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SYMMETRIC MANIFOLDS OF COMPACT TYPE ASSOCIATED TO THE JB*-TRIPLES $C_0(X, Z)$

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Abstract.

We examine the compact type symmetric Banach manifolds of infinite dimension which are associated to the JB*-triples of the form $C_0(X,Z)$, for X a locally compact Hausdorff space and Z a JB*-triple. The case of the commutative C^*-algebras receives special attention.

0. Introduction.

Symmetric manifolds of compact type are the infinite dimensional analogues of the compact Hermitian symmetric spaces. They arise as the simply connected symmetric manifolds associated via [5] to the dual triples of JB*-triples, (the dual triple of a given triple is obtained by multiplying the triple product by -1). Although "dual" in this sense to the bounded symmetric domains, the compact type symmetric manifolds are not as well understood and many of the natural questions suggested by the finite dimensional case remain to be answered.

In [7] all compact type symmetric manifolds are shown to have constant positive holomorphic curvature. The compact type manifolds associated to the JB*-triples $\mathscr{L}(H,K)$, for complex Hilbert spaces H and K, are known to be certain Grassmann manifolds (cf. [4]).

The purpose of this paper is to examine the compact type symmetric manifolds associated to JB*-triples of the form $C_0(X,Z)$, where Z is a JB*-triple and X is a locally compact Hausdorff space.

It is shown that if Z has compact type manifold M then the function space $C_0(X,M)$, consisting of all based maps from the one point compactification of X to M (sending ∞ to the base point of M), has a symmetric manifold structure on the component of the base point.

Taking universal covers then gives the compact type symmetric manifold associated to $C_0(X,Z)$.

Results of this type for compact X are given in [6].

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Section two examines the Lie algebra of derivations on $C_0(X, Z)$, for Z finite dimensional, and shows it to be exactly the Lie algebra of bounded elements in $C(X, \mathrm{aut}(Z))$. This result was already known for $Z = \mathbf{C}$.

In the final section, more precise information about the compact type symmetric manifolds of the commutative C^*-algebras, $C_0(X)$, is given. For example, it is shown that, as in the finite dimensional case, the complex structure on the symmetric manifold is induced by the action of a complex Lie group thus realising the manifold as the quotient of a complex Lie group by a complex Lie subgroup.

For background material on JB*-triples see [2] or [8].

Notation. For complex Banach spaces E and F, let $\mathscr{L}(E, F)$ denote the Banach space of all continuous linear maps: $E \to F$ and let $\mathscr{L}(E) = \mathscr{L}(E, E)$. We denote by $\mathscr{L}^k(E)$ the space of all continuous homogeneous polynomials: $E \to E$ of degree k. We say $T \in \mathscr{L}(E)$ is hermitian if $e^{i\lambda T}$ is an isometry for all $\lambda \in \mathbf{R}$. Let Z be a complex Banach space with continuous conjugate-linear mapping $*$: $Z \to \mathscr{L}^2(Z)$ and write a^* for $*(a)$. For all a, b, z in Z define

$$\{a, b, z\} := \tfrac{1}{2}(b^*(a + z) - b^*(a) - b^*(z))$$

and define $a \,\Box\, b(z) := \{a, b, z\}$, for all z in Z.
Then $(Z, *)$ is called a J*-triple if

(i) $\{\alpha, \beta, \{x, y, z\}\} = \{\{\alpha, \beta, x\}, y, z\} - \{x, \{\beta, \alpha, y\}, z\} + \{x, y, \{\alpha, \beta, z\}\}$ for all α, β, x, y and z in Z
and
(ii) $\alpha \,\Box\, \alpha \in \mathscr{L}(Z)$ is hermitian for all α in Z.

If, in addition, $\alpha \,\Box\, \alpha \geqq 0$ and $\|\alpha \,\Box\, \alpha\| = \|\alpha\|^2$ for all α in Z we call $(Z, *)$ a JB*-triple.

A manifold M modelled locally on open subsets of complex Banach spaces with biholomorphic coordinate transformations is called a complex Banach manifold. Let TM denote the tangent bundle of M.

A mapping α: $TM \to \mathbf{R}$ is called a norm on TM if the restriction of α to every tangent space T_x, $x \in M$, is a norm on T_x with the following property: there is a neighbourhood U of x in M which can be realised as a domain in a complex Banach space E such that

$$c\,\|a\| \leqq \alpha(u, a) \leqq C\,\|a\|$$

for all $(u, a) \in TU \cong U \times E$ and suitable constants $0 < c \leqq C$. We then refer to (M, α), or simply M, as a normed manifold. If $(\tilde{M}, \tilde{\alpha})$ is another complex normed manifold, we say that a holomorphic mapping ϕ: $M \to \tilde{M}$ is an isometry if for all

$$(z,v)\in TM, \quad \tilde{\alpha}(\phi(z), \phi'(z)v) = \alpha(z,v).$$

Let $\mathrm{Aut}(M)$ denote the group of all biholomorphic isometries of M.

A connected complex normed manifold M is called symmetric if for every $a \in M$ there exists an involution $s_a \in \mathrm{Aut}(M)$ having a as an isolated fixed point.

A morphism of the symmetric manifolds M and $\tilde{M}$ is a holomorphic mapping $h\colon M \to \tilde{M}$ such that $h \circ s_x = s_{h(x)} \circ h$, for all x in M. The symmetric manifolds are characterised by the following deep result in [5]: *The category of simply connected symmetric complex Banach manifolds with base point is equivalent to the category of J*-triples.*

If $(Z, *)$ is any J*-triple, the J*-triple $(Z, -*)$ is referred to as the dual triple of $(Z, *)$. In particular, if $(Z, *)$ is a JB*-triple its dual triple is no longer JB* and the simply connected symmetric manifold associated to $(Z, -*)$ is called the compact type symmetric manifold associated to $(Z, *)$. In finite dimensions these are exactly the Hermitian symmetric spaces of compact type.

The set of all continuous mappings from a topological space Y to a topological space W is denoted $C(Y, W)$. Unless otherwise stated, $C(Y, W)$ will have the compact open topology and all subspaces will have the induced topology.

Given $g \in C(Y, W)$ we define, for any topological space X, $g_*\colon C(X, Y) \to C(X, W)$ by $g_*(f) = g \circ f$.

We use the terminology of pairs of topological spaces; namely a pair (X, A) (also denoted X_A) where X is a topological space and $A \subseteq X$ is a subspace. When $A = \{x\}$ we write (X, A) simply as X_x. A mapping between pairs of topological spaces (X, A) and (Y, B) is a mapping $f\colon X \to Y$ such that $f(A) \subseteq B$. We denote the set of all such continuous mappings as $C(X_A, Y_B)$.

We may consider homotopies of such mappings relative to the pairs i.e. continuous mappings $\lambda\colon [0,1] \times X_A \to Y_B$.

As above, if $g \in C(Y_B, W_C)$, for pairs (Y, B), (W, C) then for any pair (X, A), we define $g_*\colon C(X_A, Y_B) \to C(X_A, W_C)$ by $g_*(f) = g \circ f$.

1. Symmetric manifolds of compact type for $C_0(X, Z)$.

Let Z be a J*-triple and let M be the unique simply connected symmetric manifold associated to Z with base point m_0. Let $L = \mathrm{Aut}(M)$ be the real Lie group of biholomorphic isometries of M and denote by $\ell = \mathrm{aut}(M)$ the Lie algebra of all infinitesimal isometries of M. There is a decomposition

$$\ell = k \oplus \rho$$

where k is the Lie subalgebra of all triple derivations of Z, also denoted $\mathrm{aut}(Z)$ and ρ consists of all vector fields of the form $X_\alpha = (\alpha - \{z, \alpha, z\})\dfrac{\partial}{\partial z}$, for all α in Z.

(Recall that a linear mapping $T: Z \to Z$ is a triple derivation if

$$T(\{x,y,z\}) = \{T(x),y,z\} + \{x,T(y),z\} + \{x,y,T(z)\}$$

for all x, y, $z \in Z$.)

Throughout, X will denote a locally compact Hausdorff space and $\tilde{X} = X \cup \infty$ its one point compactification. Clearly, the Banach space $C_0(X,Z)$ with the pointwise defined triple product is again a J*-triple.

The aim of this paper is to determine the form of the symmetric manifold associated to $C_0(X,Z)$.

The following lemma is immediate.

LEMMA 1.1. *The space*

$$\tilde{\ell} := C(\tilde{X}_\infty, \ell_k) = \{f \in C(\tilde{X},\ell): f(\infty) \in k\},$$

with Lie product defined pointwise on $C(\tilde{X},\ell)$ and with the supremum norm, is a real Lie algebra. Moreover $\tilde{\ell} = C(\tilde{X},k) \oplus C_0(X,\rho)$.

We look for a corresponding Lie group.

LEMMA 1.2. *The space*

$$Q = C(\tilde{X}_\infty, L_K) = \{f \in C(\tilde{X},L): f(\infty) \in K\}$$

has the structure of a real Lie group with Lie algebra $\tilde{\ell}$.

PROOF. It is well known that $C(\tilde{X},L)$ is a Lie group with Lie algebra $C(\tilde{X},\ell)$ (cf. [6]). Moreover, if $\exp: \ell \to L$ is the exponential mapping then the exponential mapping on $C(\tilde{X},\ell)$ is exactly $\exp_*: C(\tilde{X},\ell) \to C(\tilde{X},L)$.

The group Q is a closed subgroup of $C(\tilde{X},L)$ for which the Lie algebra

$$\{f \in C(\tilde{X},\ell): \exp_*(tf) \in Q \quad \text{for all} \quad t \in \mathsf{R}\}$$

is a split subspace of $C(\tilde{X},\ell)$. In fact,

$$\tilde{\ell} = \{f \in C(\tilde{X},\ell): \exp_*(tf) \in Q \quad \text{for all} \quad t \in \mathsf{R}\}$$

and satisfies $C(\tilde{X},\ell) = \tilde{\ell} \oplus \rho$, where elements of ρ are identified with constant mappings in $C(\tilde{X},\ell)$. As a consequence (cf. [8] Prop. 8.13) Q is a Lie subgroup of $C(\tilde{X},L)$ with Lie subalgebra $\tilde{\ell}$.

In fact, the charts on Q are exactly those induced by the mapping $\exp_*$. To be precise, let U and V be neighbourhoods of $0 \in \ell$ and the identity $e \in L$ respectively, such that $\exp: U \to V$ is bianalytic. Then

$$\exp_*: \underbrace{C(\tilde{X},U) \cap \tilde{\ell}}_{:=U_*} \to \underbrace{C(\tilde{X},V) \cap Q}_{:=V_*}$$

is a homeomorphism, so that $(V_*, \exp_*^{-1})$ is a chart of Q about e. Using the multiplicative structure of Q, we get a chart about every point $f \in Q$ as follows. Since Q is a topological group, the mapping: $Q \to Q$ given by: $g \mapsto fg$ is a homeomorphism and hence fV_* is an open neighbourhood of f in Q which is homeomorphic to U_* in $\tilde{\ell}$.

We define a pointwise action of $C(\tilde{X}, L)$ on $C(\tilde{X}, M)$ as follows. Let $r\colon L \times M \to M$ denote the action of L on M. Identifying $C(\tilde{X}, L \times M)$ with $C(\tilde{X}, L) \times C(\tilde{X}, M)$ in the natural way, then $r_*\colon C(\tilde{X}, L) \times C(\tilde{X}, M) \to C(\tilde{X}, M)$ where, as above,

$$r_*(g, m)(x) = r(g(x), m(x))$$

for all $g \in C(\tilde{X}, L)$ and $m \in C(\tilde{X}, M)$. It is not difficult to see that r_* is continuous.

Let $\rho\colon L \to M$ be defined by $\rho(f) = r(f, m_0)$, for all $f \in L$. Let m_0 denote the constant mapping in $C(\tilde{X}, M)$ sending $\tilde{X}$ to m_0.

Then, $\rho_*\colon C(\tilde{X}, L) \to C(\tilde{X}, M)$ and ρ_* is continuous.

The restriction of r_* to $Q \times C(\tilde{X}, M)$ and the restriction of ρ_* to Q are also continuous and are again denoted by r_* and ρ_*.

Throughout, we let $C_0(X, M) := C(\tilde{X}_\infty, M_{m_0})$, with base point the constant mapping m_0.

It is clear that $\rho_*(g) \in C_0(X, M)$ for all $g \in Q$. We examine the range of the mapping $\rho_*\colon Q \to C_0(X, M)$. The following result is Lemma 1.2 of [6].

LEMMA 1.3. *The mapping $\rho\colon L \to M$ is a locally trivial fibre bundle with fibre K.*

The path-connected component of a base point in a topological space Y is denoted by Y^0. The base point of a Lie group is its identity element.

For convenience, $\tilde{L} := C(\tilde{X}_\infty, L_K)^0$ and $\tilde{M} := C_0(X, M)^0$.

Note that, since M is simply connected, K is connected and hence path connected. Therefore, identifying elements of K with constant mappings in $C(\tilde{X}_\infty, L_K)$, we have $K \subseteq \tilde{L}$.

LEMMA 1.4. *The real Lie group $\tilde{L}$ acts transitively on $\tilde{M}$.*

PROOF. Fix $f \in \tilde{M}$ arbitrary. There exists $\lambda\colon I \times \tilde{X} \to M$ continuous with $\lambda(0, x) = m_0$ and $\lambda(1, x) = f(x)$ for all $x \in \tilde{X}$, and $\lambda(t, \infty) = m_0$ for all $t \in I = [0, 1]$. Since $\rho\colon L \to M$ is a locally trivial fibre bundle, we have a solution to the homotopy lifting problem posed by the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{e} & L \\ \downarrow{\scriptstyle i_0} & \nearrow & \downarrow{\scriptstyle \rho} \\ I \times \tilde{X} & \xrightarrow{\lambda} & M \end{array}$$

where $i_0(x) = (0, x)$ for $x \in \tilde{X}$.

Namely, there is a homotopy μ of e, $\mu: I \times \tilde{X} \to L$ such that $\rho \circ \mu = \lambda$. In particular, $\mu(t, \infty) \in \rho^{-1}(m_0) = K$, for all $t \in I$ and hence μ is a path in Q. Let $g \in C(\tilde{X}, L)$ be given by $g(x) = \mu(1, x)$ for all $x \in \tilde{X}$. Then $g \in Q^0$ and $\rho_*(g) = f$.

We denote the action of $\tilde{L}$ on $\tilde{M}$ again by r_*. For $g \in \tilde{L}$ and $m \in \tilde{M}$ we let $g(m) := r_*(g, m)$ and for $W \subseteq \tilde{M}$ we let $g(W) := \{g(w): w \in W\}$.

PROPOSITION 1.5. *$\tilde{M}$ has the structure of a symmetric complex manifold on which the real Lie group $\tilde{L}$ acts transitively as a group of biholomorphic isometries.*

PROOF. The isotropy subgroup, $\tilde{K}$, of $\tilde{L}$ at m_0 in $\tilde{M}$ is contained in $C(\tilde{X}, K)$. From Lemma 1.4 we may identify $\tilde{L}/\tilde{K}$ and $\tilde{M}$ under the mapping $g\tilde{K} \to g(m_0)$, for $g \in \tilde{L}$. We may therefore consider $\tilde{M}$ as a topological space for the quotient topology which it inherits under this identification.

The quotient topology thus described on $\tilde{M}$ is finer than the compact open topology on $\tilde{M}$ (that is, the topology induced on $\tilde{M}$ from the compact open topology on $C(X, M)$). Although the compact open topology on $\tilde{M}$ is the most natural one in this instance, we first show that $\tilde{M}$ with the quotient topology has the desired structure. It will later be an easy matter to show that this structure is also compatible with the compact open topology on $\tilde{M}$.

The following result is well known: for any Lie subgroup P of a Lie group Q the space $N := Q/P$ has a manifold structure such that the canonical projection: $Q \to N$ is an analytic submersion. Moreover, for any Banach space splitting of the Lie algebra q of Q as $q = p \oplus m$, where p is the Lie algebra of P, there is a chart (χ, U, m) of N about $n_0 := P$ such that $\chi(\exp X(n_0)) = X$, for all X in some neighbourhood of $0 \in m$.

The underlying real analytic structure of the complex manifold M is isomorphic to that of L/K and we have $\ell = k \oplus \rho$, is homeomorphic to Z via a real-linear mapping ξ.

From the above therefore, there exists a chart (ϕ, U, Z) of M about m_0 such that $\phi(\exp X(m_0)) = \xi(X)$, for all X in some neighbourhood of $0 \in \rho$.

Now, $\tilde{M}$ with the quotient topology is homeomorphic to $\tilde{L}/\tilde{K}$, so $\tilde{M}$ has the structure of a real analytic manifold such that the canonical projection: $\tilde{L} \to \tilde{M}$ is a real analytic submersion. The splitting $\tilde{\ell} = C(\tilde{X}, k) \oplus C_0(X, \rho)$ then gives rise to a chart ψ of $\tilde{M}$ about m_0 such that

$$\psi(\exp \Lambda(m_0)) = \xi_*(\Lambda)$$

for all Λ in some neighbourhood of $0 \in C_0(X, \rho)$.

For x in X, we have

$$\psi(\exp \Lambda(m_0))(x) = \xi_*(\Lambda)(x) = \xi(\Lambda(x)) = \phi(\exp \Lambda(x)(m_0)) = \phi((\exp \Lambda(m_0))x).$$

In other words, there is a neighbourhood U_* of m_0 in $\tilde{M}$ and a chart

$\psi\colon U_* \to C_0(X,Z)$ such that $\psi(f)(x) = \phi(f(x))$, for all $f \in U_*$ and all $x \in X$. In our notation $\psi = \phi_*$ and we may assume that $U_* = C(X,U) \cap \tilde{M}$, so that $(\phi_*, U_*, C_0(X,Z))$ is the real analytic chart of $\tilde{M}$ about m_0.

Charts given elsewhere are translates of this. Since $\tilde{L}$ acts real analytically on $\tilde{M}$, the mapping $\psi_g = \phi_* \circ g^{-1}$ on $g(U_*)$ is a chart of $\tilde{M}$ about $m = g(m_0)$, for $g \in \tilde{L}$.

On the other hand, it is easy to see that if $\tilde{M}$ is given the compact open topology, the mapping $\phi_*\colon U_* \to C_0(X,Z)$ described above is still a homeomorphism onto its image. Moreover, for $g \in \tilde{L}$, the mapping: $\tilde{M} \to \tilde{M}$ sending $m \to g(m)$ is a homeomorphism for the compact open topology so that $\tilde{M}$ with the compact open topology has the real analytic structure described above.

In fact, the charts just described give $\tilde{M}$ a complex manifold structure on which the elements of $\tilde{L}$ act as biholomorphic mappings.

To see this, we must show that for all g, h in $\tilde{L}$ such that $W = g(U_*) \cap h(U_*) \neq \emptyset$ then $\psi_h \circ \psi_g^{-1}\colon \psi_g(W) \to \psi_h(W)$ is a biholomorphic mapping.

It suffices to consider the case $g(U_*) \cap U_* \neq \emptyset$, for $g \in \tilde{L}$.

Fix g arbitrary in $\tilde{L}$ such that $W = U_* \cap g(U_*) \neq \emptyset$. We have the following commutative diagram

$$\begin{array}{cccc} W & f & \to & g^{-1}f \in W \\ & \downarrow \phi_* & & \downarrow \phi_* \\ \phi_*(W) & \phi_*(f) & \to^{\lambda} & \phi_*(g^{-1}f) \in \psi_g(W) \end{array}$$

where $\lambda = \psi_g \circ \phi_*^{-1} = \phi_* \circ g^{-1} \circ \phi_*^{-1}$.

We know that λ is real analytic, indeed real bi-analytic. To see that λ is biholomorphic we must simply show that $\lambda'(a)$ is a complex linear mapping for all $a \in \phi_*(W)$.

Fix $a = \phi_*(f) \in \phi_*(W)$ arbitrary and fix x arbitrary in X.

Then $g(x)U \cap U \neq \emptyset$ in M and the mapping $\xi_x = \phi \circ g(x)^{-1} \circ \phi^{-1}$ defined by the following commutative diagram is holomorphic.

$$\begin{array}{cccc} U \cap g(x)U & m & \to^{r_{g(x)^{-1}}} & g(x)^{-1} m \in U \\ & \downarrow \phi & & \downarrow \phi \\ & \phi(m) & \to^{\xi_x} & \phi(g(x)^{-1}(m)) \end{array}$$

By definition $\lambda(a)(x) = \xi_x(a(x))$.

The mapping $\varepsilon_x\colon C_0(X,Z) \to Z$, given by $\varepsilon_x(z) = z(x)$, for all $z \in C_0(X,Z)$, is complex linear.

Define a mapping $\lambda_x\colon \phi_*(W) \to Z$ by $\lambda_x(w) = \lambda(w)(x) = \xi_x \circ \varepsilon_x(w)$, for all $w \in \phi_*(W)$.

The mapping λ_x is clearly holomorphic. Moreover, for $w \in \phi_*(W)$,

$$\lim_{w\to 0}\frac{\|\lambda_x(a+w)-\lambda_x(a)-(\lambda'(a)w)x\|}{\|w\|} = \lim_{w\to 0}\frac{\|\lambda(a+w)x-\lambda(a)x-(\lambda'(a)w)x\|}{\|w\|}$$
$$\leqq \lim_{w\to 0}\frac{\|\lambda(a+w)-\lambda(a)-(\lambda'(a)w)\|}{\|w\|}$$
$$= 0.$$

By definition then, $\lambda_x'(a)(w) = (\lambda'(a)w)x$.

Since λ_x is holomorphic, $\lambda_x'(a)$ is complex linear and as this holds for all $x \in X$, $\lambda'(a)$ is a complex linear mapping. Therefore λ is holomorphic and thus clearly biholomorphic. It follows that $\tilde{M}$ has a complex manifold structure on which the elements of $\tilde{L}$ act on $\tilde{M}$ as biholomorphic mappings.

Now fix h arbitrary in $C(\tilde{X}, K)$ and fix x arbitrary in X.

The identification $T_{m_0}\tilde{M} \cong C_0(X, Z)$ identifies $h'(m_0)$ and $\lambda'(0)$ where $\lambda = \phi_* \circ h \circ \phi_*^{-1}$.

In identifying $T_{m_0}M$ and Z we identify $h(x)'(m_0)$ and $\xi_x'(0)$ where $\xi_x = \phi \circ h(x) \circ \phi^{-1}$.

In particular, for $w \in C_0(X, Z)$,

$$(\dagger) \qquad \|\lambda'(0)w\| = \sup_{x\in X}\|\lambda'(0)(w)x\| = \sup_{x\in X}\|\lambda_x'(0)w\|$$
$$= \sup_{x\in X}\|\xi_x'(0)(w(x))\|$$

Since for all $x \in X$, $h(x)$ acts as an isometry on M, $\xi_x'(0)$ is an isometry for the norm on Z and hence $\|\lambda'(0)(w)\| = \|w\|$, for all $w \in C_0(X, Z)$.

In other words, the norm on $C_0(X, Z)$ is invariant under the action of $h'(m_0)$, for all h in the isotropy subgroup $\tilde{K} \subset C(\tilde{X}, K)$ and we may therefore define a tangent norm on $\tilde{M}$ as follows.

For $(m, v) \in T\tilde{M}$, chose $g \in \tilde{L}$ with $g(m) = m_0$ and define $\alpha(m, v) = \|g'(m)v\|$. Then $\alpha\colon T\tilde{M} \to \mathrm{R}^+$ defines a tangent norm on $\tilde{M}$ which is invariant under the action of $\tilde{L}$.

In particular, $\tilde{M}$ becomes a normed complex manifold for which the elements of $\tilde{L}$ act as biholomorphic isometries.

Let $s_{m_0} \in L$ be the symmetry of M at m_0. It follows easily from ($\dagger$) above that s_{m_0}, viewed as a constant mapping in $\tilde{L}$, is the symmetry of $\tilde{M}$ at m_0. Since $\tilde{L}$ acts transitively on $\tilde{M}$ as a group of biholomorphic isometries, it follows that $\tilde{M}$ is a symmetric manifold.

PROPOSITION 1.6. *If the J^*-triple associated to M is Z then the J^*-triple associated to the symmetric manifold $\tilde{M} = C_0(X, M)^0$ is $C_0(X, Z)$ with the usual pointwise defined triple product.*

PROOF. From Proposition 1.5, $\tilde{M}$ is a symmetric manifold with local coordinates in $C_0(X,Z)$ and $\tilde{\ell} = C(\tilde{X},k) \oplus C_0(X,\rho) \subseteq \mathrm{aut}(\tilde{M})$. It is known that $\mathrm{aut}(\tilde{M}) = \tilde{\tau} \oplus \tilde{p}$ where $\tilde{\tau} = \{\Lambda \in \mathrm{aut}(\tilde{M})\colon \mathrm{Ad}(s_{m_0})(\Lambda) = \Lambda\} = \{\Lambda \in \mathrm{aut}(\tilde{M})\colon \Lambda(m_0) = 0\}$ and $\tilde{p} = \{\Lambda \in \mathrm{aut}(\tilde{M})\colon \mathrm{Ad}(s_{m_0})\Lambda = -\Lambda\}$, where Ad denotes the natural adjoint action.

It follows that $C_0(X,\rho) \subseteq \tilde{p}$. As is well-known, there are real linear homeomorphisms $\xi\colon \rho \to Z$ and $\bar{\xi}\colon \tilde{p} \to C_0(X,Z)$.

Clearly, ξ_* maps $C_0(X,\rho)$ into $C_0(X,Z)$ and is a real linear isomorphism. Hence $\tilde{p} = C_0(X,\rho)$.

We recall the following (cf. [5]).

For all a in Z there exists a unique X^a in ρ such that $X^a(m_0) = a$. Define vector fields $Y^a = \frac{1}{2}(X^a - iX^{ia})$ and $Z^a = \frac{1}{2}(X^a + iX^{ia})$ in $\rho \oplus i\rho$. Then the triple product on Z arises as follows: for all a, b, c in Z, $\{a, b, c\}$ is that element of Z with

$$Y^{\{a,b,c\}} = \tfrac{1}{2}[[Y^a, Z^b], Y^c].$$

Let $\alpha \in C_0(X,Z)$. We use P^α, Q^α and R^α respectively for the vector fields on $\tilde{M}$ corresponding to X^a, Y^a and Z^a on M. By uniqueness, $P^\alpha(x) = X^{\alpha(x)}$ and hence $Q^\alpha(x) = Y^{\alpha(x)}$ and $R^\alpha(x) = Z^{\alpha(x)}$, for all $x \in X$. In particular, for α, β, $\gamma \in C_0(X,Z)$, and $x \in X$,

$$\begin{aligned} Y^{\{\alpha,\beta,\gamma\}(x)} &= Q^{\{\alpha,\beta,\gamma\}}(x) \\ &= \tfrac{1}{2}[[Q^\alpha(x), R^\beta(x)], Q^\gamma(x)] \\ &= \tfrac{1}{2}[[Y^{\alpha(x)}, Z^{\beta(x)}], Y^{\gamma(x)}] \\ &= Y^{\{\alpha(x), \beta(x), \gamma(x)\}}. \end{aligned}$$

In other words, the triple product defined by $\tilde{M}$ on $C_0(X,Z)$ is the usual pointwise defined triple product.

As the following example shows, the symmetric manifold $C_0(X,M)^0$ is not, in general, simply connected and it is thus necessary to pass to the universal covering manifold to get uniqueness.

EXAMPLE 1.7. Let $S^n := \{x \in \mathbb{R}^{n+1}\colon \|x\| = 1\}$. For topological spaces X and Y, $[X, Y]$ denotes the space of all homotopy classes of based mappings from X to Y and $\pi_n(X) = [S^n, X]$ the nth homotopy group of X, $n \geqq 0$. Let $X = S^1$ and $Z = \mathbb{C}$. Since $M = \bar{\mathbb{C}}$ is homeomorphic to S^2 we have that $C_0(X,M)$ is homeomorphic to $C(S^1,S^2)$. The space $C(S^1,S^2)$ is path connected since its space of path components is

$$\pi_0(C(S^1,S^2)) = [S^1,S^2] = \pi_1(S^2) = 0.$$

It is not simply connected, however, as

$$\pi_1(C(S^1,S^2)) = [S^1, C(S^1,S^2)] = [S^1 \times S^1, S^2] = \pi_2(S^2) \neq 0.$$

COROLLARY 1.8. *Let M be the unique simply connected symmetric manifold associated to the J^*-triple Z.*

The unique simply connected symmetric manifold associated to $C_0(X,Z)$ is the universal covering manifold of $C_0(X,M)^0$.

COROLLARY 1.9. *If M is the compact type symmetric manifold associated to the* JB*-*triple Z then the compact type symmetric manifold associated to $C_0(X,Z)$ is the universal covering manifold of $C_0(X,M)^0$.*

COROLLARY 1.10. *If h: $M_1 \to M_2$ is a morphism of the symmetric manifolds M_1 and M_2 then h_* is a morphism of the symmetric manifolds $\tilde{M}_1 = C_0(X,M_1)^0$ and $\tilde{M}_2 = C_0(X,M_2)^0$.*

PROOF. If m_1 is the base point of M_1, we assume without loss of generality that $h(m_1)$ is the base point of M_2. Then, using the same techniques as in Proposition 1.5, we see that h_*: $\tilde{M}_1 \to \tilde{M}_2$ is holomorphic.

Fix m arbitrary in $\tilde{M}_1$ and consider the symmetry of $\tilde{M}_1$ at m, $s_m \in \tilde{L}_1$ (where $\tilde{L}_1$ is to $\tilde{M}_1$ as $\tilde{L}$ is to $\tilde{M}$ in Proposition 1.5).

Since $s_m(x) = s_{m(x)}$, for all $x \in X$ and since h: $M_1 \to M_2$ is a morphism of M_1 and M_2,

$$h \circ s_{m(x)} = s_{h(m(x))} \circ h \quad \text{for all} \quad x \in X$$

implies that $h_* \circ s_m = s_{h_*(m)} \circ h_*$ and hence h_*: $\tilde{M}_1 \to \tilde{M}_2$ is a morphism of the symmetric manifolds $\tilde{M}_1$ and $\tilde{M}_2$.

REMARKS. When X is a compact Hausdorff space, ∞ is an isolated point of X and it is not difficult to see that the results of this section then imply the results presented in section one of [6].

2. Lie algebras of vector fields on $C_0(X, Z)$, for Z finite dimensional.

We first examine the Lie algebra of all JB*-triple derivations on $C_0(X,Z)$, for Z finite dimensional. For $Z = \mathbf{C}$, the following result is already part of the folklore.

PROPOSITION 2.1. *Let Z be a finite dimensional* JB*-*triple. Then*

$$\operatorname{aut}(C_0(X,Z)) \cong C_b(X,\operatorname{aut}(Z)) \cong C(\beta X,\operatorname{aut}(Z)),$$

where $C_b(X,\operatorname{aut}(Z))$ denotes all elements of $C(X,\operatorname{aut}(Z))$ which are bounded in the supremum norm, βX denotes the Stone-Čech compactification of X, and $\cong$ denotes isometric isomorphism of Lie algebras (where $C_b(X,\operatorname{aut}(Z))$ and $C(\beta X,\operatorname{aut}(Z))$ have the pointwise defined Lie algebra product).

PROOF. Take $\lambda \in \operatorname{aut}(C_0(X,Z))$ and fix x arbitrary in X.

As before let $\varepsilon_x: C_0(X,Z) \to Z$ by $\varepsilon_x(f) = f(x)$, for all $f \in C_0(X,Z)$. Let $\lambda^x: Z \to Z$ be the mapping defined by the following commutative diagram.

$$\begin{array}{ccc} C_0(X,Z) & \xrightarrow{\varepsilon_x} & Z \\ \downarrow \lambda & & \downarrow \lambda^x \\ C_0(X,Z) & \xrightarrow{\varepsilon_x} & Z \end{array}$$

To show that λ^x is well defined, it suffices to prove that $\lambda(a)(x) = 0$ whenever $a(x) = 0$, for $a \in C_0(X,Z)$.

Fix $a \in C_0(X,Z)$ with $a(x) = 0$. The element $a^{1/3} \in C_0(X,Z)$ is well defined via the functional calculus and $a = \{a^{1/3}, a^{1/3}, a^{1/3}\}$.

Therefore

$$\lambda(a) = 2\{\lambda(a^{1/3}), a^{1/3}, a^{1/3}\} + \{a^{1/3}, \lambda(a^{1/3}), a^{1/3}\}.$$

If $a(x) = 0$ then $a^{1/3}(x) = 0$ and hence $\lambda(a)(x) = 0$ and λ^x is well defined.

It is easy to see that $\lambda^x \in \operatorname{aut}(Z)$.

Define $\hat{\lambda}: X \to \operatorname{aut}(Z)$ by $\hat{\lambda}(x) = \lambda^x$, for all $x \in X$.

To prove that $\hat{\lambda}$ is continuous we must show that $\hat{\lambda}^{-1}(U)$ is open in X for $U \subseteq \operatorname{aut}(Z)$ open.

Fix $x_0 \in \hat{\lambda}^{-1}(U)$ arbitrary.

Since U is open in $\operatorname{aut}(Z)$, there exists $\varepsilon > 0$ such that if $\mu \in \operatorname{aut}(Z)$ and $\|\mu - \lambda^{x_0}\| < \varepsilon$ then $\mu \in U$.

In particular, if $\|\lambda^y - \lambda^{x_0}\| < \varepsilon$ then $y \in \hat{\lambda}^{-1}(U)$.

Let Q be a compact neighbourhood of x_0 in X and let W be any open set in X containing Q. Since X is locally compact there is an open set V in X such that $\bar{V}$ is compact with $Q \subset V \subset \bar{V} \subset W$. By Urysohn's Lemma there exists $\alpha_Q \in C(X,[0,1])$ such that $\alpha_Q|_Q \equiv 1$ and $\alpha_Q|_{X \setminus V} \equiv 0$. Since α_Q vanishes off the compact set $\bar{V}$ then $\alpha_Q \in C_0(X,[0,1])$.

For any $z \in Z$ we define $\tilde{z} \in C_0(X,Z)$ by $\tilde{z}(x) = z\alpha_Q(x)$, for all $x \in X$. In particular, $\tilde{z}(x) = z$, for all $x \in Q$ and $\|\tilde{z}\| = \|z\|$.

Let $B := \{z \in Z: \|z\| < 1\}$ and write $\bar{B}$ for its closure. For $y \in Q$,

$$\sup_{z \in \bar{B}} \|\lambda^y(z) - \lambda^{x_0}(z)\| = \sup_{z \in \bar{B}} \|\lambda(\tilde{z})y - \lambda(\tilde{z})x_0\|.$$

Now

$$\begin{aligned} \|\lambda(\tilde{w})y - \lambda(\tilde{w})x_0\| &\leqq \|\lambda(\tilde{w})y - \lambda(\tilde{z})y\| + \|\lambda(\tilde{z})y - \lambda(\tilde{z})x_0\| + \|\lambda(\tilde{z})x_0 - \lambda(\tilde{w})x_0\| \\ &\leqq 2\,\|\lambda(\tilde{w}) - \lambda(\tilde{z})\| + \|\lambda(\tilde{z})y - \lambda(\tilde{z})x_0\|. \end{aligned}$$

Since derivations are automatically continuous (cf. [1]), the mapping: $z \to \lambda(\tilde{z})$ is continuous and there exists a neighbourhood A_z of $z \in Z$ such that for all $w \in A_z$, $\|\lambda(\tilde{w}) - \lambda(\tilde{z})\| < \varepsilon/4$. Moreover, for z fixed, $\lambda(\tilde{z})$ is continuous, and hence there exists a neighbourhood B_z of x_0 in X such that $\|\lambda(\tilde{z})y - \lambda(\tilde{z})x_0\| < \varepsilon/2$ for all $y \in B_z$. In other words, for $w \in A_z$ and $y \in B_z$ (we assume without loss of generality that $B_z \subset Q$) we have $\|\lambda(\tilde{w})y - \lambda(\tilde{w})x_0\| < \varepsilon$.

By compactness of $\bar{B}$, there exists $z_1, \ldots, z_n \in \bar{B}$ such that $\bar{B} \subset \cup_{i=1}^{n} A_{z_i}$. The set $W = B_{z_1} \cap \ldots \cap B_{z_n}$ is an open neighbourhood of x_0 in X. Then, for all $z \in \bar{B}$, $\|\lambda(\tilde{z})y - \lambda(\tilde{z})x_0\| < \varepsilon$ whenever $y \in W$.

Therefore, for all $y \in W$, $\|\lambda^y - \lambda^{x_0}\| < \varepsilon$ and hence $W \subset \hat{\lambda}^{-1}(U)$. In other words, $\hat{\lambda}$ is continuous.

As it is clear that $\|\hat{\lambda}\| \leqq \|\lambda\|$ we have that $\hat{\lambda} \in C_b(X, k)$ where $k = \mathrm{aut}(Z)$. The mapping $\lambda \to \hat{\lambda}$ therefore maps $\mathrm{aut}(C_0(X, Z))$ into $C_b(X, k)$.

In the opposite direction, we define for each $\mu \in C_b(X, k)$, $\bar{\mu}: C_0(X, Z) \to C_0(X, Z)$ by $\bar{\mu}(f)(x) = \mu(x)(f(x))$, for all $x \in X$ and $f \in C_0(X, Z)$.

It is easy to see that $\bar{\mu}$ is a continuous linear mapping on $C_0(X, Z)$ which acts as a derivation for the triple product structure and $\|\bar{\mu}\| \leqq \|\mu\|$.
The mapping: $\mu \to \bar{\mu}$ therefore sends $C_b(X, k)$ into $\mathrm{aut}(C_0(X, Z))$ and satisfies $\bar{\hat{\lambda}} = \lambda$ for all $\lambda \in \mathrm{aut}(C_0(X, Z))$. It is then easy to see that this mapping gives an isometric Lie algebra isomorphism between $\mathrm{aut}(C_0(X, Z))$ and $C_b(X, k)$.

Finally, since $\mathrm{aut}(Z)$ is finite dimensional, we may identify $C_b(X, \mathrm{aut}(Z))$ and $C(\beta X, \mathrm{aut}(Z))$.

An important complex Lie algebra, g^E, is associated to a JB*-triple E as follows (cf. [5]):

$$g^E = g^E_{-1} \oplus g^E_0 \oplus g^E_1$$

where $g^E_{-1} = \left\{\alpha \frac{\partial}{\partial z} : \alpha \in E\right\}$, $g^E_1 = \left\{q(z)\frac{\partial}{\partial z} : q \in W\right\}$, where W is the closure of E^* in $\mathscr{L}^2(E)$, and finally,

$$g^E_0 = \left\{\lambda(z)\frac{\partial}{\partial z} : \lambda \text{ is linear and } [\lambda, g^E_1] \subset g^E_1\right\}.$$

We realise the Lie algebra, $g^{C_0(X,Z)}$ as a Lie algebra of bounded continuous mappings from X into g^Z, using techniques similar to the above.

PROPOSITION 2.2.

$$g^{C_0(X,Z)} = C_0(X, g^Z_{-1}) \oplus C_b(X, g^Z_0) \oplus C_0(X, g^Z_1).$$

PROOF. Let $g := g^Z$ and let $\tilde{g} := g^{C_0(X,Z)}$. Let $g_i := g^Z_i$ and $\tilde{g}_i := g_i^{C_0(X,Z)}$ for $i \in \{-1, 0, 1\}$.

We make the following natural identifications $\tilde{g}_{-1} = C_0(X, Z) = C_0(X, g_{-1})$ and $\tilde{g}_1 = C_0(X, g_1)$.

To examine the subspace $\tilde{g}_0$ we apply the techniques of Proposition 2.1.

For $\lambda \in \tilde{g}_0$, $[\lambda, \tilde{g}_1] \subset \tilde{g}_1$ and therefore, for all $\alpha \in C_0(X, Z)$,

$$[\lambda(z), \{z, \alpha, z\}] \in C_0(X, g_1).$$

In other words,

$$\lambda(\{z, \alpha, z\}) - 2\{\lambda(z), \alpha, z\} \in C_0(X, g_1) \tag{2.3}$$

for all $\alpha \in C_0(X, Z)$.

Fix $\lambda \in \tilde{g}_0$ and x in X, arbitrary.

As before, $\lambda^x: Z \to Z$ is defined by the following commutative diagram.

$$\begin{array}{ccc} C_0(X,Z) & \xrightarrow{\varepsilon_x} & Z \\ \downarrow \lambda & & \downarrow \lambda^x \\ C_0(X,Z) & \xrightarrow{\varepsilon_x} & Z \end{array}$$

Fix a arbitrary in $C_0(X, Z)$ with $a(x) = 0$.

Letting $\alpha = a^{1/3}$ in (2.3), there exists $q \in C_0(X, g_1)$ such that

$$\lambda(\{z, a^{1/3}, z\}) = 2\{\lambda(z), a^{1/3}, z\} + q(z),$$

for all $z \in C_0(X, Z)$. In particular, for $z = a^{1/3}$, we have

$$\lambda(a) = 2\{\lambda(a^{1/3}), a^{1/3}, a^{1/3}\} + q(a^{1/3}).$$

As $a(x) = 0$, it follows that $a^{1/3}(x) = 0$ and hence $q(a^{1/3})(x) = 0$.

Therefore $\lambda(a)(x) = 0$. It follows that λ^x is well defined.

Clearly $\lambda^x \in g_0$ for each $x \in X$.

Define $\hat{\lambda}: X \to g_0$ by $\hat{\lambda}(x) = \lambda^x$, for all $x \in X$. As in Proposition 2.1, we may prove that $\hat{\lambda}$ is continuous and bounded (in fact, $\|\hat{\lambda}\| \leqq \|\lambda\|$).

The linear mapping: $\lambda \to \hat{\lambda}$ maps $\tilde{g}_0$ injectively into $C_b(X, g_0)$. Again, exactly as in Proposition 2.1, we can show that this mapping is an isometric Lie algebra isomorphism.

REMARKS. Proving Proposition 2.2 first would give Proposition 2.1 as an immediate corollary. As $k = \operatorname{aut}(Z) = \{\lambda \in g_0 : [\lambda, \rho] \subset \rho\}$ then $\operatorname{aut}(C_0(X, Z)) = \{\lambda \in \tilde{g}_0 : [\lambda, \tilde{\rho}] \subset \tilde{\rho}\}$. Since $\tilde{\rho} = C_0(X, \rho)$, it then follows by proposition 2.2 that

$$\operatorname{aut}(C_0(X,Z)) \cong \{\hat{\lambda} \in C_b(X, g_0): [\lambda^x, \rho] \subset \rho, \quad \text{for all} \quad x \in X\} = C_b(X, k).$$

3. The special case, $C_0(X)$.

We show that the structure on the compact type manifolds of the commutative C^*-algebras, $C_0(X)$, may be described by the action of a complex Lie group, as in the finite dimensional case.

Let M_2 denote the group of all complex 2×2 matrices and let I denote the

2×2 identity matrix. For any subgroup G of M_2 we use PG to denote the quotient $G/\{\pm I\}$.

The compact type symmetric manifold associated to $\mathbf{C}$ is $\bar{\mathbf{C}}$. The real Lie group $\mathrm{Aut}(\bar{\mathbf{C}})$ is identified with $\mathrm{PSU}_2(\mathbf{C})$ where $\mathrm{SU}_2(\mathbf{C}) := \{A \in M_2 : \det(A) = 1, A^* = A^{-1}\}$ and its Lie algebra $\mathrm{aut}(\bar{\mathbf{C}})$ is identified with $\mathrm{su}_2(\mathbf{C}) := \{A \in M_2 : A^* = -A, \mathrm{trace}(A) = 0\}$.

The complex Lie algebra, $g^{\mathbf{C}}$, associated to $\mathbf{C}$, is isomorphic as a Lie algebra to $\mathrm{sl}_2(\mathbf{C}) := \{A \in M_2 : \mathrm{trace}(A) = 0\}$ via the mapping

$$\phi\left((a + 2bz + cz^2)\frac{\partial}{\partial z}\right) = \begin{pmatrix} b & a \\ -c & -b \end{pmatrix}.$$

Also, $\mathrm{SL}_2(\mathbf{C}) := \{A \in M_2 : \det(A) = 1\}$ is a simply connected complex Lie group with Lie algebra $\mathrm{sl}_2(\mathbf{C})$, which acts transitively on $\bar{\mathbf{C}}$ as a group of biholomorphic mappings.

In this way, $\bar{\mathbf{C}}$ may also be realised as the quotient manifold of the complex Lie group $\mathrm{SL}_2(\mathbf{C})$ by its isotropy subgroup

$$I_{\mathrm{SL}_2} := \left\{\begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} : c \in \mathbf{C}, a \in \mathbf{C}\backslash\{0\}\right\}.$$

Moreover, the isotropy subgroup I_{SL_2} has Lie algebra

$$i_{\mathrm{sl}_2} = \left\{\begin{pmatrix} b & 0 \\ c & -b \end{pmatrix} : c, b \in \mathbf{C}\right\} = g_0^{\mathbf{C}} \oplus g_1^{\mathbf{C}}.$$

It follows from Proposition 1.5 that $\tilde{M} = C_0(X, \bar{\mathbf{C}})^0$ is a symmetric complex manifold on which the real Lie group $\tilde{L} = C(\tilde{X}_\infty, \mathrm{SU}_2(\mathbf{C})_K)^0$ acts transitively as a group of biholomorphic isometries, where

$$K := \left\{\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : a \in \mathbf{C} \quad \text{and} \quad |a| = 1\right\}.$$

For comparison, see [3], where the automorphism group of the open unit ball of $C(X)$ is determined.

From Proposition 2.2

$$g^{C_0(X)} = \left\{(a + bz + cz^2)\frac{\partial}{\partial z} : a, c \in C_0(X), b \in C_b(X)\right\},$$

where $C_b(X)$ denotes all complex-valued functions on X which are bounded in the supremum norm. Let

$$\tilde{g} = \left\{f \in C(X, \mathrm{sl}_2(\mathbf{C})) : f = \begin{pmatrix} b & a \\ c & -b \end{pmatrix} \quad \text{with} \quad b \in C_b(X), a, c \in C_0(X)\right\}.$$

Then $\tilde{g}$, with the pointwise defined Lie algebra product, is a complex Lie algebra and the mapping ψ: $g^{C_0(X)} \to \tilde{g}$ defined by

$$\psi\left((a + 2bz + cz^2)\frac{\partial}{\partial z}\right) = \begin{pmatrix} b & a \\ -c & -b \end{pmatrix}$$

is a Lie algebra isomorphism.

Using the techniques of Lemma 1.2, the group

$$\tilde{G} = \left\{ f \in C(X, \mathrm{SL}_2(\mathbf{C})) : f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad b, c \in C_0(X), a, d \in C_b(X) \right\}^0$$

may be given the structure of a complex Lie group having Lie algebra $\tilde{g}$.

Let q: $\mathrm{SL}_2(\mathbf{C}) \times \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ denote the natural action of $\mathrm{SL}_2(\mathbf{C})$ on $\bar{\mathbf{C}}$. Then q_*: $C(X, \mathrm{SL}_2(\mathbf{C})) \times C(X, \bar{\mathbf{C}}) \to C(X, \bar{\mathbf{C}})$.

Denote the restriction of q_* to $\tilde{G} \times \tilde{M}$ again by q_* and consider q_*: $\tilde{G} \times \tilde{M} \to C(X, \bar{\mathbf{C}})$.

To see that this will give an action of $\tilde{G}$ on $\tilde{M}$ we must first show that $g(m) := q_*(g, m) \in C_0(X, \bar{\mathbf{C}})$, for all $g \in \tilde{G}$ and $m \in \tilde{M}$. The transitivity of this action will then follow using the techniques of Lemma 1.4.

Proposition 3.1. *The complex Lie group $\tilde{G}$ acts transitively on the symmetric manifold $C_0(X, \bar{\mathbf{C}})^0$ as a group of biholomorphic mappings.*

Proof. Fix g arbitrary in $\tilde{G}$. Then $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in C_b(X)$ and $b, c \in C_0(X)$ and $ad - bc \equiv 1$ on X. We must show that the function $g(0) = b/d$ vanishes at infinity.

Suppose that b/d does not vanish at infinity. Then there exists $\varepsilon > 0$ such that for all $P \subset X$ compact, there exists $x_P \in X \backslash P$ with $\dfrac{|b(x_P)|}{|d(x_P)|} \geqq \varepsilon$. Since $b \in C_0(X)$ then for every $\delta > 0$, there exists $P_\delta \subseteq X$ compact, such that $|b(x)| < \dfrac{\delta\varepsilon}{2}$ for all $x \in X \backslash P_\delta$. We may assume, without loss of generality, that for all $x \in X \backslash P_\delta$, $|b(x)c(x)| < 1/2$. Moreover, for each $\delta > 0$, there exists $x_\delta \in X \backslash P_\delta$ such that $\dfrac{|b(x_\delta)|}{|d(x_\delta)|} \geqq \varepsilon$. Since $a(x_\delta)d(x_\delta) = 1 + b(x_\delta)c(x_\delta)$ then $|a(x_\delta)d(x_\delta)| > 1/2$.

In other words, for all $\delta > 0$ there exists $x_\delta \in X$ such that $|a(x_\delta)| > 1/\delta$. As a vanishes at infinity, this is impossible.

Therefore $g(0) \in C_0(X, \bar{\mathbf{C}})$, for all $g \in \tilde{G}$.

Now $\tilde{L} \subseteq \tilde{G}$ ($\tilde{L}$ as above) and it is not difficult to see that the restriction of q_* to $\tilde{L} \times \tilde{M}$ coincides with the action of $\tilde{L}$ on $\tilde{M}$ as described in section 1 (and denoted there by r_*). In particular, (Lemma 1.4), given any $m \in \tilde{M}$ there exists $f \in \tilde{L}$ such that $f(0) = m$.

Fix m arbitrary in $\tilde{M}$ and let $f \in \tilde{G}$ satisfy $f(0) = m$. Then, for $g \in \tilde{G}$ arbitrary, $g(m) = gf(0)$ and it follows from the above that $g(m) \in C_0(X, \bar{\mathbf{C}})$ and hence in $\tilde{M}$. In other words $q_*\colon \tilde{G} \times \tilde{M} \to \tilde{M}$ and the mapping is onto.

Since $\mathrm{SL}_2(\mathbf{C})$ acts on $\bar{\mathbf{C}}$ as a group of biholomorphic mappings and q_* is exactly this action applied pointwise on $C_0(X, \bar{\mathbf{C}})^0$, the techniques of Proposition 1.5 can again be used to prove that $\tilde{G}$ acts on $\tilde{M}$ as a group of biholomorphic mappings.

Remarks. We can use the action of the complex Lie group $\tilde{G}$ on $C_0(X, \bar{\mathbf{C}})^0$ to endow $C_0(X, \bar{\mathbf{C}})^0$ directly with the structure of a complex manifold and this complex manifold structure will then coincide with that induced by the action of the real Lie group $\tilde{L}$.

Denote by $\tilde{H}$ the isotropy subgroup of $\tilde{G}$ at $0 \in C_0(X, \bar{\mathbf{C}})^0$.

Then $\tilde{H}$ is a complex Lie subgroup of $\tilde{G}$ with Lie algebra

$$\tilde{h} = \left\{ f \in C(X, \mathrm{sl}_2(\mathbf{C}))\colon f = \begin{pmatrix} b & 0 \\ -c & -b \end{pmatrix} : b \in C_b(X), c \in C_0(X) \right\}.$$

In particular, identifying $g^{C_0(X)}$ and $\tilde{g}$ gives $\tilde{h} = \tilde{g}_0 \oplus \tilde{g}_1$.

Corollary 3.2. *The symmetric manifold $C_0(X, \bar{\mathbf{C}})^0$ may be realised as the quotient manifold of the complex Lie group $\tilde{G}$ by its complex Lie subgroup $\tilde{H}$.*

Corollary 3.3. *Let $\bar{M}$ be the unique simply connected compact type symmetric manifold associated to $C_0(X)$.*

Then $\bar{M}$ is the quotient manifold of a complex Lie group $\bar{G}$ and a complex Lie subgroup $\bar{H}$ such that $\bar{G}$ has $\tilde{g}$ as its Lie algebra and $\bar{H}$ has Lie algebra $\tilde{h} = \tilde{g}_0 \oplus \tilde{g}_1$, where $\tilde{g} = g^{C_0(X)}$.

Proof. By corollary 1.9 $\bar{M}$ is the universal covering manifold of $\tilde{M} = C_0(X, \bar{\mathbf{C}})^0$. Let $\pi_M\colon \bar{M} \to \tilde{M}$ be the covering projection. Let $\bar{G}$ be the universal covering manifold of $\tilde{G}$, with covering projection $\pi_G\colon \bar{G} \to \tilde{G}$. Then $\bar{G}$ is a Lie group with Lie algebra $\tilde{g}$ and the action of $\tilde{G}$ on $\tilde{M}$ (denoted by q_*) may be lifted to give an action of $\bar{G}$ on $\bar{M}$ (denoted by $\bar{q}$) which satisfies the following commutative diagram.

$$\begin{array}{ccc} \bar{G} \times \bar{M} & \xrightarrow{\bar{q}} & \bar{M} \\ {\scriptstyle \pi_G \times \pi_M} \downarrow & & \downarrow {\scriptstyle \pi_M} \\ \tilde{G} \times \tilde{M} & \xrightarrow{q_*} & \tilde{M} \end{array}$$

Since $\tilde{G}$ acts transitively on $\tilde{M}$, $\bar{G}$ acts transitively on $\bar{M}$.

Choose as base point of $\bar{M}$ a $\bar{o} \in \pi_M^{-1}(0)$. Let $\bar{H}$ be the isotropy subgroup of $\bar{G}$ at $\bar{o}$. Then $\bar{H}$ is a complex Lie subgroup of $\bar{G}$ and

$$\bar{M} = \bar{G}/\bar{H}.$$

Since $\bar{M}$ is simply connected, it follows that $\bar{H}$ is connected. Then, from the commutative diagram above, it is clear that $\bar{H} = \pi_{\bar{G}}^{-1}(\tilde{H})$.

In other words, $\bar{H}$ is a covering group for $\tilde{H}$ and $\bar{H}$ therefore has the same Lie algebra as $\tilde{H}$.

Let

$$\tilde{J} = \left\{ f \in C(X, \mathrm{SU}_2(\mathbf{C})) \colon f = \begin{pmatrix} a & b \\ -\bar{b} & -\bar{a} \end{pmatrix} \quad \text{with} \quad a \in C_b(X), b \in C_0(X) \right\}^0.$$

It is easy to see that $\tilde{J}$ is a real Lie subgroup of $\tilde{G}$ which acts transitively on $C_0(X, \bar{\mathbf{C}})^0$ with isotropy subgroup $I_{\tilde{J}} = \left\{ f \in C(X, \mathrm{SU}_2(\mathbf{C})) \colon f = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \right\}$. Note that $\tilde{J}$ contains $\tilde{L}$ as a subgroup. Again, as in Proposition 1.5, it can be shown that $\tilde{J}$ induces a complex manifold structure on $C_0(X, \bar{\mathbf{C}})^0$ which coincides with that induced by $\tilde{G}$ and $\tilde{L}$ above.

In particular, we may realise $C_0(X, \bar{\mathbf{C}})^0$ as the quotient manifold $\tilde{J}/I_{\tilde{J}}$. The Lie algebra of $\tilde{J}$ is

$$j = \left\{ f \in C(X, \mathrm{su}_2(\mathbf{C})) \colon f = \begin{pmatrix} i\beta & a \\ -\bar{a} & -i\beta \end{pmatrix}, a \in C_0(X) \quad \text{and} \quad \beta \in C_b(X, \mathbf{R}) \right\}$$

and the Lie algebra of $I_{\tilde{J}}$ is

$$i_j = \left\{ f \in \mathrm{C}(X, \mathrm{su}_2(\mathbf{C})) \colon f = \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix}, \text{with } \beta \in C_b(X, \mathbf{R}) \right\}.$$

Since $\mathrm{aut}(\bar{\mathbf{C}}) = k \oplus \rho$, where we make the identifications

$$k = \left\{ \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix} \colon \beta \in \mathbf{R} \right\} \quad \text{and} \quad \rho = \left\{ \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} \colon a \in \mathbf{C} \right\}$$

then $j = C_b(X, k) \oplus C_0(X, \rho)$ and $i_j = C_b(X, k)$. By Proposition 2.1,

$$i_j = \mathrm{aut}(C_0(X, Z)) \quad \text{and hence} \quad j = \mathrm{aut}(C_0(X, \bar{\mathbf{C}})^0)$$

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