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A SIMPLE PROOF OF THE DAUNS-HOFMANN THEOREM

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A theorem of Dauns and Hofmann (III, 5.2 and 8.16 of [5]) asserts that every bounded continuous scalar-valued function on the spectrum of a C^*-algebra multiplies the algebra (see 3 below for a more precise statement). Previously this result had been obtained by Kaplansky for the special case of a liminary C^*-algebra with separated spectrum (Theorem 3.3 of [9]). Subsequently a somewhat different proof than that of [5] appeared in [7], based on the decomposition theorem of Størmer [11] for the positive part of a sum of closed two-sided ideals. Later a quite different proof was given in [10], as an application of a theory of general (i.e. noncentral) multipliers.

Alfsen and Andersen proved in [3] a generalization of the C^*-algebra theorem stating that the Banach space of realvalued continuous affine functions on a compact convex set K is a module over the facially continuous functions on the extreme points of K. (For a definition of the facial topology see [1] or [3].) This theorem was further generalized by Alfsen and Effros in their work on the structure of real Banach spaces (theorem 4.9 of [4]).

The purpose of the present note is to give a proof of the Dauns-Hofmann theorem which is sufficiently simple that it in fact also proves the Alfsen-Effros theorem referred to above. The argument of the note follows closely the lines of the proof of Dauns and Hofmann, but instead of using approximate bounded decompositions of a sum of ideals it uses exact bounded decompositions. In the C^*-algebra setting the possibility of bounded decomposition is already known (it follows e.g. from Størmer's theorem) and in fact with a better bound (1 instead of 3) than is given in lemma 1 below. The merit of our proof of existence of bounded decompositions (see lemma 1 below) is that it is simple, and moreover uses only a very weak property of two-sided ideals, which can be decribed as orthogonality in the sense of the norm, modulo intersections. A further investigation of this property will be carried out in a subsequent publication [8].

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1. LEMMA. *Let A be a normed linear space. Let $J_0, \ldots, J_n$ be closed subspaces of A such that for each $k=1, \ldots, n$ the canonical linear isomorphism*

$$(J_0 + \ldots + J_k)/(J_0 + \ldots + J_k) \cap J_{k+1} \to (J_0 + \ldots + J_{k+1})/J_{k+1}$$

is an isometry. Let x be an element of $J_0 + \ldots + J_n$. Then there exist $x_0 \in J_0, \ldots, x_n \in J_n$ such that $x = x_0 + \ldots + x_n$ and for each $i=0, \ldots, n$, $\|x_i\| \leqq 3\|x\|$.

PROOF. We shall prove by induction the stronger version of the statement obtained by replacing the estimates $\|x_i\| \leqq 3\|x\|$ by

$$\|x_i\| \leqq (2+\varepsilon)\|x\| ,$$

where $\varepsilon > 0$ is fixed but arbitrary.

The conclusion is trivial for $n=0$. Suppose that $k=0, 1, \ldots$ and the strengthened conclusion is known for $n=k$, and suppose that the hypotheses are satisfied for $n=k+1$. Denote by φ the canonical linear map $A \to A/J_{k+1}$. Then

$$\varphi(J_0 + \ldots + J_{k+1}) = \varphi(J_0 + \ldots + J_k) ,$$

so there exists $y \in J_0 + \ldots + J_k$ such that $\varphi(y) = \varphi(x)$, that is, $x - y \in J_{k+1}$. Moreover, since the canonical isomorphism of $(J_0 + \ldots + J_{k+1})/J_{k+1}$ with

$$(J_0 + \ldots + J_k)/(J_0 + \ldots + J_k) \cap J_{k+1}$$

is isometric, y may be chosen with norm less than $(1+\varepsilon')\|\varphi(x)\|$, where $\varepsilon' > 0$ will be specified later. By the inductive assumption, $y = x_0 + \ldots + x_k$ with $x_0 \in J_0, \ldots, x_k \in J_k$ and

$$\|x_0\| \leqq (2+\varepsilon'')\|y\|, \quad \ldots, \quad \|x_k\| \leqq (2+\varepsilon'')\|y\| ,$$

where $\varepsilon'' > 0$ will be specified. Moreover, with $x_{k+1} = x - y$,

$$x_{k+1} \in J_{k+1} \quad \text{and} \quad \|x_{k+1}\| \leqq \|x\| + \|y\| .$$

Thus, $x = y + (x-y) = x_0 + \ldots + x_{k+1}$, and

$$\|x_0\| \leqq (2+\varepsilon'')(1+\varepsilon')\|x\|, \quad \ldots, \quad \|x_k\| \leqq (2+\varepsilon'')(1+\varepsilon')\|x\|,$$
$$\|x_{k+1}\| \leqq (2+\varepsilon')\|x\| .$$

With ε' and ε'' sufficiently small that $2\varepsilon' + \varepsilon'' + \varepsilon''\varepsilon' \leqq \varepsilon$, it follows that $x_0, \ldots, x_{k+1}$ verify the strengthened conclusion for $n = k+1$.

2. PROBLEM. The case in which A is two-dimensional with unit ball a parallelogram shows that the bounds $\|x_i\| \leqq 3\|x\|$ (actually $\|x_i\| \leqq (2+\varepsilon)\|x\|$)

cannot be made sharper than $\|x_i\| \leqq 2\|x\|$. Can these "best possible" bounds be realized? If the orthogonality hypothesis is strengthened to be symmetric in $J_0, \ldots, J_n$, it seems reasonable to ask for the bounds $\|x_i\| \leqq \|x\|$ (or at least $\|x_i\| \leqq (1+\varepsilon)\|x\|$).

3. THEOREM (Dauns-Hofmann). *Let A be a C^*-algebra, let x be an element of A, and let f be a bounded continuous scalar-valued function on* Prim$A$, *the space of primitive ideals of A endowed with Jacobson topology* (3.1 *of* [6]). *Then there exists a unique element fx of A such that*

$$(fx)(t) = f(t)x(t), \quad t \in \operatorname{Prim} A\ ,$$

where $x(t)$ for $t \in \operatorname{Prim} A$ denotes the canonical image of x in A/t.

PROOF. We may suppose that $0 \leqq f \leqq 1$. Fix $n = 1, 2, \ldots$. Let

$$f^{-1}(](i-1)n^{-1}, (i+1)n^{-1}[) = U_i, \quad i = 0, \ldots, n\ .$$

Set $\operatorname{Prim} A = X$. Set

$$\bigcap\nolimits_{t \in X \setminus U_i} t = J_i, \quad i = 0, \ldots, n\ .$$

Then, since $\bigcup_{i=0}^n U_i = X$, we have $\sum_{i=0}^n J_i = A$ (if $\sum_{i=0}^n J_i \neq A$ then by 1.8.4 and 2.9.7 (ii) of [6] there exists $t \in X$ such that $\sum_{i=0}^n J_i \subseteqq t$; that is, $t \in \bigcap_{i=0}^n X \setminus U_i = \emptyset$). Hence by lemma 1 (in which $J_0, \ldots, J_n$ satisfy the orthogonality condition by 1.8.4 of [6]), $x = \sum_{i=0}^n x_i$ with $\|x_i\| \leqq 3\|x\|$, $i = 0, \ldots, n$. Set

$$\sum\nolimits_{i=0}^n i n^{-1} x_i = y_n\ .$$

Fix $t \in X$. Then for each $i = 0, \ldots, n$, either $t \in U_i$, in which case $|f(t) - in^{-1}| \leqq n^{-1}$, or $t \in X \setminus U_i$, in which case $x_i(t) = 0$. Moreover, the first possibility can happen for at most $i = i_0$ and $i = i_0 + 1$ for some $i_0 = 0, \ldots, n$. Hence

$$\begin{aligned} \|f(t)x(t) - y_n(t)\| &= \|\textstyle\sum_{i=0}^n (f(t) - in^{-1})x_i(t)\| \\ &\leqq \textstyle\sum_{i=0}^n |f(t) - in^{-1}|\, \|x_i(t)\| \\ &\leqq n^{-1} \textstyle\sum_{i=i_0}^{i_0+1} \|x_i(t)\| \leqq 6n^{-1}\|x\|\ . \end{aligned}$$

Since $\|y\| = \sup_{t \in X} \|y(t)\|$ for any y in A (2.7.3 of [6]), the sequence y_n is Cauchy, and by the preceding inequality the limit satisfies the requirements for fx.

4. THEOREM (Alfsen-Effros). *Let A be a real Banach space, let x be an element of A, and let f be a bounded continuous scalar-valued function on* Prim$A$, *the space of primitive M-ideals of A* (*see* II, *section* 3 *of* [4])

endowed with the structure topology (II, *proposition* 3.2 *of* [4]). *Then there exists a unique element* fx *of* A *such that*

$$(fx)(t) = f(t)x(t), \quad t \in \operatorname{Prim} A ,$$

where $x(t)$ *for* t *in* $\operatorname{Prim} A$ *denotes the canonical image of* x *in* A/t.

PROOF. (For notation, the reader is referred to [4]). The proof is a word-to-word translation of the preceding one. That the intersection of M-primitives forming a closed set $(X \setminus U_i)$ is an M-ideal follows from the fact that each M-ideal is an intersection of M-primitives (use II, proposition 3.5a of [4] and the observation made directly from the definition of a primitive M-ideal that the norm of an element is the supremum of its norm in primitive quotients). Thus the J_i's are in fact M-ideals, and it follows from (II, corollary 2.4 of [4]) that their sum is also an M-ideal and that they satisfy the orthogonality condition of lemma 1.

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