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CLASSIFICATION OF JBW*-TRIPLE FACTORS AND APPLICATIONS

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1. Introduction.

The algebraic structure, known today as JB*-triples, which include C*-algebras, JB*-algebras and certain Lie algebras, was introduced by Koecher to classify finite dimensional Bounded Symmetric Domains. They were studied later by different authors. Kaup showed the equivalence between Bounded Symmetric Domains and JB*-triples for infinite dimensional cases, and obtained an analogue of the Riemann mapping theorem (see [9]). On the other hand JB*-triples occur in functional analysis by considering contractive projections on C*-algebras and their generalizations. It was shown that the category of JB*-triple is stable under contractive projections (see [10], [5]).

Recently, Dineen, Barton and Timoney showed the bidual M^{**} of a JB*-triple M is a JBW*-triple, i.e., a dual JB*-triple, whose triple product is a w^*-continuous extension of the triple product on M. JBW*-triples were introduced and studied earlier by Friedman, Russo, and Horn [3], [6]. Horn also defined and classified the so called type I JBW*-triple. His classification reduces the classification of JBW*-triples of type I to the corresponding classification of type I JBW*-algebras, the proof of which was quite complicated. As a corollary he obtained that the type I triple factors are exactly the Cartan factors. The abovementioned results were used to obtain the Gelfand-Naimark theorem for JB*-triples (see [4]).

Following is a brief description of the Cartan factors of various types.

Let H, K be m and n dimensional complex Hilbert spaces (m, n can be uncountable).

TYPE 1. $C^1_{mn} := \mathscr{B}(H, K)$, the bounded operators from H to K.

For the next two types, assume H is equipped with a conjugation $j: H \to H$; then for any $z \in \mathscr{B}(H)$, we can define its transpose as $z^t = jz^*j$.

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TYPE 2. $C_m^2 := \{z \in \mathscr{B}(H); z^t = -z\}$.

TYPE 3. $C_m^3 := \{z \in \mathscr{B}(H); z^t = z\}$.

The triple product on the above Cartan factors is defined by $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$. Moreover, C_m^2 and C_m^3 are, up to isomorphism, independent of the conjugation j on H.

TYPE 4 (also called spin factor). These are JB*-triple M which can be equipped with an inner product, and a conjugation $*$ such that

(i) The original norm on M is equivalent to the Hilbert space norm induced by the inner product;
(ii) The triple product satisfies

$$\{x, y, z\} = \tfrac{1}{2}(\langle x|y\rangle z + \langle z|y\rangle x - \langle x|z^*\rangle y^*).$$

TYPE 5. $M_{1,2}(\mathcal{O}) :=$ the space of all 1×2 matrices over $\mathcal{O}$, the complex Cayley division algebra. The triple product on $M_{1,2}(\mathcal{O})$ is $\{x, y, z\} = \frac{1}{2}[x(y^*z) + z(y^*x)]$.

TYPE 6. $H_3(\mathcal{O}) :=$ the space of all 3×3 hermitian matrices over $\mathcal{O}$. The triple product on $H_3(\mathcal{O})$ is defined via the Jordan product

$$\{x, y, z\} = (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y.$$

(The Jordan product is defined as $x \circ y = \frac{1}{2}(xy + yz)$. For a more detailed description, see [6, §6] or [8].)

There is another approach to the classification of factors. This method was used by Jordan, von Neumann, and Wigner to classify the finite dimensional Jordan algebra factors [7]. Their idea was to "build" these factors from the basic "building blocks" of the algebraic structure—the minimal projections in this case. These "building blocks" could be obtained through a process of "analysis," which decomposed the factor into subspaces of lower dimension. Such decompositions are given by the so called Peirce decompositions associated with an arbitrary projection, which decompose the Jordan algebra factor into three subspaces. Two of these subspaces are still Jordan algebras, and therefore the same "analysis" can be applied to them until one reaches minimal subspaces. Unfortunately, the third space is no longer a Jordan algebra. To deal with this space, deep and formal results from algebra were used. However, if we consider the same problem in the more general category of JB*-triples, all the subspaces occurring in the Peirce decomposition remain JB*-triples, and therefore the "analysis" can be carried out systematically. In Section 2 we will do this to obtain a new proof of the classification of type I JBW*-triple factors, of finite or infinite dimension.

Even though this result implies the Jordan algebra factors classification as a particular case, its proof is simpler since, as mentioned earlier, this "analysis" does not take us out of the category. Moreover, this classification shed light on the occurrence of the exceptional triple factors in this theory.

The "synthesis," more precisely, the construction of the factor from its building blocks is similar to the one used to construct the ranges of contractive projections on C_1 and C_∞ in [1]. This "synthesis" was also used for constructing different types of orthonormal grids by McCrimmon and Neher [12], [15], [16].

In Section 3, we will use this classification of type I triple factors, or equivalently, the w^*-closed ideal generated by a minimal tripotent to prove the abovementioned Gelfand-Naimark theorem for JB*-triples, and some other results.

Recall that a JB*-triple M is a complex Banach space M with a triple product $(x, y, z) \to \{x, y, z\}$ from $M \times M \times M$ to M, which is symmetric, linear in the outer variables x, z and conjugate linear in the inner variable y. Moreover, the operator $D: M \to \mathscr{B}(M)$ defined by $D(u)x = \{u, u, x\}$ for every u in M satisfies

(1.1) $D(u)$ is hermitian (i.e., $\exp(itD(u))$ is an isometry for every real t) with positive spectrum.

(1.2) $iD = iD(u)$ is a derivation of the triple product, i.e., $iD(\{x, y, z\}) = \{iDx, y, z\} + \{x, iDy, z\} + \{x, y, iDz\}$.

(1.3) $\|D(u)\| = \|u\|^2$

for all u in M. It is known that (1.2) is equivalent to

$$\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\} + \{\{u, v, z\}, y, x\} - \{x, \{v, u, y\}, z\} \tag{1.4}$$

for arbitrary $x,y,z,u,v \in M$, and (1.3) is equivalent to

$$\|\{z, z, z\}\| = \|z\|^3. \tag{1.5}$$

For convenience, we will write z^3 for $\{z, z, z\}$. A nonzero element e of M is called a *tripotent* if $e^3 = e$. For each $x \in M$, we can define a mapping $Q(x): M \to M$ by $Q(x)y = \{x, y, x\}$. If e is a tripotent, we have the corresponding Peirce decomposition of M into subspaces $M_k(e)$, $k \in \{0, 1, 2\}$, which are the ranges of the Peirce projections $P_k(e)$ defined as follows

$$\begin{aligned} P_2(e) &= Q(e)^2 \\ P_1(e) &= 2(D(e) - Q(e)^2 \\ P_0(e) &= \mathrm{Id} - 2D(e) + Q(e)^2. \end{aligned} \tag{1.6}$$

The following properties of these projections are well-known: $P_k(e)$ are contractive projections and $M_k(e)$ are JB*-subtriples of M, satisfying

$$P_j(e)P_k(e) = 0, \quad \text{for } j \neq k; \tag{1.7}$$

$$\sum_{j=0}^{2} P_j(e) = \mathrm{Id}; \tag{1.8}$$

and

$$M_k(e) \text{ is the } \frac{k}{2} \text{ eigenspace of } D(e), \quad k \in \{0, 1, 2\}. \tag{1.9}$$

It follows from (1.9) that if N is a JB*-triple and $e \in M \subset N$, then $M_k(e) = N_k(e) \cap M$. Moreover, the following Pierce rules express the way the triple products acts on different subspaces of the Pierce decomposition. They will play an important role throughout.

$$\{M_2(e), M_0(e), M\} = \{M_0(e), M_2(e), M\} = \{0\}. \tag{1.10}$$

$$\{M_i(e), M_j(e), M_k(e)\} \subseteqq M_{i-j+k}(e), \tag{1.11}$$

where $M_{i-j+k}(e) = \{0\}$ if $i-j+k \notin \{0, 1, 2\}$.

Formula (1.10) leads us to the definition of orthogonality. Two elements x, y in M are *orthogonal* if there is a tripotent e such that $x \in M_2(e)$ and $y \in M_0(e)$; in this case (1.10) implies $\{x, y, z\} = \{y, x, z\} = 0$ for any z in M. The $M_2(e)$ part of this Pierce decompositon is known to also be a JB*-algebra with identity e under the product $x \circ y = \{x, e, y\}$, and the involution $x \mapsto Q(e)x$ (see [8], [14]). The following property of $M_2(e)$, which is easy to verify, will be used.

REMARK 1.1. For each tripotent e, the map $Q(e)$ restricted to $M_2(e)$ is an antilinear bijection of $M_2(e)$ preserving the triple product.

For arbitrary tripotents e and f of a JB*-triple M, their Peirce projections $\{P_k(e), P_j(f)\}$ defined by (1.6), generally, do not commute, which makes it difficult to consider their joint Peirce decompositions. However, in certain cases their Peirce projections do commute, i.e., $P_j(e)P_k(f) = P_k(f)P_j(e)$ for all $j,k \in \{0, 1, 2\}$. Two such tripotents are said to be *compatible*. As it was shown (see [3], [13]), two tripotents e and f are compatible if $e \in M_k(f)$ for some k. The families of tripotents that we consider in Section 2 will satisfy this property, and therefore we will freely exchange the order in the products of their Peirce projections. Moreover for compatible tripotents e and f, the product $P_j(e)P_k(f)$ is the projection onto $M_j(e) \cap M_k(f)$. Thus, for a finite family of mutually compatible tripotents, we can define its joint Peirce decomposition in a natural way.

In general, JB*-triples may not have any tripotents at all. In order to carry out the analysis we must have "enough" tripotents. This can be achieved if we ask our JB*-triple to be a dual Banach space. Thus, we are led to the definition of JBW*-triples. More precisely, a JB*-triple U is called a JBW*-triple if it has a predual (denoted by U_*), and the triple product is separately w^*-continuous. For every element x in a JBW*-triple there is a spectral decomposition $x = \int \lambda dv_\lambda$, where v_λ are tripotents. It is known that from this, Proposition 1.2 follows.

PROPOSITION 1.2. *The set of tripotents is norm total in a* JBW*-*triple. More precisely, each element in the* JBW*-*triple can be approximated in norm by a finite linear combination of mutually orthogonal tripotents.*

If $\{e_\alpha\}_{\alpha \in A}$ *is a family of mutually orthogonal tripotents in a* JBW*-*triple* U, *then the sum* $\sum_{\alpha \in A} e_\alpha$ *converges in the* w^*-*topology* [6, Lemma 3.17]. *Moreover,* $p = \sum_{\alpha \in A} e_\alpha$ *is a tripotent and*

$$U_2(p) + U_1(p) = \overline{\text{span}}^{w^*} \{U_2(e_\alpha) + U_1(e_\alpha)\}. \tag{1.12}$$

A JBW*-triple is called a factor if it is not the sum of two orthogonal w^*-closed subtriples. To describe JBW*-factors, we will construct a basis consisting of tripotents. Therefore let us take a closer look at the set of tripotents.

REMARK 1.3. If e and f are two tripotents with $e \in U_2(f)$, then $U_2(e) \subseteqq U_2(f)$, and $U_0(f) \subseteqq U_0(e)$. In particular, if e and f are *equivalent*, i.e., $e \in U_2(f)$ and $f \in U_2(e)$, then $U_k(e) = U_k(f) =$ for all $k \in \{0, 1, 2\}$.

A tripotent v is said to be *minimal* in a JB*-triple M if $M_2(v)$ is one-dimensional

PROPOSITION 1.4. *A tripotent is minimal in a* JBW*-*triple iff it is not a sum of two orthogonal tripotents.*

Since an orthogonal family of tripotents generates necessarily a commutative JB*-triple, we must also consider tripotents which are not orthogonal to each other. More precisely, we will be interested in the relation between a minimal tripotent v and a tripotent u which is minimal in $U_1(v)$. As will be shown in Proposition 2.1, such u, v must satisfy one of the following two relations:

(a) u is *colinear* to v, denoted by $u \top v$, if $u \in U_1(v)$ and $v \in U_1(u)$;
(b) u *governs* v, denoted by $u \vdash v$, if $u \in U_1(v)$ and $v \in U_2(u)$.

REMARK 1.5. Let u_1, u_2, u_3 be mutually colinear minimal tripotents. Since $\{u_1, u_2, u_3\} \in U_2(u_1) \cap U_2(u_2) = \{0\}$ by (1.11), $\{u_1, u_2, u_3\} = 0$.

It will also be shown that if v is a minimal tripotent in U, then rank $U_1(v) \leqq 2$, that is, $U_1(v)$ cannot contain more than two mutually orthogonal tripotents. If a JBW*-triple factor U is of tank 1, then it is the norm closed span of a mutually colinear family of tripotents. More generally by Proposition of Case 1, section 2, any mutually colinear family $\{u_\alpha\}$ of minimal tripotents spans a Hilbert space H, and $\sum_\alpha P_2(u_\alpha)$, converging in norm, is a contractive projection of U onto H. If the rank of the factor is at least 2, the tripotents in the basis will form "building blocks," namely, trangles and quadrangles defined as follows:

DEFINITION. An ordered triplet $(v, u, \tilde{v})$ of tripotents is called a *trangle* if $v \perp \tilde{v}$, $u \vdash v$, $u \vdash \tilde{v}$, and $\tilde{v} = Q(u)v$.

PROPOSITION 1.6. *If $u \vdash v$ (in which case we say (v, u) form a pretrangle), let $\tilde{v} = Q(u)v$. Then $(v, u, \tilde{v})$ form a trangle. Moreover, $v = Q(u)\tilde{v}$, and the tripotent $v + \tilde{v}$ is equivalent to u.*

PROOF. From (1.11) it follows that $\{u, v, u\} \in M_{1-2+1}(v) = M_0(v)$. Thus $v \perp \tilde{v}$. Since $Q(u)u = u$, $Q(u)v = \tilde{v}$, $\{u, u, v\} = v$, $\{v, v, u\} = \frac{1}{2}u$, Remark 1.1 and (1.9) imply

$$\{u, u, \tilde{v}\} = \{Q(u)u, Q(u)u, Q(u)v\} = Q(u)\{u, u, v\} = Q(u)v = \tilde{v}.$$

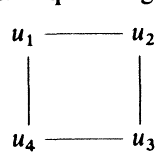
Thus $\tilde{v} \in M_2(u)$. Similarly one shows that $\tilde{v}$ is a tripotent, $\{\tilde{v}, \tilde{v}, u\} = \frac{1}{2}u$, that is, $u \vdash \tilde{v}$, and $v = Q(u)\tilde{v}$.

Since v and $\tilde{v}$ are orthogonal, $v + \tilde{v}$ is a tripotent. From (1.10), $\{v+\tilde{v}, v+\tilde{v}, u\} = \{v, v, u\} + \{\tilde{v}, \tilde{v}, u\} = u$, implying $u \in M_2(v+\tilde{v})$. Obviously $v + \tilde{v} \in M_2(u)$. Thus u and $v + \tilde{v}$ are equivalent.

DEFINITION. An ordered quadruple (u_1, u_2, u_3, u_4) of tripotents is called a *quadrangle* if $u_1 \perp u_3$, $u_2 \perp u_4$, $u_1 \top u_2 \top u_3 \top u_4 \top u_1$ and

$$u_4 = 2\{u_1, u_2, u_3\}. \tag{1.13}$$

From (1.4) it follows that (1.13) is still true if the indices are permutated cyclically, e.g., $u_1 = 2\{u_2, u_3, u_4\}, \ldots$ etc. Thus cyclic permutations of a quadrangle remain quadrangles. A quadrangle can be represented as follows:



In this diagram, two tripotents are orthogonal if they are on a diagonal, and are colinear if they are on an edge. The product of any three tripotents (on consecutive vertices) will give $\frac{1}{2}$ the remaining tripotent. For instance, $\{u_1, u_4, u_3\} = \frac{1}{2}u_2$, $\{u_1, u_1, u_2\} = \frac{1}{2}u_2$, $\{u_1, u_3, u_2\} = 0, \ldots$ etc.

PROPOSITION 1.7. *Suppose u_1, u_2, u_3 are tripotents with $u_1 \perp u_3$, and*

$u_1 \top u_2 \top u_3$ *(in which case we say* (u_1, u_2, u_3) *form a prequadrangle). Let* $u_4 = 2\{u_1, u_2, u_3\}$. *Then* u_4 *is a tripotent and* (u_1, u_2, u_3, u_4) *is a quadrangle. Moreover,* $u_1 + u_3$ *and* $u_2 + u_4$ *are equivalent tripotents.*

PROOF. Consider the map $Q(u_1+u_3)$ on $U_2(u_1+u_3)$ which is antilinear and preserves the triple product. Obviously $Q(u_1+u_3)$ fixes both u_1 and u_3. Moreover, from (1.11), $\{u_1, u_2, u_1\} = 0$ and $\{u_3, u_2, u_3\} = 0$, implying

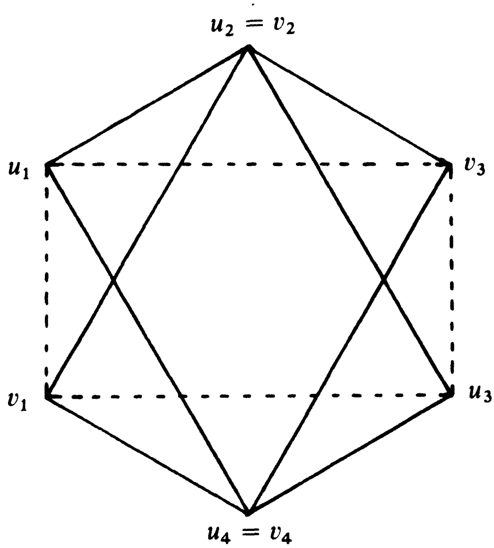
$$Q(u_1+u_3)(u_2) = 2\{u_1, u_2, u_3\} = u_4.$$

The same argument as in the proof of Proposition 1.6 shows u_4 is a tripotent satisfying $u_1 \top u_4 \top u_3$. Moreover (1.11) shows $u_4 = 2\{u_1, u_2, u_3\} \in U_0(u_2)$. Thus (u_1, u_2, u_3, u_4) form a quadrangle.

Since u_2 is colinear to both u_1, u_3, and u_1, u_3 are orthogonal, $\{u_1+u_3, u_1+u_3, u_2\} = u_2$, implying $u_2 \in U_2(u_1+u_3)$. Similarly $u_4 \in U_2(u_1+u_3)$. Thus $u_2+u_4 \in U_2(u_1+u_3)$. By the argument, $u_1+u_3 \in U_2(u_2+u_4)$, showing that u_1+u_3 and u_2+u_4 are equivalent tripotents.

By using both trangles and quadrangles as building blocks, we can construct the Cartan factor of type 3, as will be shown in Case 2 of section 2. The remaining Cartan factors are constructed mainly from quadrangles. How do we obtain the remaining five different types of factor from the same building blocks (quadrangles)? The answer is that we have two construction techniques for gluing the quadrangles together.

CONSTRUCTION A. Two quadrangles (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) are said to be "glued" by Construction A, or "glued" by diagonals if $u_2 = v_2$ and $u_4 = v_4$. The diagram of this construction is as follows:



This diagram suggests that (u_1, v_1, u_3, v_3) form a quadrangle. However, this is not true in general; further assumption is needed.

LEMMA 1.8. *Let (u_1, u_2, u_3, u_4) and (v_1, u_2, v_3, u_4) be quadrangles glued together along the diagonal (u_2, u_4). If all of the vertices are minimal tripotents, and it is known that one of u_1, u_3 is colinear to some one of the v_1, v_3, then $(u_1, v_1, u_3, -v_3)$ form a quadrangle.*

PROOF. Without loss of generality, let us assume $u_1 \top v_1$. To show $(u_1, v_1, u_3, -v_3)$ form a quadrangle, by Proposition 1.7 it suffices to show that $v_1 \top u_3$ and $\{u_1, v_1, u_3\} = -\frac{1}{2}v_3$. From (1.11), $u_3 = 2\{u_4, u_1, u_2\} \in U_1(v_1)$. Since v_1 is a minimal tripotent, $v_1 \in U_1(u_3) \cup U_2(u_3)$ [3, Lemma 2.1]. But $v_1 \notin U_2(u_3)$ since from minimality of u_3 follows $U_2(u_3) = \mathbf{C}u_3$. Thus $v_1 \in U_1(u_3)$, and $v_1 \top u_3$. To show $\{u_1, v_1, u_3\} = -\frac{1}{2}v_3$, we have from (1.4),

$$\begin{aligned}\{u_1, v_1, u_3\} &= 2\{u_1, v_1, \{u_4, u_1, u_2\}\} \\ &= 2(\{\{u_1, v_1, u_4\}, u_1, u_2\} + \{\{u_1, v_1, u_2\}, u_1, u_4\} - \{u_4, \{v_1, u_1, u_1\}, u_2\}).\end{aligned}$$

Since u_1, v_1, u_4 are mutually colinear minimal tripolents, Remark 1.5 implies $\{u_1, v_1, u_4\} = 0$. Similarly $\{u_1, v_1, u_2\} = 0$. Thus

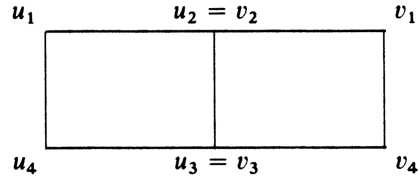
$$\{u_1, v_1, u_3\} = -2\{u_4, \{v_1, u_1, u_1\}u_2\} = -\{u_4, v_1, u_2\} = -\tfrac{1}{2}v_3.$$

Let us call a quadruple (u_1, u_2, u_3, u_4) an *odd quadrangle* if $(u_1, u_2, u_3, -u_4)$ is a quadrangle. Thus odd quadrangles have all the properties of quadrangles described earlier, except that the triple product of three consecutive tripotents will give $-\frac{1}{2}$ of the remaining one. In order to preserve the symmetry of our construction, we replace the original quadrangles by odd ones, and obtain the following proposition.

PROPOSITION 1.9. *If (u_1, u_2, u_3, u_4) and (v_1, u_2, v_3, u_4) are odd quadrangles consisting of minimal tripotents such that one of u_1, u_3 is colinear to some one of v_1, v_3, then (u_1, v_1, u_3, v_3) form an odd quadrangle.*

Thus as we shall see, if we start out with an appropriate family of odd quadrangles and apply Construction A, we will obtain a spin grid consisting of odd quadrangles pairwise "glued" together diagonally. To such grid we may add one trangle. These grids form bases for Cartan factors of type 4. (See section 2, Case 3.)

CONSTRUCTION B. Two quadrangles (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) are glued together by Construction B, or glued side by side if $u_2 = v_2$ and $u_3 = v_3$. The diagram of the construction is as follows:



PROPOSITION 1.10. *Let* (u_1, u_2, u_3, u_4) *and* (v_1, u_2, u_3, v_4) *be quadrangles of tripotents. If* $u_1 \top v_1$, *then* (u_1, v_1, v_4, u_4) *form a quadrangle.*

PROOF. Because of Proposition 1.7, it suffices to show $u_4 \perp v_1$ and $\{v_1, u_1, u_4\} = \frac{1}{2}v_4$. From (1.11) follows $u_4 = 2\{u_1, u_2, u_3\} \in U_0(v_1)$, that is, $u_4 \perp v_1$. From (1.4) follows

$$\begin{aligned}\{v_1, u_1, u_4\} &= 2\{v_1, u_1, \{u_1, u_2, u_3\}\} \\ &= 2(\{\{v_1, u_1, u_1\}, u_2, u_3\} + \{\{v_1, u_1, u_3\}, u_2, u_1\} \\ &\quad - \{u_1, \{u_1, v_1, u_2\}, u_3\}).\end{aligned}$$

Since $u_1 \perp u_3$, and u_1, v_1, u_2 are mutually colinear tripotents, $\{v_1, u_1, u_3\} = 0$ and $\{u_1, v_1, u_2\} \in U_2(u_1)$, implying $\{u_1, \{u_1, v_1, u_2\}, u_3\} = 0$. Thus we have

$$\{v_1, u_1, u_4\} = 2\{\{v_1, u_1, u_1\}, u_2, u_3\} = \{v_1, u_2, u_3\} = \tfrac{1}{2}v_4.$$

REMARK. In the previous proposition, if we had started put with odd quadrangles, we would have still obtained a quadrangle. Thus, it is more appropriate to apply Construction B to a family of quadrangles.

Cartan factors of type 1 are obtained by applying Construction B to an appropriate family of quadrangles, as will be shown in Case 4 of section 2.

To construct the remaining three Cartan factors we will need to use both constructions. Basically, each of these factors consists of spin grids of the same dimension, (partially) glued together by Construction B. As will be shown in section 2, the dimensions of these spin grids can only be 6, 8, or 10, which give rise to the Cartan factors of types 2, 5, and 6, respectively. Moreover, for each minimal tripotent v in these factors, the space $U_1(v)$ is a JBW*-triple constructed from spin grids of two dimensions less then the dimension of the original grids. For factors involving spin grids of dimension 6, the space $U_1(v)$ are of rank 2 and are Cartan factors of type 1, which consist of all $2 \times k$ matrices for arbitrary k. Therefore we have Cartan factors of type 2 of arbitrary size. However, if we wish to build up a factor from spin grids of dimension 8, $U_1(v)$ must be a rank 2 Cartan factor of type 2, which is not a spin grid. Such a factor is unique—the 5×5 anti-symmetric matrices. Thus there is only one Cartan factor of type 5, and the factor consisting of 5×5 anti-symmetric matrices paves the way from the special to the exceptional factors. Note that this factor as well as the Cartan factor of type 5 constructed from it, which is the first exceptional factor, are not Jordan algebras. On the other hand, the next factor, which is constructed from spin grids of dimension 10, is the exceptional Jordan algebra, the Cartan factor of type 6. Since this one is already of rank 3, no factor other than type 4 can be constructed from spin grids of dimensions higher than 10.

Finally, in order to extend homomorphism to the w^*-closure, we will need the following extension lemma.

DEFINTITON 1.11. Let M be a JB*-triple, a functional f of M^*, is said to be an atom if there is a minimal tripotent v of M such that

$$P_2(v)x = \langle f, x\rangle v \tag{1.14}$$

for any x in M. In this case we will write $f = f_v$.

DEFINITION 1.12. A JB*-triple M is nuclear if any f in M^* has a decomposition $f = \sum \alpha_i f_{v_i}$ where $\{v_i\}$ is an orthogonal family of minimal tripotents in M and $\sum|\alpha_i| = \|f\| < \infty$.

REMARK 1.13. If M is a JBW*-triple and v is a minimal tripotent of M, then since the left hand side of (1.14) is w^*-continuous, we have that f_v is in M_*.

LEMMA 1.14. *Let M_1 be a w^*-dense* JB*-*subtriple of a* JBW*-*triple U_1. Let M_2 be a nuclear* JB*-*triple and $\phi: M_1 \to M_2$ be a triple isomorphism. Then, this isomorphism extends to an isomorphism $\tilde{\phi}: U_1 \to M_2^{**}$.*

PROOF. Let f_v be an atom of M_2^* and let $v = \phi(w)$. Then, since ϕ is an isomorphism, we have

$$\langle \phi^* f_v, x\rangle v = \langle f_v, \phi(x)\rangle v = P_2(v)\phi(x) = \phi(P_2(w)x) = \langle f_w, x\rangle \phi(w)$$

for any $x \in M_1$, proving $\phi^*(f_v)$ is an atom of M_1^*. It is known that each isomorphism is an isometry. Thus ϕ^* is an isometry, implying M_1 is nuclear.

Let w be a minimal tripotent of M_1. Since $P_2(w)$ is w^*-continuous, we have $P_2(w)M_1 = \mathbf{C}w$ is w^*-dense in $P_2(w)U_1$; implying $P_2(w)U_1 = \mathbf{C}w$. Thus every minimal tripotent of M_1 is also a minimal tripotent of U_1. By Remark 1.13 above, (1.14), defines f for all x in U_1. Thus each atom of M_1^* can be identified with an atom of U_{1*}. This identification induces a natural embedding $i: M_1^* \to U_{1*}$, with $i(f)|M = f$ for any f in M_1^*. Since M_1 is w^*-dense in U_1, we have $i(g|M) = g$ for any g in U_{1*}, implying that the map i is surjective. Thus, the map $\tilde{\phi} = (i \circ \phi^*)^*$ is an isomorphism from U_1 to M_2^{**}. Standard argument shows that $\tilde{\phi}$ is a w^*-continuous extension of ϕ.

The general plan of our classification of a JBW*-triple-factor U containing a minimal tripotent, say v, is as follows. Use Peirce decomposition to decompose U into subtriples $U_2(v)$, $U_1(v)$ and $U_0(v)$. $U_2(v)$ is 1-dimensional. We describe $U_1(v)$ by picking up a maximal ortho-colinear family F_1 of tripotents in $U_1(v)$. In most cases, this description reduces to the previous case of the classification. Note that $U_2(v)+U_1(v)$ is no longer a JBW*-triple. We need to complete F_1 by adjoining elements obtained by completing pretrangles and prequadrangles (x, v, y) where x, y are from F_1, using Propositions 1.6 and 1.7. These new elements together with F_1 and v form a family F_2 of ortho-colinear-governing tripotents which is closed under triple products. We then show that F_2 is w^*-total in U. To achieve this, we show that F_1 is norm-total in $U_1(v)$ and that each element u of F_2 can be exchanged with v via an automorphism, and then apply (1.12).

Next we determine the product rules on F_2 to verify that they are the same as the product rules on the corresponding Cartan factor. Here, the main tools used are Constructions A and B described earlier. The same verification was done in [12] and [16]. Neher also obtained a full classification of maximally connected grids. Since our verification is significantly shorter, we include it here for completeness.

Finally, the product rules on F_2 determine a natural isomorphism from the norm closure of span F_2 into the Cartan factor, which can be extended by the previous lemma to U.

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2. Ideal classification.

Our goal in this section is to classify the w^*-closed ideals generated by a minimal tripotent in a JBW*-triple, and factors containing a minimal tripotent. This classification is similar to the one in [1], and generalizes the classification of Jordan, Von Neumann, and Wigner [7] of finite dimensional JB*-algebra factors to infinite dimensional JBW*-triple factors. The main result is the following.

MAIN THEOREM. *Let U be a JBW*-triple, v be a minimal tripotent in U, then $J(v)$, the w^*-closed ideal generated by v, is a Cartan factor of one of the* 6 *types, and $J(v)$ is a summand in U. That is, $U = J(v) \oplus K$, where K is a w^*-closed ideal in U orthogonal to $J(v)$.*

To prove the above theorem, we will first classify $J(v)$ into different cases, then discuss each case separately. Our main tools for the classification will be the following Propositions 2.1 and 2.3.

PROPOSITION 2.1. ("Tripe system analyzer (TSA)"). *Let U be a JBW*-triple* *system containing a minimal tripotent v. Let u be a tripotent in $U_1(v)$; then exactly one of the following 3 cases will occur:*

(i) *u is minimal in U. This occurs if and only if u and v are colinear.*
(ii) *u is not minimal in U but is minimal in $U_1(v)$. In this case $u \vdash v$. Let $\tilde{v} = \{u, v, u\}$; then $\tilde{v}$ is a minimal tripotent of U, and $(v, u, \tilde{v})$ form a trangle.*
(iii) *u is not minimal in $U_1(v)$ (thus u is not minimal in U either). In this case $u = u_1 + \tilde{u}_1$, where u_1, $\tilde{u}_1$ are 2 orthogonal minimal tripotents of U, contained in $U_1(v)$. Let $\tilde{v} = \{u, v, u\}$. Then $\tilde{v}$ is a minimal tripotent of U, and $(v, u_1, \tilde{v}, \tilde{u}_1)$ form a quadrangle.*

PROOF. From the definition of minimality, u must be either (i) minimal in U; or (ii) minimal in $U_1(v)$, but not minimal in U; or (iii) not minimal in $U_1(v)$. It is easy to show that since v is a minimal tripotent of U and $u \in U_1(v)$, $v \in U_2(u) \cup U_1(u)$ (cf. [3, Lemma 2.1]).

(i) If u is a minimal tripotent of U, then $v \notin U_2(u)$. Thus $v \in U_1(u)$; that is, $v \top u$. Conversely, if $v \top u$ then by the Colinear Exchange Theorem [12, p. 1501], the automorphism $B(v+u, v+u)$ defined there exchanges u and v. Since any automorphism maps minimal tripotents to minimal tripotents, u is a minimal tripotent of U.

(ii) Now, assume u is not minimal in U but is minimal in $U_1(v)$. From (i) above, $v \notin U_1(u)$, implying $v \in U_2(u)$, that is, $u \vdash v$. From Proposition 1.6, $(v, u, \tilde{v})$ forms a trangle. Since the map $Q(u)$ exchanges v with $\tilde{v}$, and $U_2(v)$, $U_2(\tilde{v})$ are both contained in $U_2(u)$. Remark 1.1 implies that $\tilde{v}$ is minimal tripotent of U.

(iii) If u is not minimal in $U_1(v)$, then $u = u_1 + \tilde{u}_1$, where $u_1, \tilde{u}_1$ are 2 orthogonal tripotents in $U_1(v)$. If $v \in U_2(u_1)$ then $U_2(v) \subseteqq U_2(u_1)$, and $\tilde{u}_1 \perp u_1$ implies $\tilde{u}_1 \perp v$, contradicting $\tilde{u}_1 \in U_1(v)$. Thus $v \in U_1(u_1)$. Similarly, $v \in U_1(\tilde{u}_1)$. Therefore by (i), $u_1, \tilde{u}_1$ are minimal tripotents of U, and $(u_1, v, \tilde{u}_1)$ form a prequadrangle. From Proposition 1.7, it follows that $(u_1, v, \tilde{u}_1, \tilde{v})$ form a quadrangle. Since $\tilde{v} \top u_1$, part (i) implies that $\tilde{v}$ is minimal in U.

COROLLARY 2.2. *Let v be a minimal tripotent in a* JBW*-*triple U; then* rank $U_1(v) \leqq 2$.

PROPOSITION 2.3. *Let v, $\tilde{v}$, u_1, $\tilde{u}_1$ be minimal tripotents of a JBW*-triple U such that $(v, u_1, \tilde{v}, \tilde{u}_1)$ form a quadrangle and $U_1(v+\tilde{v}) \neq 0$. Then* $m = \dim U_2(v+\tilde{v})$ *is even and* $4 \leqq m \leqq 10$.

We will need the following lemma in the proof of Proposition 2.3.

LEMMA 2.4. *Let v and $\tilde{v}$ be orthogonal tripotents of a* JBW*-*triple U. Then $U_1(v+\tilde{v})$ and $U_2(v+\tilde{v})$ can be decomposed into*

(2.1) $$U_1(v+\tilde{v}) = [U_1(v) \cap U_0(\tilde{v})] \oplus [U_1(\tilde{v}) \cap U_0(v)];$$

(2.2) $$U_2(v+\tilde{v}) = U_2(v) \oplus U_2(\tilde{v}) \oplus [U_1(v) \cap U_1(\tilde{v})].$$

PROOF. By the generalized Peirce decomposition [11] relative to the orthogonal family $\{v, \tilde{v}\}$, U has a decomposition.

$$U = U_2(v) \oplus U_2(\tilde{v}) \oplus U_1(v) \cap U_1(\tilde{v}) \oplus U_1(v) \cap U_0(\tilde{v}) \oplus$$
$$\oplus U_1(\tilde{v}) \cap U_0(v) \oplus U_0(v) \cap U_0(\tilde{v})$$

which implies (2.1) and (2.2).

PROOF OF PROPOSITION 2.3. Let $\tilde{U} = U_1(v) \cap U_1(\tilde{v})$. Note that in (2.1), $U_1(v) \cap U_0(\tilde{v}) = U_1(v) \cap U_1(v+\tilde{v})$ and similarly, $U_1(\tilde{v}) \cap U_0(v) = U_1(\tilde{v}) \cap U_1(v+\tilde{v})$. Thus we can assume without loss of generality that $U_1(v) \cap U_1(v+\tilde{v}) \neq \{0\}$. Now since $u_1+\tilde{u}_1$ and $v+\tilde{v}$ are equivalent tripotents, by applying (2.1) once more to the JBW*-triple $U_1(v)$ and orthogonal tripotents $u_1, \tilde{u}_1$, we may assume without loss of generality that $U_1(\tilde{u}_1) \cap U_1(v+\tilde{v}) \cap U_1(v) \neq \{0\}$. Let z be a tripotent in this subspace. Since z and u_1 are two orthogonal tripotents in $U_1(v)$, z is a minimal tripotent of U and by Proposition 2.1, this implies that $P_2(z)\tilde{U} = \{0\}$. Thus, denoting $P_k(z)\tilde{U}$ by $\tilde{U}_k(z)$ $(k = 0, 1)$, we have $\tilde{U} = \tilde{U}_1(z) + \tilde{U}_0(z)$. Using (1.11) and that $v \top z \perp \tilde{v}$, it is easy to check that the mapping $Q(v+\tilde{v}) = 2Q(v, \tilde{v})$ is a bijection from $\tilde{U}_1(z)$ onto $\tilde{U}_0(z)$. Therefore $\dim \tilde{U}_1(z) = \dim \tilde{U}_0(z)$; and by (2.2) we have

(2.3) $$m = \dim U_2(v+\tilde{v}) = 2+\dim \tilde{U} = 2+2\dim \tilde{U}_1(z)$$

is even.

To show that $m \leqq 10$, let us assume that $m = \dim U_2(v+\tilde{v}) \geqq 12$. Since $\tilde{U}_0(z)$ is a JBW*-triple of rank 1, $\dim \tilde{U}_0(z) \geqq 5$, and $u_1 \in \tilde{U}_0(z)$, we can choose u_2, u_3, u_4, u_5 in $\tilde{U}_0(z)$ such that $\{u_i\}_{i=1}^5$ is a colinear family of tripotents. Since u_i and u_1 are colinear, each u_i is a minimal tripotent of U. Thus, $(v, u_i, \tilde{v})$ form prequadrangles, which can be completed to quadrangles $(v, u_i, \tilde{v}, \tilde{u}_i)$ with $\tilde{u}_i = 2\{v, u_i, \tilde{v}\}$. Using $\tilde{u}_i$ are minimal in U, and (1.11), we have $u_i \top \tilde{u}_j$ for $i \neq j$, implying $(z, \tilde{u}_i, u_j)$ form prequadrangles. Define

$$u_{23} = 2\{z, \tilde{u}_2, u_3\}$$
$$u_{45} = 2\{z, \tilde{u}_4, u_5\}.$$

The family $\{u_1, u_{23}, u_{45}\}$ is obviously in $U_1(v)$. We will show it is an orthogonal one and thus contradicting rank $U_1(v) \leqq 2$ (Corollary 2.2).

By (1.11), u_1 is orthogonal to both u_{23} and u_{45}.

By (1.11), $u_{23} \perp u_5$ and $u_{23} \in U_1(\tilde{u}_4)$. Since u_{23} is minimal in U, Proposition

2.1 gives $\tilde{u}_4 \in U_1(u_{23})$. Thus by (1.11) again $u_{45} \in U_0(u_{23})$, that is, u_{23} and u_{45} are orthogonal.

Let v be a minimal tripotent of a JBW*-triple U. Let $J(v)$ be the w^*-closed ideal generated by v. We will classify $J(v)$ according to the following scheme.

Classification scheme. From Corollary 2.2, rank $U_1(v)$ is either 0, 1, or 2.

CASE 0. Rank $U_1(v) = 0$, that is, $U_1(v) = \{0\}$. Then obviously $J(v) = \mathbf{C}c \simeq \mathbf{C}$.

A. If rank $U_1(v) = 1$: Let u be a tripotent in $U_1(v)$. From Proposition 2.1, exactly one of the following two cases may happen.

CASE 1. Rank $U_1(v) = 1$ and $u \top v$. We will show that in this case, $J(v)$ is a Hilbert space, which is a special case of a Cartan factor of type 1.

CASE 2. Rank $U_1(v) = 1$ and $u \vdash v$. We will show that for this case $J(v)$ is a Cartan factor of type 3, the "symmetric matrices."

B. If rank $U_1(v) = 2$. Let u be a nonminimal tripotent of $U_1(v)$. By Proposition 2.1(iii), $u = u_1 + \tilde{u}_1$. Let $\tilde{v} = \{u, v, u\}$. Then $\tilde{v}$ is a minimal tripotent of U, and $(v, u_1, \tilde{v}, \tilde{u}_1)$ form a quadrangle. Exactly one of the following five cases will occur.

CASE 3. $U_1(v+\tilde{v}) = \{0\}$. We will show in this case $J(v)$ is a Cartan factor of type 4, the spin factor.

If $U_1(v+\tilde{v}) \neq \{0\}$, Proposition 2.3 can be applied giving $m = \dim U_2(v+\tilde{v})$ is even, and $4 \leqq m \leqq 10$. Thus we have:

CASE 4. $U_1(v+\tilde{v}) \neq \{0\}$, $\dim U_2(v+\tilde{v}) = 4$. We will show $J(v)$ is a Cartan factor of type 1, the "full matrices."

CASE 5. $U_1(v+\tilde{v}) \neq \{0\}$, $\dim U_2(v+\tilde{v}) = 6$. We will show $J(v)$ is a Cartan factor of type 2, the "anti-symmetric matrices."

CASE 6. $U_1(v+\tilde{v}) \neq \{0\}$, $\dim U_2(v+\tilde{v}) = 8$. We will show $J(v)$ is a Cartan factor of type 5, the exceptional JB*-triple-factor of dimension 16.

CASE 7. $U_1(v+\tilde{v}) \neq \{0\}$, $\dim U_2(v+\tilde{v}) = 10$. We will show $J(v)$ is a Cartan factor of type 6, the exceptional JB*-triple-factor of dimension 27.

We now discuss each case in detail.

Case 1.

PROPOSITION. *Let v be a minimal tripotent in a* JBW*-*triple U such that* rank $U_1(v) = 1$, *and there is a tripotent u in $U_1(v)$ with $u \top v$. Then $J(v)$ is isometric to a Hilbert space, and is a summand in U.*

The proof of the above proposition follows from the lemma below, which we will state in a more general setting.

LEMMA. *Let $\{u_i\}_{i\in I}$ be an arbitrary family of mutually colinear minimal tripotents in a* JB*-*triple* M, *then*

(i) $W := \overline{\text{span}\{u_i\}_{i\in I}}$ *is a* JBW*-*triple, and is isometric to a Hilbert space with* $\{u_i\}_{i\in I}$ *as an orthonormal basis.*

(ii) $P := \sum_{i\in I} P_2(u_i)$ *converges strongly, and does not depend on the order of the summation. Moreover,* P *is contractive projection from* M *onto* W.

PROOF. (i) First, we show if $\{u_i\}_{i=1}^n$ is a finite set of mutually colinear minimal tripotents; then for all $\alpha_i \in \mathbf{C}$,

$$\left\| \sum_{i=1}^{n} \alpha_i u_i \right\|^2 = \sum_{i=1}^{n} \|\alpha_i u_i\|^2. \tag{2.4}$$

From Remark 1.5, it follows that $\{u_i, u_j, u_k\} = 0$ unless it is of the form u_i^3, or $\{u_i, u_i, u_k\}$. Thus,

$$\begin{aligned} \left(\sum_{i=1}^{n} \alpha_i u_i\right)^3 &= \sum_{i,j,k} \alpha_i \bar{\alpha}_j \alpha_k \{u_i, u_j, u_k\} \\ &= \sum_{i=1}^{n} |\alpha_i|^2 \alpha_i u_i + 2 \sum_{i} \sum_{k \neq i} |\alpha_i|^2 \alpha_k \{u_i, u_i, u_k\} \qquad (2.5) \\ &= \sum_i |\alpha_i|^2 \sum_k \alpha_k u_k. \end{aligned}$$

Therefore,

$$\left\| \sum_{i=1}^{n} \alpha_i u_i \right\|^3 = \left\| \left(\sum_{i=1}^{n} \alpha_i u_i \right)^3 \right\| = \left(\sum_{i=1}^{n} |\alpha_i|^2 \right) \left\| \sum_{k=1}^{n} \alpha_k u_k \right\|,$$

proving (2.4). It follows from (2.4) that W is isometric to a Hilbert space with $\{u_i\}_{i\in I}$ as an orthonormal basis. Moreover, since the family $\{u_i\}_{i\in I}$ is closed under the triple product, W is a JBW*-triple.

(ii) First, we show that if $\{u_i\}_{i=1}^n$ is an arbitrary family of mutually colinear minimal tripotents in M, then for all $\alpha_i \in \mathbf{C}$,

$$Q\left(\sum_{i=1}^{n} \alpha_i u_i\right) = Q\left(\sum_{i=1}^{n} \alpha_i u_i\right) \sum_{i=1}^{n} P_2(u_i). \tag{2.6}$$

It is obvious that

$$Q\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i^2 Q(u_i) + 2 \sum_{i \neq j} \alpha_i \alpha_j Q(u_i, u_j)$$

where the operator $Q(u_i, u_j)$ is defined by $Q(u_i, u_j)z := \{u_i, z, u_j\}$. It is easy to show $Q(u_i) = Q(u_i)P_2(u_i)$. Thus to show (2.6), it suffices to show that

$$Q(u_i, u_j) = Q(u_i, u_j)(P_2(u_i) + P_2(u_j)) \quad \text{for } i \neq j.$$

Let z be an element of M, then

$$Q(u_i, u_j)z = Q(u_i, u_j) \sum_{k=0}^{2} P_k(u_i) \sum_{k=0}^{2} P_k(u_j)z.$$

Obviously, $P_2(u_i)P_1(u_j) = P_2(u_i)$, $P_2(u_j)P_1(u_i) = P_2(u_j)$, $P_2(u_i)P_2(u_j) = P_2(u_i)P_0(u_j) = 0$. On the other hand, (1.11) shows $Q(u_i, u_j)$ vanishes on the remaining terms, giving $Q(u_i, u_j) = Q(u_i, u_j)(P_2(u_i) + P_2(u_j))$, and thus proves (2.6).

Next, let F be a finite subset of I. From (2.5), $\sum_{i \in F} \alpha_i u_i$ is a tripotent iff $\sum |\alpha_i|^2 = 1$. It follows that for each element z in M, there are a complex number λ and a tripotent v in the span of $\{u_i\}_{i \in F}$ such that $\sum_{i \in F} P_2(u_i)z = \lambda v$. Since $P_2(v) = Q(v)^2$, from (2.6) we have

$$P_2(v)z = P_2(v) \sum_{i \in F} P_2(u_i)z = P_2(v)(\lambda v) = \lambda v = \sum_{i \in F} P_2(u_i)z.$$

Thus for any finite subset F of I, $\sum_{i \in F} P_2(u_i)$ is a contractive projection. From this and (2.4), statement (ii) of the lemma follows easily.

Proof of the Proposition. By Zorn's lemma, there is an index set I containing 1 and a maximal colinear family $\{u_i\}_{i \in I}$ of tripotents in $U_1(v)$ with $u_1 = u$. Since $u_i \top u_1$ it follows from Proposition 2.1 that each u_i is a minimal tripotent of U and $u_i \top v$. If $U_1(v) \cap [\bigcap_{i \in I} U_1(u_i)] \neq \{0\}$, let w be a tripotent in that space. Since every tripotent in the rank one JBW*-triple $U_1(v)$ is minimal, Proposition 2.1 implies $w \top u_i$ for all $i \in I$, contradicting the maximality of $\{u_i\}_{i \in I}$. Thus $U_1(v) \cap [\bigcap_{i \in I} U_1(u_i)] = \{0\}$.

If z is an element of $U_1(v)$, then $\sum_{i \in I} P_2(u_i)z$ converges in norm by the above lemma, implying

$$z - \sum_{i \in I} P_2(u_i)z \in U_1(v) \cap \left[\bigcap_{i \in I} U_1(u_i)\right] = \{0\}.$$

Thus $U_1(v) = \overline{\text{span}\{u_i\}_{i \in I}}$, implying

$$U_2(v) + U_1(v) = \overline{\text{span}\{v, u_i\}_{i \in I}},$$

and is a JBW*-triple isometric to a Hilbert space. Moreover, $U_2(v)+U_1(v)$ is clearly w^*-closed in U. To show $J(v) = U_2(v)+U_1(v)$ and is a summand, we only need to show $U_0(v)$ is orthogonal to u_i for every $i \in I$. Obviously,

$$U_0(v) = P_2(u_i)U_0(v)+P_1(u_i)U_0(v)+P_0(u_i)U_0(v).$$

Since u_i is minimal in U, $P_2(u_i)U_0(v) = \{0\}$. If $P_1(u_i)U_0(v) \neq \{0\}$, let $\tilde{v}$ be a tripotent in that space. Since v, $\tilde{v}$ are two orthogonal tripotents in $U_1(u_i)$, Proposition 2.1 gives that $(v, u_i, \tilde{v})$ form a prequadrangle which can be completed to a quadrangle with $\tilde{u}_i = 2\{v, u_i, \tilde{v}\}$, which is orthogonal to u_i and is in $U_1(v)$, contradicting rank $U_1(v) = 1$. Thus $U_0(v) = P_0(u_i)U_0(v)$, proving $U_0(v) \perp u_i$.

Corollary. *If U is a JBW*-triple of* rank 1, *then it is isometric to a Hilbert space. Moreover, every maximal colinear family of tripotents in U form an orthonormal basis.*

Case 2. The following defintion is adopted from [12]:

Definition. Let $F = \{u_{ij} | i,j \in I\}$ be a family of tripotents in a Jordan triple system U, for some index set I. F is a hermitian grid if:

(i) For every i, j, k, l in I, we have

$$\begin{aligned} &u_{ij} = u_{ji}; \\ &u_{ij} \perp u_{kl} \qquad \text{if } \{i, j\} \cap \{k, l\} = \emptyset; \\ &u_{ii} \dashv u_{ij} \top_{jk} \quad \text{if } i, j, k \text{ are different.} \end{aligned}$$

(ii) Every triple product among elements of F which cannot be brought to the form $\{u_{ij}, u_{jk}, u_{kl}\}$ vanishes.

(iii) For arbitrary i, j, k, l triple products involving at least two different elements satisfy

$$\{u_{ij}, u_{jk}, u_{kl}\} = \tfrac{1}{2}u_{il}, \quad \text{for } i \neq l, \tag{2.7}$$

$$\{u_{ij}, u_{jk}, u_{ki}\} = u_{ii}. \tag{2.8}$$

It is easy to see that each Cartan factors of type 3 has a hermitian grid, whose span is w^*-dense.

Proposition. *Let v be a minimal tripotent of a JBW*-triple U. If* rank $U_1(v) = 1$, *and there is a tripotent u in $U_1(v)$ such that $u \vdash v$, then $J(v)$ is a Cartan factor of type* 3 *(and is a summand).*

Proof. By Zorn's lemma, there is in $U_1(v)$ a maximal colinear family of tripotents $\{u_{1i}\}_{i \in I}$, which includes u; where I is some index set not containing 1. Since $u \top u_{1j}$ (if $u_{1j} \neq u$), Proposition 2.1 implies $u_{1i} \vdash v$ for all $i \in I$. Let $\tilde{I} = I \cup \{1\}$, $u_{11} = v$ and define

$$(2.9) \qquad \begin{aligned} u_{i1} &= u_{1i} \\ u_{ij} &= u_{ji} = 2\{u_{1i}, u_{11}, u_{1j}\} \quad \text{for } i \neq j, \\ u_{ii} &= \{u_{1i}, u_{11}, u_{1i}\}. \end{aligned}$$

We will show the family $F = \{u_{ij}\}_{i,j \in \tilde{I}}$ forms a hermitian grid.

First, we show that the family F satisfies property (i) of hermitian grid. Let i, j be distinct indices. Since $u_{11} \in U_2(u_{1i})$, by (1.9) and Remark 1.1, $u_{ii} = Q(u_{1i})u_{11}$ is a minimal tripotent of U, and moreover, u_{1i} governs both u_{11} and u_{ii}. On the other hand, by (1.11)

$$u_{ii} = \{u_{1i}, u_{11}, u_{1i}\} \in U_0(u_{1j});$$

i.e., the two tripotents u_{ii} and u_{1j} are orthogonal. Thus $(u_{1j}+u_{ii})$ is a tripotent, and the map $Q(u_{1j}+u_{ii})$ is an antilinear homomorphism of order 2 on $U_2(u_{1j}+u_{ii})$.

Claim 1. For distinct indices 1, i, j, the map $Q(u_{1j}+u_{ii})$ exchanges u_{11} with u_{jj}, u_{1i} with u_{ij} and fixes u_{1j} as well as u_{ii}. That is, it exchanges 1 with j in the set of indices $\{1, i, j\}$.

The action of the map can be represented as the symmetric reflection along the diagonal $u_{1j}+u_{ii}$ of the following table:

u_{11}	u_{1i}	u_{1j}
u_{1i}	u_{ii}	u_{ij}
u_{1j}	u_{ij}	u_{jj}

Since the map is of order 2 in order to prove that $Q(u_{1j}+u_{ii})$ exchanges u_{1i} with u_{ij}, it suffices to show that $Q(u_{1j}+u_{ii})u_{1i} = u_{ij}$. But

$$\begin{aligned} Q(u_{1j}+u_{ii})u_{1i} &= \{u_{1j}+u_{ii}, u_{1i}, u_{1j}+u_{ii}\} \\ &= \{u_{ii}, u_{1i}, u_{ii}\} + \{u_{1j}, u_{1i}, u_{1j}\} + 2\{u_{1j}, u_{1i}, u_{ii}\} \end{aligned}$$

By (1.11) the first two terms are in $U_3(u_{ii})$ and $U_3(u_{1j})$, respectively, thus vanish. On the other hand, by first applying (1.4) and then (1.11) to $\{u_{1j}, u_{1i}, u_{ii}\} = \{u_{1j}, u_{1i}, \{u_{1i}, u_{11}, u_{1i}\}\}$, we get $2\{u_{1j}, u_{1i}, u_{ii}\} = u_{ij}$. The remaining statements of the claim are obvious.

From Claim 1 above, it follows that u_{ij} is a minimal tripotent of $U_1(u_{ji})$ and u_{ij} governs u_{jj}. On the other hand, from (1.11) follows $u_{ij} \perp u_{kl}$ if $\{i, j\} \cap \{k, l\} = \emptyset$, and $u_{ij} \top u_{jk}$ if i, j, k are distinct.

Next, we show that the family F satisfy properties (ii) and (iii). Since any two elements of F are orthogonal if they do not share a common index, every nonvanishing (triple) product among elements of F must have one of the following forms.

(2.10) $\{u_{ij}, u_{jk}, u_{kl}\}$ i, j, k, l are arbitrary.

(2.11) $\{u_{ij}, u_{jk}, u_{jl}\}$, where i, j, l are arbitrary, $k \notin \{i, j, l\}$.

Since these formulas involves at most 4 indices, consider first:

CASE A. The 4 indices i, j, k, l are distinct. Since u_{ij}, u_{jk}, u_{jl} are mutually colinear minimal tripotents of the JBW*-triple $U_1(u_{jj})$, $\{u_{ij}, u_{jk}, u_{jl}\} = 0$. Thus it remains to show $\{u_{ij}, u_{jk}, u_{kl}\} = \frac{1}{2}u_{il}$, which is equivalent to showing that $(u_{ij}, u_{il}, u_{kl}, u_{kj})$ form a quadrangle.

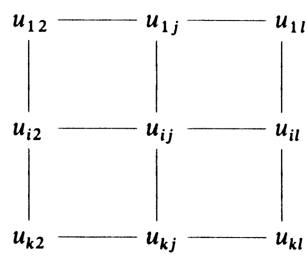
Let i, l be any two distinct indices with $i, l \notin \{1, 2\}$. By applying (1.4) and (1.11) to $\{u_{12}, u_{1l}, u_{il}\} = 2\{u_{12}, u_{1l}, \{u_{1i}, u_{11}, u_{1l}\}\}$, we obtain $\{u_{12}, u_{1l}, u_{il}\} = \frac{1}{2}u_{i2}$. Thus by Proposition 1.7, $(u_{12}, u_{1l}, u_{il}, u_{i2})$ form a quadrangle (for any 4 distinct indices $1, 2, i, l$). Now we can show $(u_{ij}, u_{il}, u_{kl}, u_{kj})$ form a quadrangle as follows: Apply Proposition 1.10 to

(a) the quadrangles $(u_{1j}, u_{12}, u_{i2}, u_{ij})$ and $(u_{1l}, u_{12}, u_{i2}, u_{il})$ to obtain that $(u_{1j}, u_{1l}, u_{il}, u_{ij})$ is a quadrangle;

(b) the quadrangles $(u_{12}, u_{1j}, u_{kj}, u_{k2})$ and $(u_{12}, u_{1l}, u_{kl}, u_{k2})$ to obtain that $(u_{1j}, u_{1l}, u_{kl}, u_{kj})$ is a quadrangle;

(c) the quadrangles $(u_{1j}, u_{1l}, u_{il}, u_{ij})$ and $(u_{1j}, u_{1l}, u_{kl}, u_{kj})$ to obtain that $(u_{ij}, u_{il}, u_{kl}, u_{kj})$ is a quadrangle;

The above construction can be visualized by the diagram below:



It remains to verify products of the form (2.10) and (2.11) for the following case.

CASE B. The number of distinct indices of the set $\{i, j, k, l\}$ is at most 3. By using the following claim we can assume that $j = 1$.

CLAIM 2. For distinct indices 1, i, j, k, the map $Q(u_{1j}+u_{ii}+u_{kk})$ exchanges the indices j with 1 in the set of indices $\{1, i, j, k\}$.

The proof of this claim follows immediately from Claim 1.

By assuming $j = 1$ in triple products of the form (2.11), cardinality of $\{1, i, k, l\}$ is at most 3 and $k \notin \{i, 1, l\}$, we can only have the possibility (a)

$i = 1$; (b) $l = 1$; (c) $i = l$. By (1.11) the product (2.11) vanishes in each case. Similarly, by assuming $j = 1$ in the triple products of the form (2.11), and cardinality of $\{1, i, k, l\}$ is at most 3, we have the possibilities (a) $1 = i$; (b) $1 = k$; (c) $1 = l$; (d) $i = k$; (e) $i = l$; (f) $k = l$. From (2.9) product in case (b) satisfies (iii) of the definition of hermitian grid. From (i) of the definition follows that products in cases (c) and (d) which are of the form $D(x)y$ satisfy (2.7). Cases (a), (e), and (f) are treated as follows. Replace the last term in the product by its definition from (2.9), then apply (1.4) to reduce it to products calculated earlier. Thus the family $\{u_{ij}\}$ defined by (2.9) forms a hermitian grid.

Let $p = \sum_{i \in \tilde{I}} u_{ii}$. First, we show that $J(v) = w^*$-closed span $\{u_{ij}\}$. It suffices to show $U_2(p) = w^*$-closed span $\{u_{ij}\}$ and $U_1(p) = \{0\}$. From [6, Lemma 3.17],

$$U_2(p) = \overline{\text{span}}^{w^*} \{U_2(u_{ii}), U_1(u_{ii}) \cap U_1(u_{jj}); i, j \in \tilde{I}\}.$$

Since u_{ii} is minimal in U, $U_2(u_{ii}) = \mathbb{C}u_{ii}$. On the other hand, since u_{ij} governs both u_{ii} and u_{jj}, the tripotents u_{ij} and $u_{ii}+u_{jj}$ are equivalent, giving $U_2(u_{ij}) = U_2(u_{ii}+u_{jj})$. By (2.2),

$$U_1(u_{ii}) \cap U_1(u_{jj}) \subseteq U_2(u_{ij}).$$

Using u_{ij} as minimal in $U_1(u_{ii})$ (in $U_1(u_{jj})$ as well), we conclude that $U_1(u_{ii}) \cap U_1(u_{jj}) = \mathbb{C}u_{ij}$. Thus

$$U_2(p) = \overline{\text{span}}^{w^*} \{u_{ij}\}_{i,j \in \tilde{I}}.$$

For $U_1(p) = \{0\}$ it suffices to show that $U_1(u_{ii}) \subseteq U_2(p)$ for all $i \in \tilde{I}$. For arbitrary i, let $w = \frac{1}{2}(u_{ii}+u_{11}+u_{1i})$. A straightforward calculation reveals w a tripotent. By [3, Lemma 1.1], the map $T = P_2(w) - P_1(w) + P_0(w)$ is an automorphism of U with $T^2 = \text{Id}$. Obviously, T leaves the space $U_2(p)$ invariant. Moreover, a straightforward calculation show T exchanges u_{11} with u_{ii}. But from the Corollary in Case 1, $U_1(u_{11}) \subseteq U_2(p)$. Thus, $U_1(u_{ii}) \subseteq U_2(p)$.

Finally we show that $J(v)$ is a Cartan factor of type 3. Let H be a Hilbert space with an orthonormal basis $\{\xi_i; i \in \tilde{I}\}$. Let us equip H with a conjugation J defined as $J(\sum \lambda_i \xi_i) = \sum \bar{\lambda}_i \xi_i$. For $i, j \in \tilde{I}$, let $S_{ii} = e_{ii}$, and $S_{ij} = e_{ij} + e_{ji}$ for $i \neq j$ where e_{ij} and e_{ji} are matrix units corresponding to the basis $\{\xi_i\}$. The family $\{S_{ij}\}$ is then a hermitian grid. Let M_2 be its norm-closed span. Then M_2 is a nuclear JB*-triple, and moreover, M_2^{**} is a Cartan factor of type 3 (as described in section 1). Let M_1 = norm-closed span $\{u_{ij}\}_{i,j \in \tilde{I}}$, and define a linear map $\phi : M_1 \to M_2$ by $\phi(u_{ij}) = S_{ij}$. Obviously ϕ is a triple isomorphism. Thus by Lemma 1.14, it extends to an isomorphism $\tilde{\phi} : J(v) \to M_2^{**}$, proving that $J(v)$ is a Cartan factor of type 3.

Case 3.

PROPOSITION. *Let v and $\tilde{v}$ be minimal tripotents of a* JBW*-*triple U such that* rank $U_1(v) = 2$ *and* $U_1(v+\tilde{v}) = \{0\}$. *Then $J(v)$ is a Cartan factor of type* 4 *and is a summand.*

PROOF. It is obvious that $J(v) = U_2(v+\tilde{v})$, and is a summand. Let $u_1 = v$, $\tilde{u}_1 = \tilde{v}$. By Zorn's lemma there is a maximal colinear family $\{u_i\}_{i\in I}$ of minimal tripotents in U, including u_1 (where I is some index set). For $i \in I$, with $i \neq 1$, let $\tilde{u}_i = -2\{u_1, u_i, \tilde{u}_1\}$. Then the $(u_1, u_i, \tilde{u}_1, \tilde{u}_i)$ are odd quadrangles glued together along the diagonal $(u_1, \tilde{u}_1)$. It follows (from Proposition 1.9) that $(u_1, u_i, \tilde{u}_1, \tilde{u}_i)$ form odd quadrangles for $i \neq j$.

If $\bigcap_{i\in I} U_1(u_i) \neq \{0\}$, let u_0 be a tripotent in that JBW*-triple. From Proposition 2.1 and the maximality of the family $\{u_i\}_{i\in I}$, it follows that u_0 is a minimal tripotent of $\bigcap_{i\in I} U_1(u_i)$, and u_0 governs both $u_i, \tilde{u}_i$ for all $i \in I$, implying

$$\bigcap_{i\in I} U_1(u_i) = P_2(u_0)\left(\bigcap_i U_1(u_i)\right) = \mathbf{C}u_0.$$

By multiplying u_0 with an appropriate scalar λ if necessary, we can assume without loss of generality that u_0 is a tripotent satisfying $\{u_0, u_1, u_0\} = -\tilde{u}_1$. From (1.4) and (1.11), it follows that $\{u_0, u_i, u_0\} = -\tilde{u}_i$, and $\{u_0, \tilde{u}_i, u_0\} = -u_i$ for any $i \in I$. From the lemma of Case 1, the summations $\sum_{i\in I} P_2(u_i)z$ and $\sum_{i\in I} P_2(\tilde{u}_i)z$ converge in norm for all $z \in U$. Thus $J(v)$ is the norm closed span of either $\{u_i, \tilde{u}_i\}_{i\in I}$ or $\{u_i, \tilde{u}_i, u_0\}_{i\in I}$.

We define an inner product on $J(v)$ as

$$\left\langle \sum_{i\in I} \alpha_i u_i + \tilde{\alpha}_i \tilde{u}_i + \beta u_0 \,\middle|\, \sum_{i\in I} \lambda_i u_i + \tilde{\lambda}_i \tilde{u}_i + \delta u_0 \right\rangle = \sum_{i\in I} \alpha_i \bar{\lambda}_i + (\tilde{\alpha}_i)\overline{(\tilde{\lambda})_i} + 2\beta\bar{\delta}.$$

Denote the inner product norm by $\|\cdot\|_2$ and the original norm on U by $\|\cdot\|$. We will show that these two norms are equivalent on $J(v)$. Let

$$z = \sum_{i\in I} \lambda_i u_i + \tilde{\lambda}_i \tilde{u}_i + \beta u_0$$

be an arbitrary element of $J(v)$. Using Lemma 2 and Bessel's inequality we have

$$\begin{aligned} \|z\| &\leq \left\|\sum_{i\in I} \lambda_i u_i\right\| + \left\|\sum_{i\in I} \tilde{\lambda}_i \tilde{u}_i\right\| + \|\beta u_0\| \\ &\leq \left(\sum_{i\in I} |\lambda_i|^2\right)^{1/2} + \left(\sum_{i\in I} |\tilde{\lambda}_i|\right)^{1/2} + |\beta| \leq 3\|z\|_2. \end{aligned}$$

On the other hand,

$$\|z\|_2^2 = \sum_{i\in I} |\lambda_i|^2 + \sum_{i\in I} |\tilde{\lambda}_i|^2 + 2|\beta|^2 = \left\|\sum_{i\in I} P_2(u_i)z\right\|^2 + \left\|\sum_{i\in I} P_2(\tilde{u}_i)z\right\|^2 + 2\|\beta u_0\|.$$

By the lemma of Case 1,

$$\|\beta u_0\| = \left\|z - \sum_{i\in I} P_2(u_i)z - \sum_{i\in I} P_2(\tilde{u}_i)z\right\| \leqq 3\|z\|$$

and $\|z\|_2^2 \leqq 8\|z\|^2$. Therefore the original norm $\|\cdot\|$ and the inner product norm $\|\cdot\|_2$ are equivalent on $J(v)$, proving that $J(v)$ is isomorphic to a Hilbert space.

Finally let $*$ be a conjugation on $J(v)$ defined by $(u_i)^* = \tilde{u}_i$, $(u_0)^* = u_0$. Then the triple product on $J(v)$ can be expressed in terms of the inner product by

$$2\{a, b, c\} = \langle a|b\rangle c + \langle c|b\rangle a - \langle a|c^*\rangle b^*.$$

Simple verification shows this identity holds on the basis elements $\{u_i, \tilde{u}_i, u_0\}_{i\in I}$ of $J(v)$. Then by passing to the limit we conclude that it holds for arbitrary a, b, c in $J(v)$.

We will need the following corollary later:

Corollary. *If U is a spin factor* (i.e., *a Cartan factor of type* 4), *then U is the norm closed span of a family consisting of minimal tripotents $\{u_i, \tilde{u}_i\}_{i\in I}$ and possibly a tripotent u_0 such that*

(i) $u_0 \vdash u_i$, $u_0 \vdash \tilde{u}_i$, $Q(u_0)u_i = -\tilde{u}_i$, $Q(u_0)\tilde{u}_i = -u_i$, *for all* $i \in I$;
(ii) $(u_i, u_j, \tilde{u}_i, \tilde{u}_j)$ *are odd quadrangles for* $i \neq j$;
(iii) $U = U_2(u_i + \tilde{u}_i)$ *for all* $i \in I$.

Such a family $\{u_i, \tilde{u}_i, u_0\}$ will be called a spin grid.

Case 4.

Definition (see [12]). Let I, J denote some index sets. A family of minimal tripotents $\{u_{ij}\}_{i\in I, j\in J}$ is a rectangular grid if

(i) u_{jk}, u_{il} are colinear if they share a common row index ($j = i$) or a column index ($k = l$), and are orthogonal otherwise;
(ii) $(u_{jk}, u_{jl}, u_{il}, u_{ik})$, $j \neq i$, $k \neq l$, is a quadrangle;
(iii) all other types of products (i.e., not of the form $D(x)y$ or $\{x, y, z\}$, where (x, y, z) form a prequadrangle) vanish.

It is easy to see that each Cartan factor of type 1 has a rectangular grid whose span is w^*-dense.

PROPOSITION. *Let v be a minimal tripotent in a* JBW*-*triple* U. *Suppose there are minimal tripotents* $u_1, \tilde{u}_1, \tilde{v}$ *such that* $(v, u_1, \tilde{v}, \tilde{u}_1)$ *form a quadrangle, and* $\dim U_2(v+\tilde{v}) = 4$; *then* $J(v)$ *is a Cartan factor of type* 1 *and is a summand in* U.

We will need the following lemma which gives the structure of $U_1(v)$ in this case. Note that by the corollary in Case 3, the hypothesis $\dim U_2(v+\tilde{v}) = 4$ is equivalent to $U_1(v) \cap U_1(u_1) \cap U_1(\tilde{u}_1) = \{0\}$.

LEMMA. *Let* v, u_1, w_1 *be minimal tripotents in a JBW*-triple* U *such that* (u_1, v, w_1) *form a prequadrangle and* $U_1(v) \cap U_1(u_1) \cap U_1(w_1) = \{0\}$. *Then there are maximal colinear families of tripotents* $\{u_i\}_{i\in I}$ *and* $\{w_j\}_{j\in J}$ *(with* $1 \in I \cap J$*) such that*

(i) $U_1(v) = \overline{\operatorname{span}\{u_i\}_{i\in I}} \oplus \overline{\operatorname{span}\{w_j\}_{j\in J}} := \mathscr{H}_1 \oplus \mathscr{H}_2$;
(i) $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *are orthogonal*, i.e., *every tripotent in* $\mathscr{H}_1$ *is orthogonal to every tripotent in* $\mathscr{H}_2$;
(iii) $P_1(v) = \sum_{i,j}(P_2(u_i)+P_2(w_j))$ *(where the summation converges strongly).*

PROOF. Since $U_1(v)$ is a rank 2 JBW*-triple containing u_1, w_1 as two orthogonal tripotents and $U_1(v) = (P_2(u_1+w_1)+P_1(u_1+w_1))U_1(v)$. Let

$$M = P_1(u_1)P_0(w_1)U_1(v)$$
$$\tilde{M} = P_1(w_1)P_0(u_1)U_1(v).$$

Then by (2.1) and (2.2),

$$P_1(u_1+w_1)U_1(v) = M \oplus \tilde{M}$$
$$P_2(u_1+w_1)U_1(v) = P_2(u_1)U_1(v) \oplus P_2(w_1)U_1(v) \oplus P_1(u_1)P_1(w_1)U_1(v).$$

From the hypothesis, $P_1(u_1)P_1(w_1)U_1(v) = \{0\}$, giving

$$P_2(u_1+w_1)U_1(v) = P_2(u_1)U_1(v) \oplus P_2(w_1)U_1(v)$$
$$= \mathbf{C}u_1 \oplus \mathbf{C}w_1.$$

Thus, $U_1(v) = \mathbf{C}u_1 \oplus \mathbf{C}w_1 \oplus M \oplus \tilde{M}$.

Since M, $\tilde{M}$ are orthogonal to w_1 and u_1, respectively, they are of rank 1 (or 0). By the corollary of Case 1, they are the norm closed spans of maximal colinear families of tripotents. By Proposition 2.1, every tripotent in M is colinear to u_1 and every tripotent in $\tilde{M}$ is colinear to w_1. Thus there are index sets I, J (with $1 \in I \cap J$); and maximal colinear families $\{u_i\}_{i\in I}$, $\{w_j\}_{j\in J}$ of tripotents, satisfying part (i) of the lemma.

Since $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces spanned by $\{u_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$, part (iii) of the lemma follows from the lemma of Case 1.

For part (ii), it suffice to show that $u_i \perp w_j$ for all $(i,j) \in I \times J - \{(1,1)\}$.

By (1.11), $\{u_i, u_i, w_j\} \in \tilde{M}$. Since rank $\tilde{M} = 1$, if $\{u_i, u_i, w_j\} \neq 0$, there is a minimal tripotent $e \in \tilde{M}$ and a scalar λ such that $\{u_i, u_i, w_j\} = \lambda e$. We will show (u_1, u_i, e) form a prequadrangle. It is known that $u_1 \top u_i$, and $u_1 \perp e$. On the other hand, by (1.11),

$$e = P_2(u_i)e + P_1(u_i)e = \mu u_i + P_1(u_i)e.$$

Since $e = P_0(u_1)e$, we have $e \in U_1(u_i)$, implying $e \top u_i$ (by Proposition 2.1). Thus (u_1, u_i, e) form a prequadrangle which can be completed with $f = 2\{u_1, u_i, e\}$. By (1.11), $f \in U_1(v) \cap U_1(u_1) \cap U_1(w_1)$, contradicting $U_1(v) \cap U_1(u_1) \cap U_1(w_1) = \{0\}$.

If u_1, u_2 are two colinear minimal tripotents, then $(u_1 + u_2)/\sqrt{2}$ is a tripotent and the map

$$T(u_1, u_2) := P_2\left(\frac{u_1 + u_2}{\sqrt{2}}\right) - P_1\left(\frac{u_1 + u_2}{\sqrt{2}}\right) + P_0\left(\frac{u_1 + u_2}{\sqrt{2}}\right)$$

is an isometry on M, i.e., an isometric automorphism with $T(u_1, u_2)^2 = \mathrm{Id}$. Straightforward calculation shows that $T(u_1, u_2)$ exchange u_1 with u_2, and

$$T(u_1, u_2) = \mathrm{Id} + Q(u_1 + u_2)^2 - 2D(u_1 + u_2, u_1 + u_2), \tag{2.12}$$

which is exactly the colinear exchange map in [12, Section 1.1]. As will be shown later, if u_1, u_2 are two colinear elements from a rectangular or sympletic grid, then roughly speaking $T(u_1, u_2)$ will exchange two "rows" or two "columns" of the grid.

Proof of the Proposition. Let $u_{11} = v$, $u_{12} = u_1$, and $u_{21} = \tilde{u}_1$. From the above lemma, there are maximal colinear families of tripotents $\{u_{i1}, i \in I\}$, $\{u_{1j}, j \in J\}$ in $U_1(u_{11})$ with $2 \in I \subseteqq J$ and $1 \notin I$ such that

$$P_1(u_{11}) = \sum_{i \in I} P_2(u_{i1}) + \sum_{j \in J} P_2(u_{1j}) \tag{2.13}$$

where the summation converges strongly. Also (u_{i1}, u_{11}, u_{1j}) form a prequadrangle; thus, define $u_{ij} = 2\{u_{i1}, u_{11}, u_{1j}\}$, $(i, j) \in I \times J$.

Let $\tilde{I} = I \cup \{1\}$, $\tilde{J} = J \cup \{1\}$. By Proposition 2.1, u_{ij} is a minimal tripotent of U for any $(i, j) \in \tilde{I} \times \tilde{J}$. We will show the family $\{u_{ij}; (i, j) \in \tilde{I} \times \tilde{J}\}$ form a rectangular grid. Properties (i) and (iii) of the definition of rectangular grid follow from (1.11). To show (iii), i.e., $(u_{jk}, u_{jl}, u_{il}, u_{ik})$ form a quadrangle (for $i \neq j$ and $k \neq l$), note that the four quadrangles $(u_{j1}, u_{11}, u_{1k}, u_{jk})$, $(u_{j1}, u_{11}, u_{1l}, u_{jl})$, $(u_{i1}, u_{11}, u_{1l}, u_{il})$ and $(u_{i1}, u_{11}, u_{1k}, u_{ik})$ can be glued together according to Proposition 1.10 to yield the quadrangle $(u_{jk}, u_{jl}, u_{il}, u_{ik})$. The process is suggested in the diagram below and is similar to the one in Case 2:

$$\begin{array}{ccccc}
u_{11} & — & u_{1k} & — & u_{1l} \\
| & & | & & | \\
u_{j1} & — & u_{jk} & — & u_{jl} \\
| & & | & & | \\
u_{i1} & — & u_{ik} & — & u_{il}
\end{array}$$

For $(i, j) \in \tilde{I} \times \tilde{J}$, define the symmetries $T(u_{11}, u_{1j})$ and $T(u_{1j}, u_{ij})$ as in (2.12). Straightforward calculations shows:

$$\begin{aligned}
T(u_{11}, u_{1j})\colon\ & u_{11} \to u_{1j} \\
& u_{k1} \to -u_{kj} \quad (k \neq 1) \\
& u_{1j} \to u_{11} \\
& u_{1l} \to -u_{1l} \quad (l \neq 1, j)
\end{aligned}$$

and

$$\begin{aligned}
T(u_{1j}, u_{ij})\colon\ & u_{1j} \to u_{ij} \\
& u_{1l} \to -u_{il} \quad (l \neq j) \\
& u_{ij} \to u_{1j} \\
& u_{kj} \to -u_{kj} \quad (k \neq 1, i).
\end{aligned}$$

By taking composition of the above maps if necessary, for each $(i, j) \in \tilde{I} \times \tilde{J}$ we can assume there is an isometric automorphism ψ such that $\psi(u_{11}) = u_{ij}$ and

$$\begin{aligned}
\psi(u_{1l}) &= \pm u_{il} \quad l \in J - \{j\} \\
\psi(u_{1j}) &= \pm u_{i1} \\
\psi(u_{k1}) &= \pm u_{kj} \quad k \in I - \{i\} \\
\psi(u_{i1}) &= \pm u_{1j}.
\end{aligned}$$

Since ψ is an automorphism, for each tripotent e, we have

$$P_k(\psi(e)) = \psi P_k(e) \psi^{-1} \quad (k = 0, 1, 2).$$

Thus, using (2.13) and that ψ is isometric, we have

$$
\begin{aligned}
P_1(u_{ij}) &= P_1(\psi(u_{11})) = \psi P_1(u_{11})\psi^{-1} \\
&= \sum_{k\in I} \psi P_2(u_{k1})\psi^{-1} + \sum_{l\in J} \psi P_2(u_{1l})\psi^{-1} \\
&= \sum_{k\in I} P_2(\psi(u_{k1})) + \sum_{l\in J} P_2(\psi(u_{1l})) \\
&= \sum_{k\in \tilde{I}-\{i\}} P_2(u_{kj}) + \sum_{l\in \tilde{J}-\{j\}} P_2(u_{il}).
\end{aligned} \tag{2.14}
$$

Let $p = \sum_{i\in \tilde{I}} u_{ii}$. Since $U_2(p)+U_1(p)$ is the w^*-closed span of

$$\{U_2(u_{ii}), U_1(u_{ii}) : i \in \tilde{I}\},$$

(2.14) implies $U_2(p)+U_1(p)$ is the w^*-closed span of $\{u_{ij}; (i,j)\in \tilde{I}\times\tilde{J}\}$. On the other hand, it is obvious that $P_2(u_{ij})U_0(p) = \{0\}$, and moreover from (2.14) follows $P_1(u_{ij})U_0(p) = \{0\}$. Thus, $P_0(u_{ij})U_0(p) = U_0(p)$ for arbitrary i, j, proving $U_0(p)$ is orthogonal to $U_2(p)+U_1(p)$. That is $J(v)$ is the w^*-closed span of $\{u_{ij}; (i,j)\in \tilde{I}\times\tilde{J}\}$, and is a summand.

To show that $J(v)$ is a Cartan factor of type 1, let H, K be Hilbert spaces of appropriate dimensions, and $\{e_{ij}; (i,j)\in \tilde{I}\times\tilde{J}\}$ be a system of matrix units of $\mathscr{B}(H,K)$. The family $\{e_{ij}\}$ is a rectangular grid. Let M_2 be the norm-closed span of $\{e_{ij}\}$. Then M_2 is a nuclear JB*-triple and $M_2^{**} = \mathscr{B}(H,K)$ is a Cartan factor of type 1.

Let M_1 be the norm-closed span of $\{u_{ij}\}$ and define a linear map $\phi: M_1 \to M_2$ by $\phi(u_{ij}) = e_{ij}$. Then ϕ is a triple isomorphism and thus has an extension to an isomorphism $\tilde{\phi}: J(v) \to M_2^{**}$ by Lemma 1.14.

Case 5.

DEFINITION. A family $F = \{u_{ij}; i, j\in \tilde{I}\}$ in a JBW*-triple is called a *sympletic grid* if $u_{ii} = 0$, u_{ij} are minimal tripotents with $u_{ij} = -u_{ij}$ (for all distinct $i, j \in \tilde{I}$) and

(i) u_{ij} and u_{kl} are colinear if they share an index and are orthogonal, otherwise.
(ii) $(u_{ij}, u_{il}, u_{kl}, u_{kj})$ form a quadrangle for pairwise distint i, j, k, l.
(iii) Nonvanishing triple products among elements of the the family are one of the forms $D(x)y$ or $\{x, y, z\}$ where (x, y, z) form a prequadrangle.

Each Cartan factor of type 2 is the w^*-closed span of a symplectic grid.

PROPOSITION. *Let v be a minimal tripotent in a* JBW*-*triple U such that* $\operatorname{rank} U_1(v) = 2$. *If there is a minimal tripotent $\tilde{v}$ orthogonal to v with* $\dim U_2(v+\tilde{v}) = 6$, *then $J(v)$ is a Cartan factor of type* 2, *and is a summand.*

PROOF. Since $U_2(v+\tilde{v})$ is a 6-dimensional spin factor, the corollary of Case 3 implies $U_1(v)$ contains a quadrangle $(u_{13}, u_{14}, u_{24}, u_{23})$ consisting of minimal tripotents, and $\dim U_1(v) \cap U_2(u_{13}+u_{24}) = 4$. Let $J(u_{13})$ be the w^*-closed ideal of $U_1(v)$ generated by u_{13}. By the proposition in Case 4, $J(u_{13})$ is isomorphic to a Cartan factor of type 1, and is the norm-closed span of a rectangular grid $\{u_{1i}, u_{2i}\}_{i \in I}$, where I is some index set not containing $\{1, 2\}$. Since $J(u_{13})$ is a summand of rank 2 in the rank 2 JBW*-triple $U_1(v)$, we have $U_1(v) = J(u_{13})$. Let $u_{12} = v$. The lemma of Case 1 implies:

$$P_1(u_{12}) = \sum_{i \in I} (P_2(u_{1i}) + P_2(u_{2i})). \tag{2.15}$$

Let $\tilde{I} = I \cup \{1, 2\}$. For distinct $i, j \in I$, let $u_{i1} = -u_{1i}$, $u_{i2} = -u_{2i}$. From the definition of a rectangular grid, $u_{2i} \perp u_{1j}$, implying (u_{i2}, u_{12}, u_{1j}) form a prequadrangle. Thus define

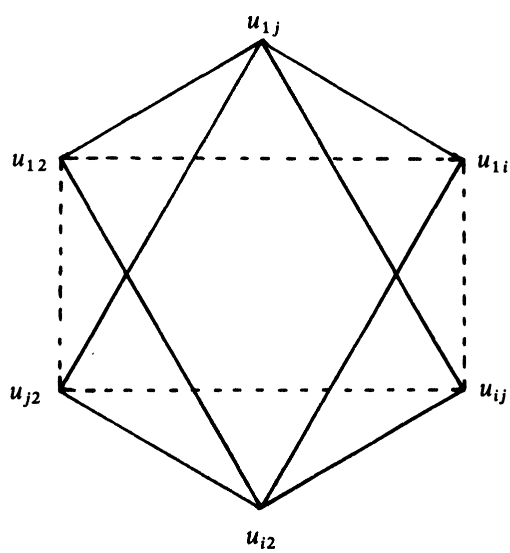
$$u_{ij} = 2\{u_{i2}, u_{12}, u_{1j}\} \quad \text{for distinct } i, j \in I$$

$$u_{ii} = 0 \qquad\qquad \text{for all } i \in \tilde{I}.$$

We will show that $\{u_{ij}; i, j \in I\}$ form a symplectic grid. To show $u_{ij} = -u_{ji}$ for $i, j \in \tilde{I}$, it suffices to consider the cases $i, j \in I$. By Lemma 1.8, the two quadrangles $(u_{1i}, u_{i2}, u_{j2}, u_{1j})$ and $(u_{ij}, u_{i2}, u_{12}, u_{1j})$ can be glued together along the diagonal (u_{i2}, u_{1j}) to give the quadrangle $(u_{1i}, u_{ij}, u_{j2}, -u_{12})$, i.e.,

$$u_{ij} = -2\{u_{j2}, u_{12}, u_{1i}\} = -u_{ji}.$$

The construction is described in the following diagram:



Property (i) of the definition of symplectic grid can be verified easily with (1.11). Property (iii) follows from (1.11) and the fact that any two distinct nonzero elements of the family $\{u_{ij}; i, j \in \tilde{I}\}$ are either orthogonal or colinear minimal tripotents. To verify (ii), note that the four quadrangles $(u_{i2}, u_{12}, u_{1j}, u_{ij})$, $(u_{i2}, u_{12}, u_{1l}, u_{il})$, $(u_{k2}, u_{12}, u_{1l}, u_{kl})$ and $(u_{k2}, u_{12}, u_{1j}, u_{kj})$ can be glued together according to Proposition 1.10 to yield the quadrangle $(u_{ij}, u_{il}, u_{kl}, u_{kj})$. This can be visualized in the following diagram:

$$\begin{array}{ccccc} u_{12} & \text{———} & u_{1j} & \text{———} & u_{1l} \\ | & & | & & | \\ u_{i2} & \text{———} & u_{ij} & \text{———} & u_{il} \\ | & & | & & | \\ u_{k2} & \text{———} & u_{kj} & \text{———} & u_{kl} \end{array}$$

For each i,j in I with $i \neq 1$, $j \neq 2$, define the symmetries $T(u_{12}, u_{1j})$ and $T(u_{1j}, u_{ij})$ as in (2.12). Straightforward calculation gives

$$\begin{aligned} T(u_{12}, u_{1j}) \colon u_{1k} &\to -u_{1k} \quad (k \neq 2, j) \\ u_{2j} &\to -u_{2j} \\ u_{2k} &\to -u_{jk} \quad (k \neq j), \end{aligned}$$

and

$$\begin{aligned} T(u_{1j}, u_{ij}) \colon u_{1k} &\to -u_{ik} \quad (k \neq i) \\ u_{1i} &\to -u_{1i} \\ u_{kj} &\to -u_{kj} \quad (k \neq 1, i). \end{aligned}$$

For distinct i,j in $\tilde{I}$, by taking composition of the above maps if necessary, we can assume there is an isometric automorphism ψ such that $\psi(u_{12}) = u_{ij}$ and

$$\begin{aligned} \psi(u_{1l}) &= \pm u_{il} \quad l \neq 2, j \\ \psi(u_{1j}) &= \pm u_{i2} \\ \psi(u_{k2}) &= \pm u_{kj} \quad k \neq 1, i \\ \psi(u_{i2}) &= \pm u_{1j}. \end{aligned}$$

By the same reason as in the proof of the proposition in Case 4, we have

$$(2.16) \qquad P_1(u_{ij}) = \sum_{k \in \tilde{I} - \{i\}} P_2(u_{kj}) + \sum_{l \in \tilde{I} - \{j\}} P_2(u_{il})$$

(in which the summations converge strongly). Let W be a maximal orthogonal subset of $\{u_{ij}; i,j \in \tilde{I}\}$ and $p = \sum_{e \in W} e$. It follows from (2.16) that $U_2(p) + U_1(p)$ is the w^*-closed span of $\{u_{ij}; i,j \in \tilde{I}\}$ and moreover $U_2(p) + U_1(p)$ is orthogonal to $U_0(p)$. The detailed argument is the same as in Case 4. Thus $J(v)$ is the w^*-closed span of u_{ij} and is a summand.

To show that $J(v)$ is a Cartan factor of type 2, let H be a Hilbert space with an orthonormal basis $\{\xi_i\}_{i \in \tilde{I}}$ and a conjugation J defined by $J(\Sigma \lambda_i \xi_i) = \Sigma \bar{\lambda}_i \xi_i$. For $i, j \in \tilde{I}$, $i < j$, let $a_{ii} = 0$ and $a_{ij} = e_{ij} - e_{ji}$ where the e_{ij} are matrix units corresponding to the basis $\{\xi_i\}$. The family $\{u_{ij}; i,j \in \tilde{I}\}$ is a symplectic grid. Let M_2 be its norm-closed span. Then M_2 is a nuclear JB*-triple, and moreover M_2^{**} is a Cartan factor of type 2.

Let M_1 be the norm-closed span of $\{u_{ij}\}$, and define a linear map $\phi: M_1 \to M_2$ by $\phi(u_{ij}) = a_{ij}$. Obviously ϕ is a triple isomorphism. Thus, by Lemma 1.14, it extends to an isomorphism $\tilde{\phi}: J(v) \to M_2^{**}$, proving that $J(v)$ is a Cartan factor of type 2.

Case 6. Throughout this case and Case 7 we will use the following notations Let $I = \{0, 1, 2, 3, 4, 5\}$; $I_i = I - \{i\}$ for any $i \in I$; and (i, j, k, l, m, n) denotes any permutation of $(0, 1, 2, 3, 4, 5)$. If φ is a permutation on I, then $\mathrm{sign}(\varphi) = 1$ or -1 depending on whether φ is even or odd.

DEFINITION. For any fixed $i \in I$, a family $\mathscr{F} = \{u_i, u_{jk}, u_n : j, k, n \in I_i\}$ is called an exceptional grid of the first type if

(i) $\{u_{jk} : j, k \in I_i\}$ form a sympletic grid;
(ii) for any j, k, l, s, t in I_i, we have

$$(2.17)\qquad \begin{array}{ll} u_i \top u_{jk},\ u_i \perp u_k & \\ u_k \perp u_{st} & \text{if } k \in \{s,t\}; \\ u_k \top u_{st} & \text{if } k \notin \{s,t\}; \\ u_k \top u_l & \text{if } k \neq l; \end{array}$$

(iii) the nonvanishing products among elements of $\mathscr{F}$ are of the form $D(x)y$ or $\{x, y, z\}$ where (x, y, z) form a prequadrangle in a quadrangle of one of the following three types $(j, k, l, m \in I_i)$:
(a) $(u_{jk}, u_{jl}, u_{ml}, u_{mk})$,
(b) $(\mathrm{sign}(\varphi)u_i, u_{jk}, u_l, u_{mn})$, where $\varphi = (i, j, k, l, m, n)$,
(c) $(u_j, u_k, -u_{jl}, u_{kl})$.

LEMMA. *Let* $\mathscr{F} = \{u_i, u_{jk}, u_n : j, k, n \in I_i\}$ *be a set of minimal tripotents in a* JB*-*triple such that*

(i) $\{u_{jk} : j, k \in I_i\}$ *is a sympletic grid,*
(ii) $u_i \top u_{jk}$ *for all distinct* $j, k \in I_i$,

(iii) $u_l = \text{sign}(\varphi)2\{u_{jk}, u_i, u_{mn}\}$ *for any* $l \in I_i$, *and* $\varphi = (i, j, k, l, m, n)$ *is a permutation with* $j < k < m < n$.

Then $\mathscr{F}$ *is an exceptional grid of the first type.*

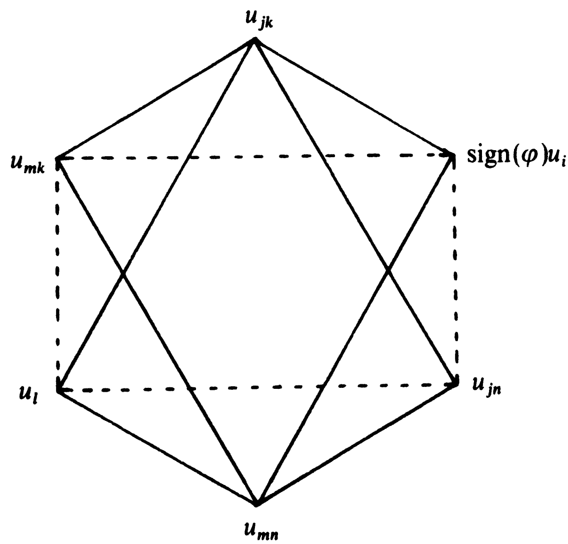
PROOF. By (1.11) we have that all relations (2.17) are satisfied and $\mathscr{F}$ is an orthocolinear family of minimal tripotents. Therefore the nonvanishing products of $\mathscr{F}$ are of the form $D(x)y$ or $\{x, y, z\}$ where (x, y, z) form a pre-quadrangle.

If the quadrangle contains only elements of the form u_{jk} with $j, k \in I_i$, it belongs to the symplectic grid and is of the form (a). If the quadrangle contains u_i, it is of the form (b). Otherwise, it must be of the form (c). Thus, it remains to show that (b) and (c) are actually quadrangles in $\mathscr{F}$.

To show that (b) are quadrangles, we will use the following:

CLAIM. Let $\varphi = (j, k, l, m, n)$ be any permutation on I_i. If $\text{sign}(\varphi)u_i, u_{jk}, u_l, u_{mn})$ form a quadrangle and τ is any permutation on the set $\{j, k, m, n\}$, then $(\text{sign}(\tau\varphi)u_i, u_{\tau(j)\tau(k)}, u_l, u_{\tau(m)\tau(n)})$ form a quadrangle.

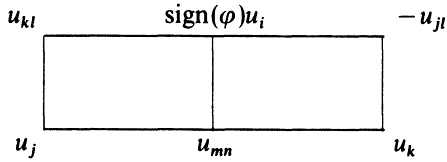
PROOF OF CLAIM: Since every permutation is a product of transpositions, i.e., permutations that exchange only two elements of the set and leave the other elements fixed, we can assume that τ is a transposition. Moreover, if τ exchanges k with j, or n with m, then the claim holds since $u_{kj} = -u_{jk}$ and $u_{nm} = -u_{mn}$. Thus we can assume further that τ exchanges j with m. By Lemma 1.8, the two quadrangles $(\text{sign}(\varphi)u_i, u_{jk}, u_l, u_{mn})$ and $(u_{mk}, u_{jk}, u_{jn}, u_{mn})$ can be glued together along the diagonal (u_{jk}, u_{mn}) to obtain the quadrangle $(-\text{sign}(\varphi)u_i, u_{mk}, u_l, u_{jn})$. That is, $(\text{sign}(\tau\varphi)u_i, u_{\tau(j)\tau(k)}, u_l, u_{\tau(m)\tau(n)})$ is a quadrangle, proving the claim. The diagram of this construction is



To show (c) form quadrangles: If $\varphi = (i,k,l,j,m,n)$ and $\varphi' = (i,j,l,k,m,n)$, then $\operatorname{sign}(\varphi) = -\operatorname{sign}(\varphi')$. Thus we have two quadrangles $(\operatorname{sign}(\varphi)u_i, u_{kl}, u_j, u_{mn})$ and

$$(-\operatorname{sign}(\varphi)u_i, u_{jl}, u_k, u_{mn}) = (\operatorname{sign}(\varphi)u_i, -u_{jl}, u_k, u_{mn})$$

which can be glued together along the side $(\operatorname{sign}(\varphi)u_i, u_{mn})$ according to Proposition 1.10, to obtain the quadrangle $(u_j, u_k, -u_{jl}, u_{kl})$. The diagram of the construction is



PROPOSITION. *Let v be a minimal tripotent in a* JBW*-*triple U. If there is a tripotent $\tilde{v}$ orthogonal to v such that* $\dim U_2(v+\tilde{v}) = 8$ *and* $U_1(v+\tilde{v}) \neq \{0\}$, *then $J(v)$ is* 16 *dimensional and spanned by an exceptional grid of the first type. Moreover, $J(v)$ is isomorphic to a Cartan factor of type* 5 *and is a summand.*

PROOF. If $v_1, \tilde{v}_1$ are orthogonal tripotents in $U_1(v) \cap U_1(\tilde{v})$, then $\dim U_1(v) \cap U_2(v_1+\tilde{v}_1) = 6$ and by Lemma 2.4, we may assume that $U_1(v_1+\tilde{v}_1) \cap U_1(v) \neq 0$. By the previous case, the ideal generated by v_1 in $U_1(v)$ is isomorphic to a Cartan factor of type 2. Since $\operatorname{rank} U_1(v) = 2$ this ideal and $U_1(v)$ must be isomorphic to C_5^2, the 5×5 antisymmetric matrices. Let $\{u_{jk} : j,k \in I_0\}$ be a sympletic grid spanning $U_1(v)$. Denote $u_0 = v$.

For any integer $l \in I_0$, define

$$u_l = \operatorname{sign}(\varphi)2\{u_{jk}, u_0, u_{mn}\}$$

where $\varphi = (0,j,k,l,m,n)$ with $j < k < m < n$.

From Proposition 2.1, each u_l is a minimal tripotent in U. By the above Lemma, the family $\mathscr{F} = \{u_0, u_{jk}, u_n : j,k,n \in I_0\}$ is an exceptional grid of the first type. To show $J(v) = \operatorname{span}\mathscr{F}$ and is a summand, let $p = u_0 + u_1$. Since $u_0 \top u_{jk} \top u_i$ for any three distinct indices i,j,k, any element of $\mathscr{F}$ can be excahnged with u_0 by an automorphism of U. Thus, for any $f \in \mathscr{F}$, we have

$$\dim U_1(f) = \dim U_1(u_0) = 10.$$

From (2.17) each element of $\mathscr{F}$ is colinear to exactly 10 other elements of the grid, implying $U_1(f) \subseteqq \operatorname{span}\mathscr{F}$. By repeating the same argument in Case 4, we have $U_2(p)+U_1(p) = \operatorname{span}\mathscr{F}$ and is orthogonal to $U_0(p)$. Thus $J(v) = \operatorname{span}\mathscr{F}$ and is a summand.

Since $M_{1,2}(\mathbb{O})$, the 16 dimensional exceptional factor, satisfies the hypothesis of this proposition, it is spanned by an exceptional grid of the first type; i.e., $J(v)$ and $M_{1,2}(\mathcal{O})$ are isomorphic.

Case 7. Let ε denote the sign $+$ or $-$ with the natural multiplication rule; i.e., $+-=-+=-$, $--=+,\ldots$ etc.

DEFINITION. A family of tripotents $\mathcal{G} = \{u_n^\varepsilon, u_{ij} : i,j,n \in I; \varepsilon = \pm\}$ is called an exceptional grid of the second type if

(i) $\{u_{ij}\}_{i,j\in I}$ is a 6×6 sympletic grid;
(ii) for any $i \in I$ and $\varepsilon = \pm$, the family $\{-\varepsilon u_i^\varepsilon, u_{jk}, u_n^{-\varepsilon} : j,k,n \in I_i\}$ is an exceptional grid of the first type;
(iii) the quadrangles of the family $\mathcal{G}$ are those determined by parts (i) and (ii) above or of the form

$$(u_j^\varepsilon, u_j^{-\varepsilon}, u_k^{-\varepsilon}, u_k^\varepsilon) \quad \text{for distinct } j,k, \in I; \tag{2.18}$$

(iv) the family $\mathcal{G}$ is ortho-colinear, and therefore all nonvanishing products among elements of $\mathcal{G}$ are either of the form $D(x)y$, or $\{x,y,z\}$ where (x,y,z) form a prequadrangle.

PROPOSITION. *Let v be a minimal tripotent in* JBW**-triple U. If there is a minimal tripotent $\tilde{v}$ such that* $\dim U_2(v+\tilde{v}) = 10$ *and* $U_1(v+\tilde{v}) \neq \{0\}$, *then $J(v)$ is spanned by an exceptional grid of the second type, and is a summand. Moreover $J(v)$ is isomorphic to $H_3(\mathcal{O})$.*

PROOF. Let u_0^- be a minimal tripotent in $U_1(v)$, $J(u_0^-)$ be the ideal in $U_1(v)$ generated by u_0^-. The same argument as in the Proposition of Case 6 shows that $J(u_0^-) = U_1(v)$, and is spanned by an exceptional grid of the first type; namely, $\{u_0^-, u_{jk}, u_n^+ : j,k,n \in I_0\}$.

Let $v = -u_0^+$ and define

$$u_l^- := \operatorname{sign}(\varphi)2\{u_{jk}, -u_0^+, u_{mn}\}$$

where $\varphi = (0,j,k,l,m,n)$ and $i < k < m < n$.

$$u_{i0} := 2\{u_{ij}, -u_0^+, u_j^+\}, \tag{2.19}$$

where $i \in I_0$, and j is some element in $I_0 \cap I_i$.

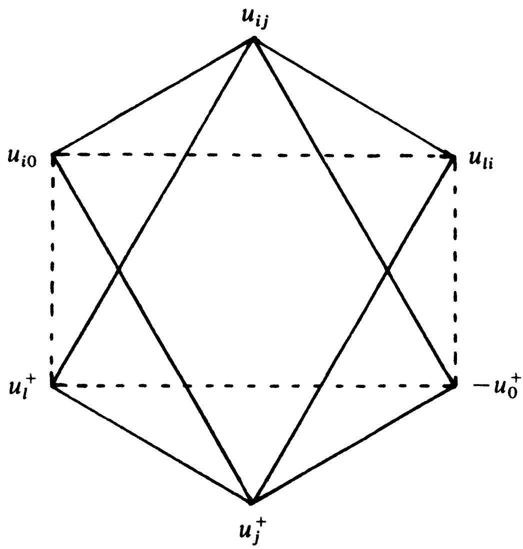
Note that for $i,l \in I_0$, u_l^- and u_{i0} are also minimal tripotents of U; and that the family $\{-u_0^+, u_{jk}, u_n^- : j,k,n \in I_0\}$ is also an exceptional grid of the first type by construction. Moreover, the definition of u_{i0} in (2.19) does not depend on the choice of j. To show this, let l be another element of $I_0 \cap I_i \cap I_j$. Since $\{u_0^-, u_{jk}, u_n^+\}$ is an exceptional grid of the first type,

$$(u_j^+, u_l^+, u_{ij}, u_{li}) = (u_j^+, u_l^+, -u_{ji}, u_{li})$$

is a quadrangle. Thus by Lemma 1.8, it can be glued to the quadrangle $(u_{i0}, u_{ij}, -u_0^+, u_j^+)$ to obtain that

$$(u_{i0}, u_{li}, -u_0^+, -u_l^+) = (u_{i0}, u_{il}, -u_0^+, u_l^+)$$

is a quadrangle, proving $u_{i0} = 2\{u_{il}, -u_0^+, u_l^+\}$. The diagram of the construction is as follows:



Let $u_{00} = 0$, $u_{0i} = -u_{i0}$. We will show that the family

$$\mathscr{G} = \{u_n^\varepsilon, u_{ij} : i, j, n \in I; \varepsilon = \pm\}$$

is an exceptional grid of the second type. Let i,j,k be arbitrary elements of I_0. From (1.11) follows that $u_{i0} \top u_{jk}$ if $i \in \{j, k\}$ and $u_{i0} \perp u_{jk}$ if $i \notin \{j, k\}$. Moreover, if i,j,k are pairwise distinct, we can glue the quadrangles $(u_{i0}, u_{ij}, -u_0^+, u_j^+)$ and $(u_{k0}, u_{kj}, -u_0^+, u_j^+)$ together according to Proposition 1.10 to obtain that $(u_{i0}, u_{ij}, u_{kj}, u_{k0})$ is a quadrangle. Thus, since the family $\{u_{ij}\}_{i,j \in I_0}$ is a 5×5 symplectic grid, the family $(u_{ij}\}_{i,j \in I}$ is a 6×6 symplectic grid, proving (i).

As noted already, part (ii) holds if $i = 0$. For $i \neq 0$ we can apply (1.11) to verify that $u_i^\varepsilon \top u_{jk}$ for all distinct $j,k \in I_i$. Thus, because of the Lemma in Case 6, it suffices to show that

$$\text{(2.20)} \qquad (\operatorname{sign} \varphi(-\varepsilon u_i^\varepsilon), u_{jk}, u_l, u_{mn})$$

are quadrangles for any $\varphi = (i, j, k, l, m, n)$. The proof of this depends on the fact that (2.18) are quadrangles; so we consider (2.18) first.

Let $\varphi = (0, i, j, k, l, m)$. Then we have two quadrangles $(-\operatorname{sign}(\varphi)u_0^-, u_{ij}, u_k^-, u_{lm})$ and $(\operatorname{sign}(\varphi)u_0^-, u_{ij}, u_k^+, u_{lm})$, which can be glued along the diagonals (u_{ij}, u_{lm}) according to Lemma 1.8 to obtain the quadrangle $(u_0^+, u_0^-, u_k^-, u_k^+)$, for arbitrary $k \neq 0$. Now, if $j \neq 0$ and $j \neq k$, then the two quadrangles $(u_0^+, u_0^-, u_j^-, u_j^+)$ and $(u_0^+, u_0^-, u_k^-, u_k^+)$ can be glued together according to Proposition 1.10 to obtain the quadrangle $(u_j^+, u_j^-, u_k^-, u_k^+)$; thus proving that (2.18) are quadrangles for any distint $j, k \in I$.

Next we show that

$$(u_0^\varepsilon, u_i^\varepsilon, u_{j0}, u_{ij}) \tag{2.21}$$

are quadrangles for any $i \in I$ and any ε. From (2.19) we obtain the quadrangle $(u_0^+, u_i^+, u_{j0}, u_{ij})$. Applying Proposition 1.10 to this and the quadrangle $(u_0^+, u_0^-, u_i^-, u_i^+)$ from (2.18) we obtain the quadrangle $(u_0^-, u_i^-, u_{j0}, u_{ij})$; thus proving (2.21) are quadrangles.

Now we are ready to verify (2.10) are quadrangles. Since $\{-\varepsilon u_0^\varepsilon, u_{jk}, u_n: j, k, n \in I_0\}$ are exceptional grids of the first type, elementary properties of permutations imply that (2.10) are quadrangles if $0 \in \{i, l\}$. Thus, it remains to consider the cases when $0 \in \{j, k, m, n\}$. By applying a transposition if necessary we can assume that $k = 0$.

Let $\sigma = (0, i, j, l, m, n)$ and $\varphi = (i, j, 0, l, m, n)$. From (2.21) we obtain the quadrangle $(\varepsilon u_i^\varepsilon, \varepsilon u_0^\varepsilon, u_{ij}, u_{j0})$. Applying Proposition 1.10 to this and the quadrangle $(\varepsilon u_0^\varepsilon, u_{ij}, u_l^{-\varepsilon}, -\operatorname{sign}(\sigma)u_{mn})$, we obtain the quadrangle $(\varepsilon u_i^\varepsilon, u_{j0}, u_l^{-\varepsilon}, -\operatorname{sign}(\sigma)u_{mn})$. It is easy to verify that $\operatorname{sign}(\sigma) = \operatorname{sign}(\varphi)$, implying $(\operatorname{sign}\varphi(-\varepsilon u_i^\varepsilon), u_{j0}, u_l^{-\varepsilon}, u_{mn})$ is a quadrangle for $\varphi = (i, j, 0, l, m, n)$. Thus (ii) is verified.

We have already verified that (2.18) are quadrangles. The remaining claim of (iii) can be verified by inspection, using (i) and the relations (2.17). Part (iv) follows from (i) and (ii). Therefore we have shown that $\mathscr{G}$ is an exceptional grid of the second type and consists of minimal tripotents. Next, observe that each element g in $\mathscr{G}$ is colinear to exactly 16 other elements of $\mathscr{G}$. Thus an argument similar to the one in Case 6 finishes the proof of this proposition.

3. Application.

The first corollary of our Classification Theorem is the decomposition of a JBW*-triple into atomic and nonatomic parts. This result was proved earlier by Friedman and Russo, and the description of the atomic part was proven by Horn.

COROLLARY 3.1. *Every* JBW*-*triple U can be decomposed into orthogonal direct sum of two w*-closed ideals $\mathscr{A}$ and $\mathscr{N}$, where $\mathscr{A}$ is a direct sum of*

Cartan factors, and is spanned by minimal tripotents; and $\mathcal{N}$ has no minimal tripotent.

PROOF. Let $\{J_i\}_{i\in I}$ be the set of distinct w^*-closed ideals generated by minimal tripotents of U. By Theorem 2.1, each of them is a Cartan factor and a summand. Denote the orthogonal components of J_i in U by $J_i^{\perp}$. Let

$$\mathcal{A} = \bigoplus_{i\in I} J_i \quad , \quad \mathcal{N} = \bigcap_{i\in I} J_i^{\perp}.$$

$\mathcal{A}$ and $\mathcal{N}$ are obviously orthogonal w^*-closed ideals in U.

For each $i\in I$, let P_i be the contractive projection from U to J_i with kernel $J_i^{\perp}$. Moreover, since their ranges, the ideals J_i are mutually orthogonal, $\sum_{i\in I} P_i(x)$ converges w^* for each $x\in U$. For $j\in I$,

$$P_j\left(x-\sum_i P_i(x)\right) = P_j(x)-\sum_i P_jP_i(x) = P_j(x)-P_j(x) = 0,$$

implying $x-\sum_i P_i(x)\in\mathcal{N}$. Thus $U = \mathcal{A}\oplus\mathcal{N}$.

Recall that in our constructions, each Cartan factor other than those of type 3 is spanned by a grid consisting of minimal tripotents. On the other hand, the Cartan factors of type 3 are spanned by hermitian grids $\{u_{ij}\}$ with u_{ii} minimal, and u_{ij} not minimal for $i\neq j$. However, it is easy to verify that such u_{ij} $(i\neq j)$ can be written as the sum of two minimal tripotents

$$u_{ij} = \tfrac{1}{2}(u_{ij}+u_{ii}+u_{jj})+\tfrac{1}{2}(u_{ij}-u_{ii}-u_{jj}).$$

Thus $\mathcal{A}$ is the w^*-closed span of the minimal tripotents.

The proof of the Gelfand-Naimark theorem below is similar to the original one in [4] and is included for completeness.

THEOREM 3.2. (Gelfand-Naimark theorem). *Every* JB*-*triple can be isometrically embedded (as triple system) into a direct sum of Cartan factors.*

PROOF. Let M be a JB*-triple. Then its bidual M^{**} is a JBW*-triple in which M can be isometrically embedded [2, Theorems 1,1 and 2.1]. Denote this embedding of M into M^{**} by π. Since M^{**} is a JBW*-triple, it has a decomposition into atomic and nonatomic parts: $M^{**} = \mathcal{A}\oplus\mathcal{N}$ as described in Corollary 3.1 above. Denote the projection from M^{**} onto $\mathcal{A}$ with kernel $\mathcal{N}$ by P. Let $T = P\circ\pi: M\to\mathcal{A}$. Obviously P is a triple homomorphism and $\|Tx\|\leqq\|x\|$, $\forall x\in M$. We will show T is isometric.

Let $x\in M$ with $\|x\| = 1$. The set $\{\varphi\in M^*: \varphi(x) = 1 = \|\varphi\|\}$ is closed and convex, hence contains an extreme point ψ. Elementary argument shows ψ is also an extreme point of M_1^*, the unit ball of M^*. Thus by [3, Proposition 4], there is a minimal tripotent $e \in M^{**}$ such that $\psi(e) = 1$ and $\psi(z) = \psi(P_2(e)(z))$, $\forall z \in M^{**}$. In particular,

$$\psi(T(x)) = \psi(P_2(e)T(x)) = \psi(P_2(e) \circ P \circ \pi(x))$$
$$= \psi(P_2(e) \circ \pi(x)) = \psi(\pi(x)) = \psi(x) = 1.$$

Thus $\|T(x)\| = 1$.

Now, we obtain some properties of the spin factors following from their construction in section 2. Let U be a Cartan factor of type 4; i.e., a spin factor, and $\{u_i, \tilde{u}_i, u_0\}_{i \in I}$ be a spin grid spanning U (cf. Corollary, Case 3). If a is any element of U, we will denote its components relative to the grid by $\{a_i, \tilde{a}_i, a_0\}$; that is, $a = \Sigma a_i u_i + \tilde{a}_i \tilde{u}_i + a_0 u_0$. Recall that U has a conjugation:

$$(3.1) \qquad (\Sigma(a_i u_i + \tilde{a}_i \tilde{u}_i) + a_0 u_0)^* = \Sigma(\bar{a}_i \tilde{u}_i + \bar{\tilde{a}}_i u_i) + \bar{a}_0 u_0$$

and inner product

$$(3.2) \qquad \langle a|b\rangle = \Sigma a_i \bar{b}_i + \tilde{a}_i \bar{\tilde{b}}_i + 2a_0 \bar{b}_0$$

and, the triple product is given by

$$(3.3) \qquad 2\{a, b, c\} = \langle a|b\rangle c + \langle c|b\rangle a - \langle a|c^*\rangle b^*$$

For any element a in U, its "determinant" is defined as

$$\det a : \sum_{i \in I} a_i \tilde{a}_i + a_0^2.$$

Note that $\det a = \frac{1}{2}\langle a|a^*\rangle$, and depends on the choice of the spin grid.

PROPOSITION 3.3. Rank $a = 1$ (i.e., *a is a multiple of minimal tripotent*) *iff* $\det a = 0$. *In this case* $\|a\| = \|a\|_2$.

PROOF. From (3.3) follows

$$Q(a)x = \langle a|x\rangle a - \tfrac{1}{2}\langle a|a^*\rangle x^*$$

for any x in U. Therefore, $Q(a)U = \mathbf{C}a$ iff $\langle a|a^*\rangle = 0$, implying rank $a = 1$ iff $\det a = 0$.

Suppose $\langle a|a^*\rangle = 0$. Then form (3.3) we have $a^3 = \|a\|_2^2 a$. Therefore $\|a\|^3 = \|a^3\| = \|a\|_2^2 \|a\|$, and $\|a\| = \|a\|_2$.

LEMMA 3.4. *If a and b are nonzero orthogonal elements in a spin factor, then* $b = \lambda a^*$ *for some* $\lambda \in \mathbf{C}$.

PROOF. Since $a \perp b$ and rank of spin factor is 2, we have rank a = rank $b = 1$. From (3.3) follows

$$0 = \{a, b, a\} = \langle a|b\rangle a - \tfrac{1}{2}\langle a|a^*\rangle b^* = \langle a|b\rangle a.$$

Thus, $\langle a|b\rangle = 0$.

On the other hand,

$$0 = 2\{a, a, b\} = \langle a|a\rangle b + \langle b|a\rangle a - \langle a|b^*\rangle a^*$$

implies $b = (\langle a|b^*\rangle / \|a\|^2) a^*$.

Now we will decompose an arbitrary element a of U into a linear combination of two orthogonal minimal tripotents. Because of Lemma 3.4, any such decomposition is of the form $a = t_1 x + t_2 x^*$ for some minimal tripotent x. This is equivalent to the following system of equations in terms of the component of a and x,

$$\begin{aligned} a_i &= t_1 x_i + t_2 \bar{\tilde{x}}_i \\ \tilde{a}_i &= t_1 \tilde{x}_i + t_2 \bar{x}_i \quad \text{for } i \in I \\ a_0 &= t_1 x_0 + t_2 \bar{x}_0 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \det x &= 0 \\ \|x\|_2 &= 1. \end{aligned} \tag{3.5}$$

Let $s_1 = |t_1|, s_2 = |t_2|$. Then from (3.4) and (3.5) we get $t_1 t_2 = \det a$. Moreover

$$\begin{aligned} s_1^2 + s_2^2 &= \|a\|_2^2 \\ s_1 s_2 &= |\det a|, \quad s_1 \geqq s_2 \geqq 0, \end{aligned} \tag{3.6}$$

which determine s_1, s_2 uniquely.

If $s_1 \neq s_2$, (3.4) has a unique solution

$$\begin{aligned} x_i &= \frac{1}{s_1^2 - s_2^2}(\bar{t}_1 a_i - t_2 \bar{\tilde{a}}_i) \\ \tilde{x}_i &= \frac{1}{s_1^2 - s_2^2}(-t_2 \bar{a}_i + \bar{t}_1 \tilde{a}_i) \\ x_0 &= \frac{1}{s_1^2 - s_2^2}(\bar{t}_1 a_0 - t_2 \bar{a}_0) \end{aligned}$$

which also satisfy (3.5).

Therefore, by letting $e = (t_1/s_1)x$, $f = (t_2/s_2)x^*$, we have a unique decomposition of a: $a = s_1 e + s_2 f$ with s_1, s_2 as nonnegative reals; and e, f as orthogonal minimal tripotents whose components are

$$e_i = \frac{1}{s_1^2 - s_2^2}\left(s_1 a_i - \frac{\det a}{s_1}\bar{\tilde{a}}_i\right)$$

(3.7)
$$\tilde{e}_i = \frac{1}{s_1^2 - s_2^2}\left(-\frac{\det a}{s_1}\bar{a}_i + s_1 \tilde{a}_i\right)$$

$$e_0 = \frac{1}{s_1^2 - s_2^2}\left(s_1 a_0 - \frac{\det a}{s_1}\bar{a}_0\right)$$

and

$$f_i = \frac{1}{s_1^2 - s_2^2}\left(-s_2 a_i + \frac{\det a}{s_2}\bar{\tilde{a}}_i\right)$$

(3.8)
$$\tilde{f}_i = \frac{1}{s_1^2 - s_2^2}\left(\frac{\det a}{s_2}\bar{a}_i - s_2 \tilde{a}_i\right)$$

$$f_0 = \frac{1}{s_1^2 - s_2^2}\left(\frac{\det a}{s_2}\bar{a}_0 - s_2 a_0\right).$$

In summary, we have the following proposition.

PROPOSITION 3.6. *For each element a in U, there is a unique set of non-negative real numbers* $\{s_1, s_2\}$ *determined by* (3.6), *such that any spectral decomposition of a has the form* $a = s_1 e + s_2 f$, *for some orthogonal minimal tripotents e and f. Moreover, if* $s_1 \neq s_2$, *then the tripotents e and f are unique, and are given by* (3.7) *and* (3.8) *above. If* $s_1 = s_2$ *any solution of* (3.4) *and* (3.5) *will give a decomposition.*

The numbers s_1 and s_2 are called the singular values of a. Note that they do not depend on the choice of the spin grid.

COROLLARY 3.7. $\|a\| = \max\{s_1, s_2\}$.

Since U is isomorphic to a Hilbert space with equivalent norm, U can be identified with its dual U^* (as well as U_*), and each functional on U is of the form $f(x) = \langle x|f\rangle$ for some element f in U. Using the one-to-one correspondence between minimal tripotents and the extreme points of the unit ball of U_*, we obtain the following corollary from Proposition 3.6.

COROLLARY 3.8.

$$\|f\|_1 = s_1 + s_2 = \sqrt{\|f\|_2 + 2|\det f|}$$

where $\|f\|_1$ *denote its norm as a linear functional, and* s_1, s_2 *are its singular values.*

Moreover, if we identify U with a JBW*-*algebra with* $u_1 + \tilde{u}_1$ *as the identity, then the states on U are identified with those elements f in U satisfying*

$$\sqrt{\|f\|_2 + 2|\det f|} = f_1 + \tilde{f}_1 = 1$$

where f_1 *and* $\tilde{f}_1$ *are the components of* f *relative to* $u_1, \tilde{u}_1$.

Finally, from our construction follows

PROPOSITION 3.9. *Any Cartan factor either has a basis consisting of an ortho-colinear family of tripotents or can be embedded as an invariant subspace of a symmetry acting on a space spanned by an ortho-colinear family.*

This follows immediately from the fact that a trangle can be embedded into a quadrangle in a natural way.

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