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LINEARIZATION OF PRODUCTS OF JACOBI POLYNOMIALS

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1. Introduction.

In dealing with eigenvalue problems one usually has to introduce an auxiliary set of orthogonal functions as closely related to the true eigenfunctions as possible. A part of the technics is to evaluate integrals over products of functions from the orthogonal set times some function appearing in the eigenvalue differential equation.

To fix the idea consider the differential equation

$$\left\{\frac{d^2}{dx^2}+k^2+Ae^{-x}/x\right\}y = 0 . \tag{1.1}$$

The function e^{-x}/x is called the Yukawa potential, and the equation constitutes in nuclear physics a very difficult scattering problem. To solve the problem one has to find solutions $y(x)$ having the properties $y(0)=0$, $y(x) \to \sin(kx+\eta)$ as $x \to \infty$, and to determine η which is called the scattering phase shift.

Consider next the equation,

$$\left\{\frac{d^2}{dx^2}-\varkappa^2+Ae^{-x}/x\right\}y = 0 , \tag{1.2}$$

with boundary conditions $y(0)=0$, $y(\infty)=0$. It constitutes an ordinary Sturm–Liouville eigenvalue problem with discrete eigenvalues if A is taken to be the eigenvalue parameter.

Moreover, if the eigenvalues $A_n(\varkappa)$ were obtainable as analytical functions of the parameter $\varkappa$, the functions $A_n(\pm ik)$ could be used for setting up an explicit formula for the scattering phase shift $\eta=\eta(k,A)$.

Finally consider a related equation

$$\left\{\frac{d^2}{dx^2}-\varkappa^2+\frac{A_n}{e^x-1}\right\}y_n = 0 , \tag{1.3}$$

with

$$A_n = (n+1)(n+1+2\varkappa), \qquad n = 0,1,2,\ldots . \tag{1.3a}$$

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The eigenfunctions $y_n(x)$ are polynomials in e^{-x} times $e^{-\varkappa x}(1-e^{-x})$. Applying the functions for the purpose of solving (1.2) one has to evaluate a double series of integrals

$$\int \frac{e^{-x}}{x} y_n y_m \, dx .$$

It is for this purpose that an expansion of products of the above polynomials in terms of the same polynomials is desireable.

Put

$$y_n = e^{-\varkappa x}(1-e^{-x})z_n, \qquad \xi = e^{-x} . \tag{1.4}$$

Then (1.3) turns into

$$\left\{\xi(1-\xi)\frac{d^2}{d\xi^2} + [1+2\varkappa-(3+2\varkappa)\xi]\frac{d}{d\xi} + n(n+2\varkappa+2)\right\} z_n = 0 . \tag{1.5}$$

Comparing with the hypergeometrical equation for Jacobi polynomials

$$x(1-x)y_n'' + [q-(p+1)x]y_n' + n(n+p)y_n = 0 , \tag{1.6}$$

it is seen that our $z_n(\xi)$ are Jacobi polynomials of degree n with parameter values $q=1+2\varkappa$ and $p=2+2\varkappa$.

The Jacobi polynomials themselves are defined for $n \geqq 0$ by the series

$$y_n(x) = F(-n,n+p,q,x) = \sum_{\nu=0}^{n} (-x)^\nu \binom{n}{\nu} \frac{\binom{n+p-1+\nu}{\nu}}{\binom{q-1+\nu}{\nu}}, \tag{1.7}$$

$$p+1 > q > 0 .$$

For convenience we define $y_{-1}=0$. Denoting the double set of product functions by

$$Y_{nm} = y_n y_m \tag{1.8}$$

the linear form

$$Y_{nm} = \sum_{k=n-m}^{n+m} c_k y_k, \qquad n \geqq m , \tag{1.9}$$

is desired.

2. Historical notes and indication of general results.

Inspecting the mathematical litterature it appears that very little can be found about the problem. In Whittaker and Watson [7, p. 331] an example is given for Legendre "coefficients", i.e., for ordinary Legendre polynomials $P_n(x)$. It refers to the work of J. C. Adams [1, p. 63] from 1878.

It is true that our problem is equivalent to evaluating certain integrals over triple products of hypergeometric functions. In this field something more has been found, however nothing of the kind equivalent to our problem. A reference list is given at the end of the present paper to show the extent of our inspections (Ferrers [2, p. 56], Gaunt [3, p. 192], Hobson [4, p. 87], Neumann [5, part 2, p. 91], Todhunter [6]).

The Legendre polynomials $P_n(\xi)$ and also the generalized polynomials $P_n^{(m)}(\xi)$ may be described in two ways as Jacobi polynomials. The simplest way is to put $\xi = 2x-1$ and consider the polynomials functions of the variable x, $0 \leqq x \leqq 1$. Then it is found that the

$$\text{Case I:} \quad p = 2q-1 ,$$

comprises all polynomials under the general name of Gegenbauer polynomials of which all ordinary and associated Legendre polynomials and even the Tschebyscheff polynomials are only particular cases.

Moreover, quite generally in the above Case I a linearization according to equ. (1.9) is possible in the elementary sense that a two term recurrence formula for the coefficients of the expansion will permit to establish the formulae in explicit form. Hence, our main problem can be solved for an extensive class of important functions.

$$\text{Case II:} \quad p = 2q ,$$

for which the same results are true, does not comprise so well-known functions. However, in many physical applications they may be of comparable importance. In particular, for $\varkappa = 0$ the polynomials of our equ. (1.5) fall into that category.

The other way of expressing Legendre or, more generally, Gegenbauer polynomials, in terms of Jacobi polynomials is to consider even polynomial functions of the new variables $x = \xi^2$ or $x = 1-\xi^2$. In this way it is found that there are still two cases:

$$\left.\begin{array}{ll}\text{Case III:} & q = \frac{1}{2} \\ \text{Case IV:} & q = p+\frac{1}{2}\end{array}\right\} \text{ no restriction on } p, \text{ apart from (1.7)}$$

for which elementary expansions are possible.

Apart from the above special cases it is found that one has to be satisfied with an implicit procedure based upon a three-term recurrence formula for the expansion coefficients of equ. (1.9).

3. General formulae for Jacobi polynomials.

In the course of the investigation we shall frequently need some general formulae. These are:

Recurrence formula:

(3.1)
$$-xy_n = \frac{(n+q)(n+p)}{(2n+p)(2n+1+p)}[y_{n+1}-y_n] - \frac{n(n+p-q)}{(2n+p)(2n-1+p)}[y_n-y_{n-1}]\,.$$

Differential formula:

$$(3.2)\quad x(1-x)y_n' = \frac{n(n+p)}{2n+p}\left\{\frac{n+q}{2n+1+p}[y_{n+1}-y_n] + \frac{n+p-q}{2n-1+p}[y_n-y_{n-1}]\right\}.$$

In the special cases I and II the formulae are seen to become considerably simpler which is not surprising from what has been said above.

In this section we shall also give the values of the lowest and highest coefficients of the expansion (1.9). The latter can be found simply from the values of the highest coefficients of y_n, y_m and y_{n+m}. The result is

$$(3.3)\qquad c_{n+m} = \frac{\binom{2n+p-1}{n}\binom{2m+p-1}{m}\binom{n+m+q-1}{n+m}}{\binom{n+q-1}{n}\binom{m+q-1}{m}\binom{2n+2m+p-1}{n+m}}\,.$$

To determine c_{n-m}, $m \leqq n$, we need the orthonormal relation

$$(3.4)\qquad \int_0^1 x^{q-1}(1-x)^{p-q}y_ny_m\,dx = N_n\delta_{nm}\,,$$

$$(3.5)\qquad N_n = \frac{n!\,[(q-1)!]^2\,(n+p-q)!}{(n+p-1)!\,(n+q-1)!\,(2n+p)}\,,$$

which is fairly easily obtainable from the equivalent definition of the polynomials

$$(3.6)\qquad y_n(x) = \frac{(q-1)!}{(n+q-1)!}\,x^{1-q}(1-x)^{q-p}\frac{d^n}{dx^n}\,x^{n+q-1}(1-x)^{n+p-q}\,.$$

From the obvious equation

$$(3.7)\qquad N_{n-m}c_{n-m} = \int_0^1 x^{q-1}(1-x)^{p-q}y_{n-m}y_my_n\,dx\,,$$

writing

$$N_n = \frac{\binom{2n+p-1}{n}}{\binom{n+q-1}{n}}\cdot\frac{n!\,(q-1)!\,(n+p-q)!}{(2n+p)!}$$

and comparing the coefficients of x^n in $y_{n-m}y_m$ and y_n, it is found that

$$(3.8)\qquad c_{n-m} = \frac{\binom{2m+p-1}{m}}{\binom{m+q-1}{m}} \cdot \frac{n!\,(2n-2m+p)!\,(n+p-q)!}{(n-m)!\,(2n+p)!\,(n-m+p-q)!}$$

$$= \frac{\binom{2m+p-1}{m}}{\binom{m+q-1}{m}} \frac{\binom{n}{m}}{\binom{2m}{m}} \frac{\binom{n+p-q}{m}}{\binom{2n+p}{2m}}.$$

In the general case of arbitrary p and q only a few additional coefficients like c_{n+m-1} and c_{n-m+1} could be determined from the above triple product integral method. We therefore turn to a widely different method.

4. The differential equation for the product function.

In addition to (1.6) we need the differentiated equation.

$$(4.1)\quad x(1-x)\,y_n''' + [q+1-(p+3)x]\,y_n'' + [n(n+p)-(p+1)]\,y_n' = 0\,.$$

Combining the 2. and 3. order equations both for y_n and y_m we arrive at the equation

$$(4.2)\qquad L_3 Y_{nm} + [n(n+p)-m(m+p)]\cdot x(1-x)[y_n y_m' - y_m y_n'] = 0\,,$$

where L_3 is a third order differential operator

$$(4.2a)\qquad \begin{aligned} L_3 y = {} & x^2(1-x)^2\,y''' + \\ & + 3[q-(p+1)x]x(1-x)\,y'' + \\ & + [q-(p+1)x][2q-1-2px]\,y' + \\ & + [2n(n+p)+2m(m+p)-(p+1)x]\,y' + \\ & + [n(n+p)+m(m+p)](2q-1-2px)\,y\,. \end{aligned}$$

The appropriate first order differential operator for removing the last term of equ. (4.2) is

(4.3)

$$L_1 = x^{2-q}(1-x)^{q-p+1}\frac{d}{dx}\,x^{q-1}(1-x)^{p-q} = x(1-x)\frac{d}{dx} + [q-1-(p-1)x]\,.$$

We shall, however, prefer the second order operator

$$(4.4)\qquad L_2 = \frac{d}{dx}L_1 = x(1-x)\frac{d^2}{dx^2} + [q-(p+1)x]\frac{d}{dx} - (p-1)\,,$$

which has the eigenvalues $-(n+1)(n+p-1)$ when applied to the polynomials $y_n(x)$.

We need write neither the fourth nor the fifth order differential equation for Y_{nm}. It is simpler to evaluate the third order differential expression $L_3 y_k$ for any of the functions entering the expansion (1.9). Then the equation to be solved with respect to the expansion coefficients c_k is

$$\sum_{k=n-m}^{n+m} c_k\{L_2 L_3 y_k + [n(n+p)-m(m+p)]^2 \frac{d}{dx}[x(1-x)y_k]\} = 0 . \tag{4.5}$$

Consider first, using (3.1) and (3.2),

$$\begin{aligned} \frac{d}{dx}x(1-x)y_k &= x(1-x)y_k' + (1-2x)y_k \\ &= y_k + \frac{(k+q)(k+p)(k+2)}{(2k+p)(2k+1+p)}(y_{k+1}-y_k) + \\ &\quad + \frac{k(k+p-q)(k+p-2)}{(2k-1+p)(2k+p)}(y_k - y_{k-1}) \\ &= \frac{(k+q)(k+p)(k+2)}{(2k+p)(2k+1+p)}y_{k+1} - \frac{(k+p-q)k(k+p-2)}{(2k-1+p)(2k+p)}y_{k-1} - \\ &\quad - \frac{(2q-1-p)(k+1)(k+p-1)}{(2k-1+p)(2k+1+p)}y_k \\ &= L_2\left\{-\frac{(k+q)y_{k+1}}{(2k+p)(2k+1+p)} + \frac{(k+p-q)y_{k-1}}{(2k-1+p)(2k+p)} + \right. \\ &\quad \left. + \frac{(2q-1-p)y_k}{(2k-1+p)(2k+1+p)}\right\}. \end{aligned} \tag{4.6}$$

The latter expression is obtained by considering the eigenvalues of L_2 with respect to y_{k+1}, y_{k-1}, y_k, equ. (4.4). Putting the operator L_2 outside in equ. (4.5), it can be dropped, since all its eigenvalues are different from zero (except for $k=0$, $p=1$ and $k=1$, $p=0$, cases which might be considered separately).

We now apply equ. (4.2a) together with (4.1) and (1.6) to y_k to obtain

$$\begin{aligned} L_3 y_k &= [2n(n+p)+2m(m+p)-k(k+p)]x(1-x)y_k' + \\ &\quad + [n(n+p)+m(m+p)-k(k+p)](2q-1-2px)y_k . \end{aligned} \tag{4.7}$$

Again, using (3.1) and (3.2)

$$(4.8)\quad L_3 y_k = [n(n+p)+m(m+p)-k(k+p)](2q-1)y_k + \\
+\frac{(k+p)^2\,(k+q)}{(2k+p)(2k+1+p)}[2(n+\tfrac{1}{2}p)^2+2(m+\tfrac{1}{2}p)^2-(k+p)^2](y_{k+1}-y_k)+ \\
+\frac{k^2(k+p-q)}{(2k-1+p)(2k+p)}[2(n+\tfrac{1}{2}p)^2+2(m+\tfrac{1}{2}p)^2-k^2](y_k-y_{k-1}) \\
=\frac{(k+p)^2(k+q)}{(2k+p)(2k+1+p)}[2(n+\tfrac{1}{2}p)^2+2(m+\tfrac{1}{2}p)^2-(k+p)^2]y_{k+1}- \\
-\frac{k^2(k+p-q)}{(2k-1+p)(2k+p)}[2(n+\tfrac{1}{2}p)^2+2(m+\tfrac{1}{2}p)^2-k^2]y_{k-1}+ \\
+\frac{(2q-1-p)(k+p)^2}{2(2k+1+p)}[2(n+\tfrac{1}{2}p)^2+2(m+\tfrac{1}{2}p)^2-(k+p)^2]y_k- \\
-\frac{(2q-1-p)k^2}{2(2k-1+p)}[2(n+\tfrac{1}{2}p)^2+2(m+\tfrac{1}{2}p)^2-k^2]y_k+ \\
+(1-p)(2q-1-p)[n(n+p)+m(m+p)+k(k+p)]y_k\,.$$

We now replace equ. (4.5) by

$$(4.9)\qquad \sum_k c_k L_2 L y_k = 0\,,$$

$$Ly_k = L_3 y_k + [n(n+p)-m(m+p)]^2 L_2^{-1}\frac{d}{dx}[x(1-x)y_k]\,,$$

and obtain by means of (4.6) and (4.8)

(4.10)

$$Ly_k = \frac{k+q}{(2k+p)(2k+1+p)}[(n+m+p)^2-(k+p)^2][(k+p)^2-(n-m)^2]y_{k+1}- \\
-\frac{k+p-q}{(2k-1+p)(2k+p)}[(n+m+p)^2-k^2][k^2-(n-m)^2]y_{k-1}+ \\
+\frac{2q-1-p}{2(2k+1+p)}[(n+m+p)^2-(k+p)^2][(k+p)^2-(n-m)^2]y_k- \\
-\frac{2q-1-p}{2(2k-1+p)}[(n+m+p)^2-k^2][k^2-(n-m)^2]y_k+ \\
+(2q-1-p)(1-p)[n(n+p)+m(m+p)-k(k+p)]y_k\,.$$

We now observe that as

$$(4.11)\qquad \begin{aligned}(k+p)^2 &= (k+1)^2+(2k+1+p)(p-1)\,,\\ k^2 &= (k-1+p)^2-(2k-1+p)(p-1)\,,\end{aligned}$$

and

(4.12)
$$n(n+p)+m(m+p)-k(k+p) = \tfrac{1}{2}[(n+m+p)^2+(n-m)^2-(k+p)^2-k^2] ,$$

Ly_k may be written in two different forms of which we shall only repeat the terms containing y_k, namely,

(4.10a)
$$\frac{2q-1-p}{2(2k+1+p)}[(n+m+p)^2-(k+p)^2][(k+1)^2-(n-m)^2]y_k - $$
$$-\frac{2q-1-p}{2(2k-1+p)}[(n+m+p)^2-(k-1+p)^2][k^2-(n-m)^2]y_k$$
$$= \frac{2q-1-p}{2(2k+1+p)}[(n+m+p)^2-(k+1)^2][(k+p)^2-(n-m)^2]y_k - $$
$$-\frac{2q-1-p}{2(2k-1+p)}[(n+m+p)^2-k^2][(k-1+p)^2-(n-m)^2]y_k .$$

From (4.9), (4.10) and (4.10a) the following recurrence formula is obtained:

(4.13)
$$\frac{2k-2+2q}{(2k-2+p)(2k-1+p)}[(n+m+p)^2-(k-1+p)^2][(k-1+p)^2-(n-m)^2]c_{k-1} - $$
$$-\frac{2k+2+2p-2q}{(2k+1+p)(2k+2+p)}[(n+m+p)^2-(k+1)^2][(k+1)^2-(n-m)^2]c_{k+1} + $$
$$+\frac{2q-p-1}{2k+1+p}[(n+m+p)^2-(k+p)^2][(k+1)^2-(n-m)^2]c_k - $$
$$-\frac{2q-p-1}{2k-1+p}[(n+m+p)^2-(k-1+p)^2][k^2-(n-m)^2]c_k = 0 ,$$

together with an alternative formula using the last expression in (4.10a).

5. Gegenbauer, Tschebyscheff and Legendre polynomials.

From one of the known coefficients c_{n-m} or c_{n+m} the other coefficients in (1.9) can be computed by means of the above 3-term recurrence formula. We first consider the simplest

Case I: $2q = p+1$, Gegenbauer type .

The recurrence formula is greatly simplified and can be written

(5.1)
$$\frac{c_{k+2}}{c_k} = \frac{(n+m+p)^2-(k+p)^2}{(n+m+p)^2-(k+2)^2} \cdot \frac{(k+p)^2-(n-m)^2}{(k+2)^2-(n-m)^2} \cdot \frac{2k+4+p}{2k+p} .$$

Since $c_{n-m-1}=0$, it follows that $c_{n-m+2k+1}=0$, and we shall therefore use it in the form

$$(5.2)\qquad \frac{c_{n-m+2k+2}}{c_{n-m+2k}} = \frac{(n+p+k)(m-k)}{(n+\frac{1}{2}p+k+1)(m+\frac{1}{2}p-k-1)} \cdot \frac{(n-m+\frac{1}{2}p+k)(k+\frac{1}{2}p)}{(n-m+k+1)(k+1)} \cdot \frac{n-m+\frac{1}{2}p+2k+2}{n-m+\frac{1}{2}p+2k}.$$

The explicit expression for the coefficients becomes

$$(5.3)\qquad c_{n-m+2k} = \frac{\binom{k+\frac{1}{2}p-1}{k}\binom{m-k+\frac{1}{2}p}{m-k}\binom{n-m+k+\frac{1}{2}p-1}{n-m+k}\binom{n+k+p-1}{n+k}}{\binom{m+p-1}{m}\binom{n+p-1}{n}\binom{n+k+\frac{1}{2}p-1}{n+k}} \cdot \frac{n-m+\frac{1}{2}p+2k}{n+\frac{1}{2}p+k}.$$

The end coefficients

$$(5.4)\qquad c_{n-m} = \frac{\binom{m+\frac{1}{2}p-1}{m}\binom{n-m+\frac{1}{2}p}{n-m}}{\binom{m+p-1}{m}\binom{n+\frac{1}{2}p}{n}},$$

$$c_{n+m} = \frac{\binom{m+\frac{1}{2}p-1}{m}\binom{n+m+p-1}{n+m}}{\binom{m+p-1}{m}\binom{n+m+\frac{1}{2}p-1}{n+m}}$$

are seen to be the same as those obtained from (3.8) and (3.4) putting $q=\frac{1}{2}(p+1)$ and using auxiliary equations of the type

$$(5.5)\qquad (2m+p-1)! = \frac{2^{2m}(p-1)!\,\bigl(m+\frac{1}{2}(p-1)\bigr)!\,(m+\frac{1}{2}p-1)!}{\bigl(\frac{1}{2}(p-1)\bigr)!\,(\frac{1}{2}p-1)!}.$$

The simplest expressions for the coefficients are obtained for even integral p, that is, for the Tschebyscheff ($p=0$) and associated polynomials:

$$(5.6)\qquad \begin{aligned} p &= 0, & c_{n-m+2k} &= \tfrac{1}{2}(\delta_{ko}+\delta_{km}), \\ p &= 2, & c_{n-m+2k} &= \frac{n-m+2k+1}{(m+1)(n+1)}. \end{aligned}$$

In both cases the equation

$$(5.7)\qquad \sum_{k=0}^{m} c_{n-m+2k} = 1$$

is easily seen to hold. For arbitrary p-values it can be shown that

$$(5.8) \qquad \sum_{k=0}^{m} c_{n-m+2k} = \sum_{k=0}^{m-1} c_{n-(m-1)+2k} = \ldots = 1$$

and, hence, that (5.7) is generally valid, but we shall omit this proof.

For odd integral p we have the next simplest case which comprises the Legendre and associated polynomials. In particular for $p=1$ we have

$$(5.9) \quad \begin{cases} P_m(\xi)P_n(\xi) = \sum_{k=0}^{m} c_{n-m+2k} P_{n-m+2k}(\xi)\,, \\ c_{n-m+2k} = \dfrac{\binom{k-\frac{1}{2}}{k}\binom{m-k-\frac{1}{2}}{m-k}\binom{n-m+k-\frac{1}{2}}{n-m-k}}{\binom{n+k-\frac{1}{2}}{n+k}} \cdot \dfrac{n-m+\frac{1}{2}+2k}{n+\frac{1}{2}+k}\,, \end{cases}$$

and this is the only linearization formula for Jacobi polynomials formerly known.

For orientation we add the case

$$(5.9\text{a}) \qquad p = 3,$$

$$c_{n-m+2k} = \frac{\binom{k+\frac{1}{2}}{k}\binom{m-k+\frac{1}{2}}{m-k}\binom{n-m+k+\frac{1}{2}}{n-m+k}}{\binom{n+k+\frac{3}{2}}{n+k+2}} \cdot \frac{n-m+\frac{3}{2}+2k}{(m+1)(m+2)(n+1)(n+2)}\,.$$

6. Simplified Jacobi polynomials of the non-symmetric type.

In this section we consider

$$\text{Case II:} \quad 2q = p\,.$$

The simplification of the recurrence formula for the Gegenbauer type of polynomials is due to their symmetry properties. The polynomials are alternatively symmetric and antisymmetric with respect to the midpoint $x=\frac{1}{2}$ of their fundamental region $0 \leqq x \leqq 1$. Therefore the products $y_m y_n$ must be expressed linearly by either symmetric or antisymmetric polynomials y_{n-m+2k}.

But this is not the only type of simplified polynomials with a two-term recurrence formula for the coefficients of the linear expansion of their products. There is another type as announced above, which is almost equally simple.

If in (4.13) we put $2q=p$, we obtain the somewhat simplified recurrence formula

(6.1a)

$$\frac{(n+m+p)^2-(k-1+p)^2}{2k-1+p}\left\{[(k-1+p)^2-(n-m)^2]c_{k-1}+[k^2-(n-m)^2]c_k\right\}-$$
$$-\frac{(k+1)^2-(n-m)^2}{2k+1+p}\left\{[(n+m+p)^2-(k+p)^2]c_k+[(n+m+p)^2-(k+1)^2]c_{k+1}\right\}=0.$$

Next we write the alternative formula as announced by equ. (4.10a), however with k replaced by $k+1$. The result is

(6.1b)

$$\frac{(k+p)^2-(n-m)^2}{2k+1+p}\left\{[(n+m+p)^2-(k+p)^2]c_k+[(n+m+p)^2-(k+1)^2]c_{k+1}\right\}-$$
$$-\frac{(n+m+p)^2-(k+2)^2}{2k+3+p}\left\{[(k+p+1)^2-(n-m)^2]c_{k+1}+[(k+2)^2-(n-m)^2]c_{k+2}\right\}=0.$$

It is easily seen that for non-vanishing pre-factors the {}-brackets of (6.1a) and (6.1b) are all zero if a single one is zero.

Now since $c_{n-m-1}=0$, the first {} of (6.1a) vanishes for $k=n-m$. Similarly since $c_{n+m+1}=0$, the last bracket of (6.1a) vanishes for $k=n+m$. We therefore obtain the alternating recurrence formulae

$$(6.2)\quad \frac{c_{k+1}}{c_k}=-\frac{(n+m+p)^2-(k+p)^2}{(n+m+p)^2-(k+1)^2},\qquad \frac{c_{k+2}}{c_{k+1}}=-\frac{(k+p+1)^2-(n-m)^2}{(k+2)^2-(n-m)^2}.$$

Replacing k by $n-m+2k$, we may write

$$(6.3a)\qquad \frac{c_{n-m+2k+1}}{c_{n-m+2k}}=-\frac{(n+p+k)(m-k)}{(n+\frac{1}{2}(p+1)+k)(m+\frac{1}{2}(p-1)-k)},$$

(6.3b)

$$\frac{c_{n-m+2k+2}}{c_{n-m+2k}}=\frac{(n+p+k)(m-k)\left(n-m+\frac{1}{2}(p+1)+k\right)\left(\frac{1}{2}(p+1)+k\right)}{\left(n+\frac{1}{2}(p+1)+k\right)\left(m+\frac{1}{2}(p-1)-k\right)(n-m+1+k)(k+1)}.$$

From these expressions and the known end coefficients the following explicit formulae are obtained

(6.4)

$$c_{n-m+2k}=\frac{\binom{k+\frac{1}{2}(p-1)}{k}\binom{m-k+\frac{1}{2}(p-1)}{m-k}\binom{n-m+k+\frac{1}{2}(p-1)}{n-m+k}\binom{n+k+p-1}{n+k}}{\binom{m+p-1}{m}\binom{n+p-1}{n}\binom{n+k+\frac{1}{2}(p-1)}{n+k}},$$

$$c_{n-m+2k+1} = -\frac{\binom{k+\frac{1}{2}(p-1)}{k}\binom{m-k-1+\frac{1}{2}(p-1)}{m-k-1}\binom{n-m+k+\frac{1}{2}(p-1)}{n-m+k}\binom{n+k+p}{n+k+1}}{\binom{m+p-1}{m}\binom{n+p-1}{n}\binom{n+k+\frac{1}{2}(p+1)}{n+k+1}}. \tag{6.4a}$$

The simplest expressions are here obtained for odd integral p. In particular

$$p = 1, \qquad c_{n-m+2k} = 1, \qquad c_{n-m+2k+1} = -1\,, \tag{6.5}$$

$$p = 3, \qquad c_{n-m+2k} = 2\frac{(k+1)(m-k+1)(n-m+k+1)(n+k+2)}{(m+1)(m+2)(n+1)(n+2)}, \tag{6.5a}$$

$$c_{n-m+2k+1} = -2\frac{(k+1)(m-k)(n-m+k+1)(n+k+3)}{(m+1)(m+2)(n+1)(n+2)}.$$

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