{and} \quad x_2 \cdot y_2 = 0 .$$

Letting

$$x = x_1 \cdot x_2 \quad \text{and} \quad y = x_2 \cdot y_1 + y_2 ,$$

we infer that $x, y \in B$, $u \leq x$, $v \leq y$ and

$$x \cdot y = x_1 \cdot x_2 \cdot (x_2 \cdot y_1 + y_2) = x_1 \cdot (x_2 \cdot y_1 + x_2 \cdot y_2) = x_1 \cdot x_2 \cdot y_1 = 0 .$$

The conclusion follows by induction.

LEMMA 2.9. *Suppose A is a perfect extension of B and n is a positive integer. If the sequence $u \in A^n$ is independent, and if $u_0, u_1, \ldots, u_{n-1}$ are finite dimensional, then there exists an independent sequence $x \in B^n$ with $u \leq x$.*

PROOF. For $j = 1, 2, \ldots, n-1$ the element $u_0+u_1+\ldots+u_{j-1}$ is finite dimensional and

$$(u_0+u_1+\ldots+u_{j-1}) \cdot u_j = 0 .$$

Hence it follows from Lemma 2.8 that there exist elements $y_j, z_j \in B$ such that

$$u_0+u_1+\ldots+u_{j-1} \leq y_j, \quad u_j \leq z_j \quad \text{and} \quad y_j \cdot z_j = 0 .$$

For $j = 0, 1, \ldots, n-1$ we let

$$x_j = z_j \cdot y_{j+1} \cdot y_{j+2} \cdot \ldots \cdot y_{n-1} ,$$

and verify that $u_j \leq x_j \in B$ and

$$(x_0+x_1+\ldots+x_{j-1}) \cdot x_j \leq y_j \cdot z_j = 0 .$$

DEFINITION 2.10. *Suppose B is a lattice and n is a positive integer. A function f on B^n to B is called a polynomial function of rank n over B if f belongs to every class $\mathcal{K}$ with the following properties:*

(i) *If $j < n$, and if φ is the function on B^n to B such that*

$$\varphi(x) = x_j \quad \textit{for} \quad x \in B^n ,$$

then $\varphi \in \mathcal{K}$.

(ii) *If $f, g \in \mathcal{K}$, then $f+g, f \cdot g \in \mathcal{K}$.*

THEOREM 2.11. *Suppose A is a perfect extension of B, and let U be the set of all finite dimensional elements of A. If n is a positive integer and f is a polynomial function of rank n over A, then*

$$f(x) = \sum_{x \geq u \in U^n} \prod_{u \leq y \in B^n} f(y) \quad \textit{for} \quad x \in A^n .$$

Let $\mathcal{K}$ be the class of all polynomial functions of rank n over A, and for $j = 0, 1, \ldots, n-1$ let φ_j be the function on A^n to A such that

$$\varphi_j(x) = x_j \quad \text{for} \quad x \in A^n .$$

We divide the proof into four parts.

PART I. *If $f \in \mathcal{K}$ and $v \in U^n$, then $f(v) \in U$.*

PROOF. For any $f \in \mathcal{K}$ we clearly have

$$f(x) \leq x_0 + x_1 + \ldots + x_{n-1} \quad \text{whenever} \quad x \in A^n .$$

Hence if $v \in U^n$, then the dimension of $f(v)$ does not exceed the sum of the dimensions of $v_0, v_1, v_2, \ldots,$ and v_{n-1}. Consequently the dimension of $f(v)$ is finite.

PART II. *If $f \in \mathcal{K}$, $v \in U^n$ and $f(v) \leq z \in B$, then there exists $y \in B^n$ such that $v \leq y$ and $f(y) \leq z$.*

PROOF. Let $\mathcal{K}_1$ be the class of all functions $f \in \mathcal{K}$ for which this statement holds. Clearly $\varphi_j \in \mathcal{K}_1$ for $j = 0, 1, \ldots, n-1$. Suppose $f, g \in \mathcal{K}_1$, and let

$$h = f + g \quad \text{and} \quad k = f \cdot g .$$

If $v \in U^n$ and $h(v) \leq z \in B$, then

$$f(v) \leq z \quad \text{and} \quad g(v) \leq z .$$

Hence there exist $y', y'' \in B^n$ such that

$$v \leq y', \quad v \leq y'', \quad f(y') \leq z \quad \text{and} \quad g(y'') \leq z .$$

Letting $y = y' \cdot y''$ we infer that $v \leq y$ and

$$h(y) = f(y) + g(y) \leq f(y') + g(y'') \leq z .$$

Now suppose $v \in U^n$ and $k(v) \leq z \in B$. Let

$$u_0 = k(v) = f(v) \cdot g(v) ,$$

and choose $u_1, u_2 \in A$ so that

$$f(v) = u_0 + u_1, \quad g(v) = u_0 + u_2 \quad \text{and} \quad u_0 \cdot u_1 = u_0 \cdot u_2 = 0 .$$

Then the sequence $\langle u_0, u_1, u_2\rangle$ is independent. Since, by Part I, u_0, u_1 and u_2 are finite dimensional, it follows by Lemma 2.9 that there exists an independent sequence $x \in B^3$ such that $u \leq x$. We may assume that $x_0 \leq z$. Since

$$f(v) \leq x_0 + x_1 \qquad \text{and} \qquad g(v) \leq x_0 + x_2\,,$$

there exist $y', y'' \in B^n$ such that

$$v \leq y', \qquad v \leq y'', \qquad f(y') \leq x_0 + x_1, \qquad \text{and} \qquad g(y'') \leq x_0 + x_2\,.$$

Letting $y = y' \cdot y''$ we infer that $v \leq y \in B^n$ and

$$k(y) = f(y) \cdot g(y) \leq f(y') \cdot g(y'') \leq (x_0 + x_1) \cdot (x_0 + x_2) = x_0 \leq z\,.$$

We have shown that if $f, g \in \mathcal{K}_1$, then $f + g \in K_1$ and $f \cdot g \in \mathcal{K}_1$. Consequently $\mathcal{K}_1 = \mathcal{K}$, and the proof of Part II is complete.

PART III. *If $f \in \mathcal{K}$, then*

$$f(v) = \prod_{v \leq y \in B^n} f(y) \quad \textit{for} \quad v \in U^n\,.$$

PROOF. Suppose $v \in U^n$. If u is an atom of A such that $u \cdot f(v) = 0$, then there exists $z \in B$ such that $f(v) \leq z$ and $u \cdot z = 0$. It follows that we can find $y \in B^n$ such that $v \leq y$ and $f(y) \leq z$, and hence $u \cdot f(y) = 0$. Thus

$$u \cdot \prod_{v \leq y \in B^n} f(y) = 0\,.$$

We infer that

$$\prod_{v \leq y \in B^n} f(y) \leq f(v) \qquad \text{for} \qquad v \in U^n\,.$$

The inclusion in the opposite direction is obvious.

PART IV. *If $f \in \mathcal{K}$, then*

$$f(x) = \sum_{x \geq v \in U^n} f(v) \qquad \textit{for} \qquad x \in A^n\,.$$

PROOF. Let $\mathcal{K}_2$ be the class of all functions $f \in \mathcal{K}$ for which the above formula holds. It is clear that $\varphi_j \in \mathcal{K}_2$ for $j = 0, 1, \ldots, n-1$, and that $f + g \in \mathcal{K}_2$ whenever $f, g \in \mathcal{K}_2$. It is therefore sufficient to show that if $f, g \in \mathcal{K}_2$ and $k = f \cdot g$, then $k \in \mathcal{K}_2$.

For any sequence $x \in A^n$ we have

$$k(x) = f(x) \cdot g(x) = \Big(\sum_{x \geq v \in U^n} f(v)\Big) \cdot \Big(\sum_{x \geq v \in U^n} g(v)\Big)\,.$$

If u is an atom of A such that $u \leq k(x)$, then

$$u \leq \sum_{x \geq v \in U^n} f(v) \qquad \text{and} \qquad u \leq \sum_{x \geq v \in U^n} g(v) .$$

Hence the set $\{v \mid x \geq v \in U^n\}$ contains a finite subset C such that

$$u \leq \sum_{v \in C} f(v) \qquad \text{and} \qquad u \leq \sum_{v \in C} g(v) .$$

Letting

$$v' = \sum_{v \in C} v ,$$

we infer that

$$x \geq v' \in U^n, \qquad u \leq f(v') \qquad \text{and} \qquad u \leq g(v') .$$

Consequently

$$u \leq k(v') \leq \sum_{x \geq v \in U^n} k(v) .$$

Thus

$$k(x) \leq \sum_{x \geq v \in U^n} k(v) \qquad \text{for} \quad x \in A^n .$$

The inclusion in the opposite direction is obvious, and we have $k \in \mathcal{K}_2$.

The conclusion of the theorem is an immediate consequence of Parts III and IV.

THEOREM 2.12. *If A is a perfect extension of B, then every identity which holds in B is also satisfied in A.*

(It is possible to give a precise mathematical definition of what is meant by an identity and what it means that a given identity holds, or is satisfied, in a certain algebra. See in this connection McKinsey-Tarski [6]. For our present purpose, the intuitive notions will be sufficient.)

PROOF OF THEOREM 2.12. Suppose n is a positive integer and f and g are polynomial functions on A^n to A, and assume that $f(x) = g(x)$ for every $x \in B^n$. Then it follows from Theorem 2.11 that $f(x) = g(x)$ for every $x \in A^n$.

THEOREM 2.13. *Suppose A is a perfect extension of B. Then A is Arguesian if and only if B is Arguesian.*

PROOF. Since every inequality in a lattice is equivalent to an identity, we see that Arguesian lattices can be characterized by means of an identity. Hence it follows from Theorem 2.12, that if B is Arguesian, then so is A. The converse is obvious.

THEOREM 2.14. *If B is a complemented modular lattice, then the following conditions are equivalent:*

(i) *B is Arguesian.*
(ii) *B is isomorphic to a lattice of commuting equivalence relations.*
(iii) *B is isomorphic to a lattice of normal subgroups of a group.*
(iv) *B is isomorphic to a lattice of subgroups of an Abelian group.*
(v) *B is isomorphic to a lattice of subspaces of an Arguesian projective space.*

PROOF. It is obvious that (iv) implies (iii), which in turn implies (ii). It is also known (Jónsson [5]) that (ii) implies (i). If (i) holds, then it follows from Theorem 2.4 that B is a regular sublattice of a projective lattice A. We then use Theorem 2.13 to infer that A is Arguesian, whence it follows by Theorem 1.7 that (v) holds.

Finally assume (v). Then it follows from Theorems 1.2 and 1.7 that B is a sublattice of an Arguesian projective lattice A. By Theorem 1.5, A is a direct product of indecomposable projective lattices A_i with i in some set I. Clearly the factors A_i of A are Arguesian; in order to prove that (iv) holds, it is sufficient to show that each A_i is isomorphic to a lattice of subgroups of an Abelian group. If the dimension of A_i does not exceed two, then this is trivial; if the dimension of A_i is three or more, then this follows from Theorems 1.5 and 1.7.

3. A counterexample. In the proof of Theorem 2.14 the assumption that B be complemented was used only in order to show that (i) implies (v). Thus we see that, for an arbitrary lattice B, each of the conditions (ii)–(v) implies all the conditions which precede it. We shall now show that the word 'complemented' cannot be dropped from the hypothesis of this theorem. In fact, we are going to construct a five dimensional modular lattice which is isomorphic to a lattice of commuting equivalence relations, but not to a lattice of normal subgroups of a group.

We shall make use of the following result, which is a special case of Lemma 4.1 in Hall-Dilworth [3]:

LEMMA 3.1. *If C_0 and C_1 are three dimensional modular lattices, then there exist a five dimensional modular lattice B and elements $a, b \in B$ with the following properties:*
(i) *$a < b$ and $B = [0, b] \cup [a, 1]$.*
(ii) *$[0, b]$ is isomorphic to C_0.*
(iii) *$[a, 1]$ is isomorphic to C_1.*

In our application of this lemma, C_0 and C_1 will be taken to be the lattices of all vector subspaces of three dimensional vector spaces over division rings D_0 and D_1 with distinct positive characteristics. The proof of the fact that in this case B cannot be represented by normal subgroups of a group will be based on the following:

LEMMA 3.2. *Suppose C is the lattice of all vector subspaces of a three dimensional vector space over a division ring D with characteristic $p > 0$, and suppose F maps C isomorphically onto a lattice of normal subgroups of a group G. If x is an atom of C, and if $\alpha \in F(x)$, then $\alpha^p \in F(0)$.*

PROOF. The set S of all atoms of C is a non-degenerate Arguesian projective plane. Choose points z and q of S such that z, q and x are distinct but collinear. Let D' be the set of all points on the line $l = z+q$, except q. With the usual definition of addition and multiplication (cf. e.g., Veblen and Young [10, pp. 141 ff.]), D' is a division ring isomorphic to D; we may let z be the zero element of D'. It follows that in the additive group of D' every element except z is of order p, so that $p \cdot y = z$ for $y \in D'$.

Since $x \leq z+q$, we have $\alpha \in F(z+q)$, and there exists $\beta \in G$ such that $\beta \in F(z)$ and $\beta\alpha^{-1} \in F(q)$. We shall show that

$$\alpha^{-1}(\beta\alpha^{-1})^{n-1} \in F(n \cdot x) \tag{1}$$

for every positive integer n. This is clearly true for $n = 1$. Assume that it holds for $n = k$. Since $(k+1)\cdot x$ is the sum (in the division ring D') of $k \cdot x$ and x, there exist points y_0, y_1, y_2, y_3, not on l, such that no three of them are collinear, and such that

$$z = l\cdot(y_0+y_1), \qquad q = l\cdot(y_0+y_2) = l\cdot(y_1+y_3)\,,$$
$$x = l\cdot(y_1+y_2), \qquad k\cdot x = l\cdot(y_0+y_3), \qquad (k+l)\cdot x = l\cdot(y_2+y_3)\,.$$

Now $\beta \in F(y_0+y_1)$, and there exists $\gamma \in G$ such that $\gamma \in F(y_0)$ and $\gamma\beta^{-1} \in F(y_1)$. Consequently,

$$\alpha^{-1}\beta\gamma^{-1} = \alpha^{-1}(\gamma\beta^{-1})^{-1} \in F(x+y_1),$$
$$\alpha^{-1}\beta\gamma^{-1} = \big(\alpha^{-1}(\beta\alpha^{-1})\alpha\big)\gamma^{-1} \in F(q+y_0)\,,$$
$$\alpha^{-1}\beta\gamma^{-1} \in F\big((x+y_1)\cdot(q+y_0)\big) = F(y_2)\,,$$
$$\gamma\alpha^{-1}(\beta\alpha^{-1})^{k-1} \in F(y_0+k\cdot x), \qquad \gamma\alpha^{-1}(\beta\alpha^{-1})^{k-1} = (\gamma\beta^{-1})(\beta\alpha^{-1})^{k} \in F(y_1+q)\,,$$
$$\gamma\alpha^{-1}(\beta\alpha^{-1})^{k-1} \in F\big((y_0+k\cdot x)\cdot(y_1+q)\big) = F(y_3)\,,$$
$$\alpha^{-1}(\beta\alpha^{-1})^{k} = (\alpha^{-1}\beta\gamma^{-1})\big(\gamma\alpha^{-1}(\beta\alpha^{-1})^{k-1}\big) \in F(y_2+y_3)\,,$$
$$\alpha^{-1}(\beta\alpha^{-1})^{k} \in F\big(l\cdot(y_2+y_3)\big) = F\big((k+1)\cdot x\big)\,.$$

Thus (1) holds by induction.

Applying (1) with $n = p$, we see that $\alpha^{-1}(\beta\alpha^{-1})^{p-1} \in F(z)$. But since $\beta \in F(z)$ and $F(z)$ is a normal subgroup of G, this implies that $\alpha^p \in F(z)$. Inasmuch as $\alpha^p \in F(x)$ and $x \cdot z = 0$, we therefore have $\alpha^p \in F(0)$.

LEMMA 3.3. *Suppose p_0 and p_1 are distinct positive primes, and for $i = 0, 1$ suppose C_i is the lattice of all vector subspaces of a three dimensional vector space over a division ring with characteristic p_i. If B, a and b satisfy the conditions of Lemma* 3.1, *then B is not isomorphic to a lattice of normal subgroups of a group.*

PROOF. Assume that there exists a function F which maps B isomorphically onto a lattice of normal subgroups of a group G. Note that the dimension of b is 3 and the dimension of a is 2. Hence there exists an atom u of $[0, b]$ such that $a \cdot u = 0$ and $a + u = b$. Since $u \nleq a$, there exists $\alpha \in G$ such that $\alpha \in F(u)$ and $\alpha \notin F(a)$. By Lemma 3.2 and the condition (ii) of Lemma 3.1 we have $\alpha^{p_0} \in F(0)$ and hence $\alpha^{p_0} \in F(a)$. On the other hand we have $u \leq b$, whence $\alpha \in F(b)$. Noting that b is an atom of the lattice $[a, 1]$, we infer from Lemma 3.2 and the condition (iii) of Lemma 3.1 that $\alpha^{p_1} \in F(a)$. Inasmuch as p_0 and p_1 are distinct primes, it follows that $\alpha \in F(a)$, contrary to our choice of α. Thus no such isomorphism F exists.

LEMMA 3.4. *Suppose B is a lattice, I is an ideal of B, J is a dual ideal of B, $B = I \cup J$ and $I \cap J \neq \Phi$. If F is a function on B to a lattice C, and if F maps both I and J homomorphically* (*isomorphically*) *into C, then F maps B homomorphically* (*isomorphically*) *into C.*

PROOF. Choose an element $c \in I \cap J$. If $x \in I$ and $y \in J$, then $(x+c) \cdot y \in I \cap J$. Hence

$$\begin{aligned} F(x+y) &= F\big(x + (x+c)\cdot y + y\big) = F\big(x + (x+c)\cdot y\big) + F(y) \\ &= F(x) + F\big((x+c)\cdot y\big) + F(y) = F(x) + F\big((x+c)\cdot y + y\big) \\ &= F(x) + F(y)\,. \end{aligned}$$

Dually, $F(x \cdot y) = F(x) \cdot F(y)$. Consequently F maps B homomorphically into C. Finally, if F is univalent on each of the sets I and J, and if the elements $x \in I$ and $y \in J$ are such that $F(x) = F(y)$, then

$$F(x) = F(x+c) \cdot F(x) = F(x+c) \cdot F(y) = F\big((x+c)\cdot y\big)\,,$$

and hence also $F(y) = F\big((x+c)\cdot y\big)$. This implies that

$$x = (x+c) \cdot y = y\,.$$

Thus F maps B isomorphically into C.

LEMMA 3.5. *For $i = 0, 1$ suppose C_i is the lattice of all vector subspaces of a three dimensional vector space over a countably infinite division ring D_i. If B, a and b are as in Lemma 3.1, then B is isomorphic to a lattice of commuting equivalence relations.*

PROOF. It follows from our assumptions that there exists a function Φ which maps the lattice $[0, b]$ isomorphically onto the lattice of all vector subspaces of a three dimensional vector space U_0 over D_0. For $x \in [0, b]$ let

$$F_0(x) = \{\langle \alpha, \beta \rangle \mid \alpha, \beta \in U_0 \text{ and } \alpha - \beta \in \Phi(x)\} .$$

Then F_0 maps $[0, b]$ isomorphically onto a lattice of commuting equivalence relations over U_0.

Clearly U_0 is countably infinite. We shall show that if $x < y \leq b$, then each $F_0(y)$ class contains infinitely many $F_0(x)$ classes. In fact, letting Y be an $F_0(y)$ class, choose $\alpha \in Y$, and choose $\gamma \in \Phi(y)$ such that $\gamma \notin \Phi(x)$. For each $d \in D_0$ we have $d\gamma \in \Phi(y)$, and hence $\alpha + d\gamma \in Y$. Furthermore, if $d, d' \in D_0$ and $d \neq d'$, then $d\gamma - d'\gamma \notin \Phi(x)$, so that $\alpha + d\gamma$ and $\alpha + d'\gamma$ must belong to different $F_0(x)$ classes. Hence Y contains infinitely many $F_0(x)$ classes.

We now consider countably many replicas $\langle F_i, U_i \rangle$, $i = 1, 2, \ldots$, of the ordered pair $\langle F_0, U_0 \rangle$. More specifically, we assume that the sets U_i are countably infinite and pairwise disjoint, and that F_i maps $[0, b]$ isomorphically onto a lattice of commuting equivalence relations over U_i in such a way that if $x < y \leq b$, then each $F_i(y)$ class contains infinitely many $F_i(x)$ classes. Letting

$$U = \bigcup_{i < \infty} U_i \quad \text{and} \quad F(x) = \bigcup_{i < \infty} F_i(x) \quad \text{for} \quad x \in [0, b] ,$$

we infer that U is countably infinite and that F maps $[0, b]$ isomorphically onto a lattice of commuting equivalence relations over U in such a way that:

(1) There are infinitely many $F(b)$ classes.
(2) Each $F(b)$ class contains infinitely many $F(a)$ classes.
(3) Each $F(a)$ class is infinite.

By exactly the same method as was used to obtain U_0 and F_0, we can find a countably infinite set V_0 and a function G_0 which maps $[a, 1]$ isomorphically onto a lattice of commuting equivalence relations over V_0 in such a way that there are infinitely many $G_0(b)$ classes, and each $G_0(b)$ class contains infinitely many $G_0(a)$ classes. With the elements α' of V_0 we now associate pairwise disjoint, countably infinite sets

$\Psi(\alpha')$, let

$$V = \bigcup_{\alpha' \in V_0} \Psi(\alpha') ,$$

and for each $x \in [a, 1]$ define the relation $G(x)$ by the condition that $\langle\alpha, \beta\rangle \in G(x)$ if and only if $\alpha, \beta \in V$ and $\langle\alpha', \beta'\rangle \in G_0(x)$ where α' and β' are the unique elements of V_0 such that $\alpha \in \Psi(\alpha')$ and $\beta \in \Psi(\beta')$. Then V is a countably infinite set, and G maps $[a, 1]$ isomorphically onto a lattice of commuting equivalence relations over V in such a way that:

(4) There are infinitely many $G(b)$ classes.
(5) Each $G(b)$ class contains infinitely many $G(a)$ classes.
(6) Each $G(a)$ class is infinite.

From (1)–(6) we see that there exists a univalent function on U onto V which maps each $F(b)$ class onto a $G(b)$ class and each $F(a)$ class onto a $G(a)$ class. We may therefore assume that

$$U = V, \qquad F(a) = G(a) \qquad \text{and} \qquad F(b) = G(b) .$$

Since a and b are the only elements common to $[0, b]$ and $[a, 1]$, it follows that there exists a function H on B such that

$$H(x) = \begin{cases} F(x) & \text{for} \quad x \in [0, b] , \\ G(x) & \text{for} \quad x \in [a, 1] . \end{cases}$$

Since H maps the ideal $[0, b]$ and the dual ideal $[a, 1]$ isomorphically into the lattice of all equivalence relations over U, it follows by Lemma 3.4 that H maps B isomorphically into the lattice of all equivalence relations over U. Furthermore, if either $x, y \in [0, b]$ or $x, y \in [a, 1]$, then $H(x)$ and $H(y)$ commute. Finally, if $x \in [0, b]$ and $y \in [a, 1]$, then

$$\begin{aligned} H(x);H(y) = H(x);H(a+y) = H(x);H(a);H(y) = H(x+a);H(y) \\ = H(x+a+y) = H(x+y) \end{aligned}$$

and, similarly,

$$H(y);H(x) = H(x+y) .$$

Thus $H(x)$ and $H(y)$ commute in this case also, and we conclude that H maps B isomorphically onto a lattice of commuting equivalence relations over U.

THEOREM 3.6. *There exists a five dimensional modular lattice B which is isomorphic to a lattice of commuting equivalence relations, but not to a lattice of normal subgroups of a group.*

PROOF: by Lemmas 3.1, 3.3 and 3.5.

Since the division rings D_0 and D_1 in Lemma 3.5 were assumed to be countably infinite, we see that the lattice B in Theorem 3.6 will also be countably infinite. However, the prime fields of D_0 and D_1 are finite, and applying Lemma 3.1 to them we obtain a finite lattice B' which is isomorphic to a sublattice of B, and therefore isomorphic to a lattice of commuting equivalence relations. On the other hand it follows from Lemma 3.3 that B' is not isomorphic to a lattice of normal subgroups of a group. Thus we see that even for finite modular lattices the conditions (ii) and (iii) of Theorem 2.14 are not equivalent.

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