MATH. SCAND. 81 (1997), 35-68

L^p MAXIMAL ESTIMATES FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

PER SJÖLIN

Abstract.

L^p maximal estimates are considered for solutions to an initial value problem for the Schrödinger equation.

1. Introduction.

Let f belong to the Schwartz space $S(\mathsf{R}^n)$ and set

$$S_t f(x) = u(x,t) = (2\pi)^{-n} \int_{\mathsf{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \hat{f}(\xi) d\xi, \qquad x \in \mathsf{R}^n,\ t \in \mathsf{R},$$

where $a > 1$. Here $\hat{f}$ denotes the Fourier transform of f, defined by

$$\hat{f}(\xi) = \int_{\mathsf{R}^n} e^{-i\xi\cdot x} f(x)\, dx.$$

It then follows that $u(x,0) = f(x)$ and in the case $a = 2$ u is a solution to the Schrödinger equation $i\partial u/\partial t = \Delta u$.

We shall here consider the maximal function

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|, \qquad x \in \mathsf{R}^n.$$

We also introduce Sobolev spaces H_s by setting

$$H_s = \{f \in \mathscr{S}'; \| f \|_{H_s} < \infty\}, \qquad s \in \mathsf{R}$$

where

$$\| f \|_{H_s} = \left(\int_{\mathsf{R}^n} (1 + |\xi|^2)^s\ |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

Received April 26, 1995.

We shall here study the local estimate

$$\| S^*f \|_{L^q(B)} \leq C_B \| f \|_{H_s}, \tag{1}$$

where B is an arbitrary ball in R^n, and the global estimate

$$\| S^*f \|_{L^q(\mathsf{R}^n)} \leq C \| f \|_{H_s} . \tag{2}$$

Here $1 \leq q \leq \infty$ and $s \in \mathsf{R}$. These inequalities have implications for the existence almost everywhere of $\lim_{t \to 0} u(x, t)$ for solutions u to the Schrödinger equation.

The estimates (1) and (2) and related questions have been studied in several papers, e.g. L. Carleson [4], A. Carbery [3], M. Cowling [5], B.E.J. Dahlberg and C.E. Kenig [6], C.E. Kenig and A. Ruiz [9], L. Vega [19], [20], P. Sjölin [15], [16], [17], P. Sjögren and P. Sjölin [13], C.E. Kenig, G. Ponce and L. Vega [7], [8], E. Prestini [12] and J. Bourgain [2]. We mention some known results.

For $n = 1$ the local estimate

$$\left(\int_B |S^*f(x)|^2 dx \right)^{1/2} \leq C_B \| f \|_{H_s} \tag{3}$$

holds for $s = 1/4$ and $1/4$ can not be replaced by a smaller number. In the case $n = 2$ (3) holds with $s = 1/2$ and in the special case $n = 2$, $a = 2$, (3) is also known to hold for some $s < 1/2$.

If $n \geq 3$ then (3) has been proved for $s > 1/2$.

In the case $n = 1$ one has the global estimate

$$\left(\int_{\mathsf{R}} |S^*f(x)|^4 dx \right)^{1/4} \leq C \| f \|_{H_{1/4}} .$$

It is also known that in the case $n = 1$ the global estimate

$$\left(\int_{\mathsf{R}} |S^*f(x)|^2 dx \right)^{1/2} \leq C \| f \|_{H_s}$$

holds for $s > a/4$ and does not hold for $s < a/4$.

We shall here consider the case $n = 1$ and the case $n \geq 2$ and f radial. We have the following results.

THEOREM 1. *Assume* $n = 1$.
If $s < 1/4$ *then* (1) *holds for no* q.

If $1/4 \leq s < 1/2$ *then* (1) *holds if and only if* $q \leq 2/(1-2s)$.
If $s = 1/2$ *then* (1) *holds if and only if* $q < \infty$.
If $s > 1/2$ *then* (1) *holds for all* q.

THEOREM 2. *Assume* $n = 1$. *Assume that the conditions in Theorem* 1 *hold and that* $q > 4(a-1)/(4s+a-2)$ *for* $s > 1/4$ *and also that* $q \geq 2$. *Then the global estimate* (2) *holds. If the condition in Theorem* 1 *does not hold or if* $q < 4(a-1)/(4s+a-2)$ *or if* $q < 2$, *then* (2) *does not hold.*

Theorem 2 implies that we have decided for which pairs (s, q) the global estimate (2) holds, except in the case $q = 4(a-1)/(4s+a-2)$, $1/4 < s \leq a/4$.

THEOREM 3. *Assume* $n \geq 2$ *and* f *radial.*
If $s < 1/4$ *then* (1) *holds for no* q.
If $1/4 \leq s < n/2$ *then* (1) *holds if and only if* $q \leq 2n/(n-2s)$.
If $s = n/2$ *then* (1) *holds if and only if* $q < \infty$.
If $s > n/2$ *then* (1) *holds for all* q.

THEOREM 4. *Assume* $n \geq 2$ *and* f *radial. Assume that the conditions in Theorem* 3 *hold and that* $q > 4(a-1)n/(4s+a(2n-1)-2n)$ *for* $s > 1/4$ *and also that* $q \geq 2$. *Then the global estimate* (2) *holds. If the condition in Theorem* 3 *does not hold or if* $q < 4(a-1)n/(4s+a(2n-1)-2n)$ *or if* $q < 2$, *then* (2) *does not hold.*

Theorem 4 means that we have decided for which pairs (s, q) the global estimate (2) holds, except in the case $q = 4(a-1)n/(4s+a(2n-1)-2n)$, $1/4 < s \leq a/4$.

THEOREM 5. *Assume* $n \geq 2$, $a = 2$ *and that the local estimate* (1) *holds for arbitrary* $f \in \mathcal{S}$ *(not necessarily radial). Then*

$$s + \frac{n-1}{2q} \geq \frac{n}{4}. \tag{4}$$

In particular if $s = 1/4$ *then* $q \leq 2$.

It follows from Theorem 5 that the estimates for radial functions in Theorems 3 and 4 do not hold for general functions.

2. Proofs.

PROOF OF THEOREM 1. We shall first prove that if $1/4 \leq s < 1/2$ and $q = 2/(1-2s)$ then the global estimate (2) holds. The sufficiency of the conditions in Theorem 1 then follows from this fact.

Let $t(x)$ denote a measurable function on $\mathbf{R}$ with $0 < t(x) < 1$ and set

$$Rf(x) = \int_{\mathbf{R}} e^{ix\xi} e^{it(x)|\xi|^a} |\xi|^{-s} \hat{f}(\xi) d\xi, \qquad f \in \mathcal{S}.$$

It is then sufficient to prove that

$$\| Rf \|_q \leq C \| f \|_2, \tag{5}$$

where the norms are taken over R.

Let $\rho \in C_0^\infty(\mathbf{R})$ be real-valued and assume that $\rho(x) = 1$, $|x| \leq 1$, and $\rho(x) = 0$, $|x| \geq 2$. Set $\rho_N(x) = \rho(x/N)$ and

$$R_N f(x) = \rho_N(x) \int_{\mathbf{R}} e^{ix\xi} e^{it(x)|\xi|^a} |\xi|^{-s} \rho_N(\xi) \hat{f}(\xi) d\xi, \qquad N > 2.$$

To prove (5) it then suffices to prove that

$$\| R_N f \|_q \leq C \| f \|_2, \qquad N > 2. \tag{6}$$

On the other hand (6) follows from the inequality

$$\| R_N^* f \|_2 \leq C \| f \|_{q'}, \qquad N > 2, \tag{7}$$

where R_N^* denotes the adjoint of R_N and $1/q + 1/q' = 1$.

To prove (7) we first observe that it follows from P. Sjölin [15], pp. 707--709, that

$$\int |R_N^* g(x)|^2 dx \leq 2\pi \int_{\mathbf{R}} \int_{\mathbf{R}} |K_N(y,z)| \, |g(y)| \, |g(z)| dy \, dz, \tag{8}$$

where

$$K_N(y,z) = \rho_N(y)\rho_N(z) \int_{\mathbf{R}} e^{i[(z-y)\xi + (t(z)-t(y))|\xi|^a]} \mu_N(\xi) |\xi|^{-2s} d\xi,$$

and $\mu = \rho^2$, $\mu_N(\xi) = \mu(\xi/N)$.

We shall prove that

$$|K_N(y,z)| \leq C|y-z|^{-(1-2s)}. \tag{9}$$

We set $d = t(z) - t(y)$ and may assume $0 < d < 1$. Performing a change of variable $\eta = d^{1/a}\xi$ we obtain

$$K_N(y,z) = \rho_N(y)\rho_N(z) \int e^{i(v\eta + |\eta|^a)} |\eta|^{-2s} \mu(\eta/L) d\eta \, d^{(2s-1)/a}, \tag{10}$$

where $v = d^{-1/a}(z-y)$ and $L = Nd^{1/a}$.

Now let

$$m(\eta) = e^{i|\eta|^a}|\eta|^{-2s}, \qquad \eta \in \mathsf{R},$$

and write $m = m_1 + m_2$, where $m_1 = m\psi, m_2 = m\rho$ and $\psi = 1 - \rho$. Also define kernels K, K_1 and K_2 by setting $\hat{K} = m, \hat{K}_1 = m_1$ and $\hat{K}_2 = m_2$.

It is proved in A. Miyachi [10] that $K_1 \in C^\infty(\mathsf{R})$ and

$$|K_1(u)| \leq C|u|^{(2s-1+a/2)/(1-a)}, \qquad |u| \geq 1.$$

It follows from the inequalities $a > 1$ and $s \geq 1/4$ that

$$\frac{2s - 1 + a/2}{a - 1} \geq 1 - 2s$$

and we therefore conclude that

$$|K_1(u)| \leq C|u|^{-(1-2s)}, \qquad |u| \geq 1.$$

We have

$$\hat{m}_2(u) = \int e^{-iu\eta}(e^{i|\eta|^a} - 1)|\eta|^{-2s}\rho(\eta)d\eta + \int e^{-iu\eta}|\eta|^{-2s}\rho(\eta)d\eta = G(u) + H(u).$$

Introducing an auxiliary function $\gamma(x) = 1/(1 + x^2)$, $x \in \mathsf{R}$, it is easy to see that

$$\int_{\mathsf{R}} |u|^{-(1-2s)}\gamma(x - u)du \leq C|x|^{-(1-2s)} \tag{11}$$

(cf. [15], pp. 711--712). Using (11) and the fact that $|\eta|^{-2s}$ has Fourier transform $C|x|^{-(1-2s)}$ we conclude that

$$|H(u)| \leq C|u|^{-(1-2s)}.$$

To estimate $G(u)$ we observe that

$$G(u) = \lim_{\varepsilon \to 0} G_\varepsilon(u),$$

where

$$G_\varepsilon(u) = \int e^{-iu\eta}(e^{i|\eta|^a} - 1)\eta|^{-2s}\rho(\eta)\psi(\eta/\varepsilon)d\eta, \qquad \varepsilon > 0.$$

An integration by parts shows that

$$|G_\varepsilon(u)| \leq C\frac{1}{|u|}\int_{-2}^{2} |\eta|^{a-1-2s}d\eta \leq C\frac{1}{|u|}$$

and hence

$$|G(u)| \le C|u|^{-(1-2s)}, \qquad |u| \ge 1.$$

It therefore follows that $|K_2(u)| \le C$, $|u| \le 1$, and $|K_2(u)| \le C|u|^{-(1-2s)}$, $|u| > 1$.

It follows that

$$|K(u)| \le C|u|^{-(1-2s)}$$

and invoking (10) and (11) we obtain

$$\begin{aligned}
|K_N(y,z)| &\le Cd^{(2s-1)/a}\left|\int K(v+u)L\hat{\mu}(Lu)du\right| \\
&= Cd^{(2s-1)/a}\left|\int K(u)L\hat{\mu}(Lu-Lv)du\right| \\
&\le Cd^{(2s-1)/a}\int |u|^{-(1-2s)}L\gamma(Lv-Lu)du \\
&= Cd^{(2s-1)/a}\int |t|^{-(1-2s)}\gamma(Lv-t)dtL^{1-2s} \\
&\le Cd^{(2s-1)/a}L^{1-2s}|Lv|^{-(1-2s)} \\
&= Cd^{(2s-1)/a}|d^{-1/a}(z-y)|^{-(1-2s)} = C|y-z|^{-(1-2s)}.
\end{aligned}$$

Hence (9) is proved and using (8) we conclude that

$$\begin{aligned}
\| R_N^* g\|_2^2 &\le C\int\int |y-z|^{-(1-2s)}|g(y)|\,|g(z)|dy\,dz \\
&\le \int |g(y)|I_{2s}(|g|)(y)dy \le C\,\|g\|_{q'}\|I_{2s}(|g|)\|_q \\
&\le C\,\|g\|_{q'}^2,
\end{aligned}$$

where I_{2s} denotes the Riesz potential operator of order $2s$. Here we used the fact that

$$\frac{1}{q} = \frac{1}{q'} - 2s.$$

We have now proved that the global estimate (2) holds for $1/4 \le s < 1/2$ and $q = 2/(1-2s)$. The sufficiency of the conditions in Theorem 1 follows if we also invoke the trivial inequality

$$\| S^* f\|_\infty \le C\,\|f\|_{H_s}, \qquad s > 1/2.$$

It remains to prove that the conditions in Theorem 1 are also necessary. It is well-known that there is no local estimate for $s < 1/4$. This follows from the proof of Theorem 4 in P. Sjölin [15].

We shall then prove that if $1/4 \le s < 1/2$ and the local estimate holds then

$q \leq 2/(1-2s)$. Let $\varphi \in C_0^\infty(\mathsf{R})$ be even and non-negative. Assume that $\operatorname{supp} \varphi \subset \{\xi; 1 < |\xi| < 2\}$ and that $\varphi(\xi) = 1$ for $5/4 \leq |\xi| \leq 7/4$. Then choose f so that $\hat{f}(\xi) = \varphi(\xi/N)$, $N > 1$. We have

$$\| f \|_{H_s}^2 \leq C \int_{N \leq |\xi| \leq 2N} N^{2s} d\xi \leq C N^{1+2s}$$

and

$$\| f \|_{H_s} \leq C N^{1/2+s}.$$

Setting $\pi_n = (2\pi)^{-n}$ one also has

$$S_0 f(x) = \pi_1 \int_{\mathsf{R}} e^{ix\xi} \varphi(\xi/N) d\xi = \pi_1 \int_{\mathsf{R}} e^{ixN\eta} \varphi(\eta) d\eta N = \pi_1 N \hat{\varphi}(Nx).$$

It follows that

$$S^* f(x) \geq c_0 N$$

for $|x| \leq \delta/N$, where δ is a small positive number.

If the local estimate holds we therefore necessarily have

$$\left(\int_{|x| \leq \delta/N} N^q dx \right)^{1/q} \leq C N^{1/2+s}$$

and hence

$$N^{1-1/q} \leq C N^{1/2+s}.$$

Letting N tend to infinity we conclude that $1 - 1/q \leq 1/2 + s$, which implies that $q \leq 2/(1-2s)$.

It now remains only to observe that the estimate

$$\| S^* f \|_{L^\infty(B)} \leq C_B \| f \|_{H_{1/2}} \tag{12}$$

does not hold, and this follows from the well-known fact that the estimate

$$\| f \|_{L^\infty(B)} \leq C_B \| f \|_{H_{1/2}}$$

is not valid.

The proof of Theorem 1 is complete.

Proof of Theorem 2. In the proof of Theorem 1 we proved that the global estimate (2) holds for $s = 1/4$, $q = 4$, and in C.E. Kenig, G. Ponce and L. Vega [8] and P. Sjölin [17] it is proved that (2) holds for $s = b/4$, $q = 2$, if

$b > a$. Interpolation (J. Bergh and J. Löfström [1], pp. 152--153) between these results shows that (2) holds for the pair (s, q) if $0 < \theta < 1$,

$$s = (1 - \theta)/4 + \theta b/4$$

and

$$1/q = (1 - \theta)/4 + \theta/2.$$

We obtain $\theta = (4s - 1)/(b - 1)$ and

$$\frac{1}{q} = \frac{1}{4} + \frac{4s - 1}{4(b - 1)} = \frac{4s + b - 2}{4(b - 1)}.$$

Using the condition $b > a$ we can conclude that (2) holds for $1/4 < s \leq a/4$ and

$$\frac{4s + a - 2}{4(a - 1)} - \varepsilon(s) < \frac{1}{q} < \frac{4s + a - 2}{4(a - 1)},$$

where $\varepsilon(s) > 0$ is small. Combining this with the global estimates in the proof of Theorem 1 we obtain the sufficiency part of Theorem 2.

We shall then prove the necessity part of Theorem 2. We shall first prove that if $1/4 \leq s \leq a/4$ and (2) holds then $q \geq 4(a - 1)/(4s + a - 2)$. Using a counter-example from P. Sjölin [17] we first define f by setting

$$\hat{f}(\xi) = \varphi(N^{a/2-1}\xi + N^{a/2}), \qquad N > 1,$$

where $\varphi \in C_0^\infty(\mathsf{R})$ and $\operatorname{supp} \varphi \subset (-1, 1)$. Following [17], pp. 112--113, we obtain

$$\| f \|_{H_s} \leq CN^{s+1/2-a/4}$$

and the estimate

$$S^*f(x) \geq cN^{1-a/2}, \qquad (1 - \varepsilon)N^{a-1} \leq x \leq N^{a-1},$$

where $\varepsilon > 0$ is small.

It follows that

$$\| S^*f \|_q \geq cN^{1-a/2}N^{(a-1)/q},$$

and if (2) holds one obtains

$$N^{1-a/2+(a-1)/q} \leq CN^{s+1/2-a/4}.$$

Letting $N \to \infty$ we conclude that

$$1-\frac{a}{2}+\frac{a-1}{q}\leq s+\frac{1}{2}-\frac{a}{4}$$

and

$$\frac{a-1}{q}\leq s-\frac{1}{2}+\frac{a}{4}.$$

The inequality $q\geq 4(a-1)/(4s+a-2)$ now follows.

It remains to prove that we can never have $q<2$ in the global estimate (2). To see this we construct a function f in the following way. Let $\psi\in C^\infty(\mathbf{R})$ and assume that $\psi(x)=0$, $x\leq 2$, and $\psi(x)=1$, $x\geq 3$. Set $f(x)=0$, $x\leq 2$, and

$$f(x)=\frac{1}{x^{1/2}\log x}\psi(x), \qquad x>2.$$

Then $f\in H_s$ for every s but $f\notin L^q$ if $q<2$, and f can be used to give a counter-example.

The necessity part of Theorem 2 now follows if we combine the above two counter-examples with the necessary conditions in Theorem 1.

This completes the proof of Theorem 2.

Proof of Theorem 3. Assume $n\geq 2$. We shall first prove that the global estimate (2) holds for radial functions if $1/4\leq s<n/2$ and $q=2n/(n-2s)$.

Let $t(x)$ be measurable and radial, $0<t(x)<1$, and set

$$Tf(x)=S_{t(x)}f(x), \qquad f\in\mathcal{S}.$$

It suffices to study T instead of S^*.

If f is radial we have

$$Tf(u)=c_n u^{1-n/2}\int_0^\infty J_{n/2-1}(ru)e^{it(u)r^a}\hat{f}(r)r^{n/2}dr, \qquad u>0,$$

where $J_{n/2-1}$ denotes a Bessel function (see E.M. Stein and G. Weiss [18], p. 155). Here we write $Tf(u)=Tf(x)$ if $u=|x|$ and $\hat{f}(r)=\hat{f}(\xi)$ if $r=|\xi|$.

We have to show that

$$\left(\int_0^\infty |Tf(u)|^q u^{n-1}du\right)^{1/q}\leq C\left(\int_0^\infty |\hat{f}(r)|^2(1+r^2)^s r^{n-1}dr\right)^{1/2}.$$

One has

$$Tf(u)u^{(n-1)/q} = c_n u^{n/q-1/q+1-n/2} \int_0^\infty J_{n/2-1}(ru)e^{it(u)r^a}\hat{f}(r)r^{n/2}dr$$

$$= c_n u^{n/q-1/q+1-n/2} \int_0^\infty J_{n/2-1}(ru)e^{it(u)r^a}g(r)(1+r^2)^{-s/2}r^{1/2}dr,$$

where $g(r) = \hat{f}(r)(1+r^2)^{s/2}r^{(n-1)/2}$, and setting

$$Pg(u) = u^{n/q-1/q+1-n/2} \int_0^\infty J_{n/2-1}(ru)e^{it(u)r^a}g(r)(1+r^2)^{-s/2}r^{1/2}dr$$

we conclude that

$$Tf(u)u^{(n-1)/q} = c_n P_g(u).$$

We therefore have to prove that

$$\left(\int_0^\infty |Pg(u)|^q du\right)^{1/q} \leq C\left(\int_0^\infty |g(r)|^2 dr\right)^{1/2}. \tag{13}$$

Now set

$$P^*g(r) = (1+r^2)^{-s/2}r^{1/2} \int_0^\infty J_{n/2-1}(ru)e^{-it(u)r^a}u^{n/q-1/q+1-n/2}g(u)du,$$

$$r > 0, \quad g \in C_0^\infty(0,\infty).$$

Then

$$\int_0^\infty Pf(u)\overline{g(u)}du = \int_0^\infty f(r)\overline{P^*g(r)}dr,$$

if $g \in C_0^\infty(0,\infty)$ and $f \in L^2(0,\infty)$ and f has suitable decay at infinity. To prove (13) it is therefore sufficient to prove that

$$\| P^*g \|_2 \leq C \| g \|_p, \qquad g \in C_0^\infty(0,\infty), \tag{14}$$

where $1/p + 1/q = 1$ and the norms are taken over the interval $(0,\infty)$.

We shall write

$$P^*g(r) = (1+r^2)^{-s/2} \int_0^\infty (ru)^{1/2} J_{n/2-1}(ru) e^{-it(u)r^a} u^{-\gamma} g(u)\,du,$$

where $\gamma = (n-1)(1/2 - 1/q)$.

We choose an auxiliary non-negative and even function $\varphi \in C_0^\infty(\mathsf{R})$ so that $\operatorname{supp} \varphi \subset \{\xi; 1/2 < |\xi| < 2\}$ and

$$\sum_{-\infty}^{\infty} \varphi(2^{-k}\xi) = 1, \qquad \xi \neq 0,$$

and set

$$\varphi_0(\xi) = 1 - \sum_1^\infty \varphi(2^{-k}\xi)$$

and

$$\psi(\xi) = \sum_1^\infty \varphi(2^{-k}\xi).$$

We then have $\varphi_0 \in C_0^\infty(\mathsf{R})$ and $\varphi_0 + \psi = 1$.

It follows from the properties of Bessel functions (see [18]) that

$$t^{1/2} J_{n/2-1}(t) = \varphi_0(t)\mathcal{O}(t^{n/2-1/2}) + \psi(t)(b_1 e^{it} + b_2 e^{-it} + \mathcal{O}(1/t)), \qquad t > 0,$$

where b_1 and b_2 denote constants. Hence

$$P^*g(r) = S(r) + B_1(r) + B_2(r) + C(r),$$

where

$$|S(r)| \leq C(1+r^2)^{-s/2} \int_0^{2/r} (ru)^{n/2-1/2} u^{-\gamma} |g(u)|\,du,$$

$$B_1(r) = b_1(1+r^2)^{-s/2} \int_0^\infty \psi(ru) e^{iru} e^{-it(u)r^a} u^{-\gamma} g(u)\,du,$$

$$B_2(r) = b_2(1+r^2)^{-s/2} \int_0^\infty \psi(ru) e^{-iru} e^{-it(u)r^a} u^{-\gamma} g(u)\,du$$

and

$$|C(r)| \leq C(1+r^2)^{-s/2} \int_{1/r}^{\infty} (ru)^{-1} u^{-\gamma} |g(u)| du.$$

We shall first study $S(r)$ for $0 < r < 1$. Invoking Hölder's inequality one has

$$|S(r)| \leq Cr^{n/2-1/2} \int_0^{2/r} u^{n/2-1/2-\gamma} |g(u)| du$$

$$\leq Cr^{n/2-1/2} \left(\int_0^{2/r} u^{(n/2-1/2-\gamma)q} du \right)^{1/q} \|q\|_p .$$

Now $n/2 - 1/2 - \gamma = (n-1)/q$ and therefore the integral in the above right hand side is majorized by

$$\int_0^{2/r} u^{n-1} du = Cr^{-n}.$$

Hence,

$$|S(r)| \leq Cr^{n/2-1/2-n/q} \|g\|_p, \qquad 0 < r < 1,$$

and one obtains

$$\left(\int_0^1 |S(r)|^2 dr \right)^{1/2} \leq C \left(\int_0^1 r^{n-1-2n/q} dr \right)^{1/2} \|g\|_p \leq C \|g\|_p$$

since $q > 2$.

For $r > 1$ we have

$$|S(r)| \leq Cr^{-s+n/2-1/2} \int_0^{2/r} u^{n/2-1/2-\gamma} |g(u)| du.$$

Setting

$$M(t) = \frac{1}{t} S\left(\frac{2}{t}\right), \qquad 0 < t < 2,$$

one has

$$\int_1^\infty |S(r)|^2 dr = 2\int_0^2 |M(t)|^2 dt$$

and we shall estimate $M(t)$. We get

$$|M(t)| \leq c\frac{1}{t}t^{s-n/2+1/2}\int_0^t u^{n/2-1/2-\gamma}|g(u)|du = C\int_0^t \frac{u^{n/2-1/2-\gamma}}{t^{-s+n/2+1/2}}|g(u)|du.$$

We then choose a number α so that $\alpha > 0$ and $s - n/2 + 1/2 \leq \alpha < 1/2$. It then follows that $-s + n/2 - 1/2 + \alpha \geq 0$ and

$$\begin{aligned}|M(t)| &\leq C\int_0^t \frac{u^{n/2-1/2-\gamma}}{t^{-s+n/2+1/2-1+\alpha}}\frac{1}{t^{1-\alpha}}|g(u)|du \\ &\leq C\int_0^t \frac{1}{t^{1-\alpha}}u^{s-\gamma-\alpha}|g(u)|du \leq CI_\alpha(u^{s-\gamma-\alpha}|g|)(t).\end{aligned}$$

Here we set $g(u) = 0$ for ≤ 0.

Using Plancherel's formula we get

$$\int_0^2 |M(t)|^2 dt \leq C\int |I_\alpha(u^{s-\gamma-\alpha}|g|)|^2 dt = C\int |\xi|^{-2\alpha}|(u^{s-\gamma-\alpha}|g|)^\wedge(\xi)|^2 d\xi.$$

We now invoke Pitt's inequality for Fourier transforms (see for instance B. Muckenhoupt [11]), which states that

$$\left(\int_{\mathbf{R}} |\hat{f}(\xi)|^2|\xi|^{-2\alpha}d\xi\right)^{1/2} \leq C\left(\int_{\mathbf{R}} |f(x)|^p|x|^{\alpha_1 p}dx\right)^{1/p}, \tag{15}$$

if $\alpha_1 = \alpha + 1/2 - 1/p$,

$$0 \leq \alpha_1 < 1 - 1/p \tag{16}$$

and

$$0 \leq \alpha < 1/2. \tag{17}$$

Here we have chosen p as above and we also choose α as above. Then (17) holds and (16) is equivalent to $1/p - 1/2 \leq \alpha < 1/2$. Choosing α close to $1/2$ we may assume that this inequality holds.

Using (15) we then obtain

$$\left(\int_1^\infty |S(r)|^2 dr\right)^{1/2} \le C\left(\int (u^{s-\gamma-\alpha}|g|)^p u^{\alpha_1 p} du\right)^{1/p}$$

$$= C\left(\int |g(u)|^p u^{sp-\gamma p-\alpha p} u^{\alpha p+p/2-1} du\right)^{1/p}$$

$$= C\left(\int |g(u)|^p u^{sp-\gamma p-1+p/2} du\right)^{1/p}.$$

However, we have

$$sp - \gamma p - 1 + p/2 = s\frac{q}{q-1} - (n-1)\left(\frac{1}{2} - \frac{1}{q}\right)\frac{q}{q-1} - 1 + \frac{q}{2(q-1)}$$

$$= \frac{q}{q-1}\left(s - \frac{n}{2} + \frac{n}{q}\right) = \frac{q}{q-1}\left(s - \frac{n}{2} + \frac{n-2s}{2}\right) = 0$$

and thus

$$\left(\int_1^\infty |S(r)|^2 dr\right)^{1/2} \le C\, \|g\|_p,$$

and we have proved that

$$\|S\|_2 \le C\, \|g\|_p\,. \tag{18}$$

We shall then study $C(r)$. One has

$$|C(r)| \le Cr^{-s}\int_{1/r}^\infty (ru)^{-1}u^{-\gamma}|g(u)|du = Cr^{-1-s}\int_{1/r}^\infty u^{-1-\gamma}|g|du.$$

Setting

$$M(t) = \frac{1}{t}C\left(\frac{1}{t}\right)$$

we have $\|C\|_2 = \|M\|_2$ and

$$|M(t)| \le Ct^s\int_t^\infty u^{-1-\gamma}|g|du \le C\int_t^\infty u^{s-1-\gamma}|g|du$$

$$= C\int_t^\infty \frac{1}{u^{1-\alpha}}u^{s-\gamma-\alpha}|g|du \le C\int_t^\infty \frac{1}{(u-t)^{1-\alpha}}u^{s-\gamma-\alpha}|g|du$$

$$\le CI_\alpha(u^{s-\gamma-\alpha}|g|)(t),$$

where α is chosen as above. We can then use the above argument to conclude that

(19) $$\| C \|_2 \leq C \, \| g \|_p \, .$$

It remains to estimate B_1 and B_2. We set

$$A(\xi) = (1+\xi^2)^{-s/2} \int_0^\infty \psi(\xi u) e^{i\xi u} e^{-it(u)|\xi|^a} u^{-\gamma} g(u) du, \qquad \xi \in \mathsf{R},$$

and shall prove that

(20) $$\left(\int_{\mathsf{R}} |A(\xi)|^2 d\xi \right)^{1/2} \leq C \, \| g \|_p,$$

from which the required estimates for B_1 and B_2 follow.

We write

$$A(\xi) = \varphi_0(\xi) A(\xi) + \psi(\xi) A(\xi) = A_0(\xi) + A_1(\xi)$$

and shall first study A_0. We have

$$A_0(\xi) = \varphi_0(\xi)(1+\xi^2)^{-s/2} \int_0^\infty \psi(\xi u) e^{i\xi u} e^{-it(u)|\xi|^a} u^{-\gamma} g(u) du$$

and changing the order of integration we obtain

$$\int_{\mathsf{R}} |A_0(\xi)|^2 d\xi = \int_0^\infty \int_0^\infty I(u,v) u^{-\gamma} g(u) v^{-\gamma} \overline{g(v)} \, du \, dv.$$

where

$$I(u,v) = \int_{\mathsf{R}} e^{i[(u-v)\xi - (t(u)-t(v))|\xi|^a]} \varphi_0(\xi)^2 (1+\xi^2)^{-s} \psi(\xi u) \psi(\xi v) d\xi.$$

It follows from the definition of the functions φ_0 and ψ that $I(u,v) = 0$ if $0 < u < 1/2$ or $0 < v < 1/2$. Hence

$$\| A_0 \|_2^2 \leq C \int_{1/2}^\infty \int_{1/2}^\infty |I(u,v)| \, |g(u)| \, |g(v)| du \, dv$$

since $\gamma > 0$.

It is clear that $|I(u,v)| \leq C$ and an integration by parts also shows that

$$|I(u,v)| \leq C\frac{1}{|u-v|}, \qquad |u-v| > 1.$$

Here we use the fact that

$$\int u|\psi'(\xi u)|d\xi \leq C.$$

We set

$$K(u) = \frac{1}{|u|+1}, \qquad u \in \mathsf{R},$$

and $Rf = K * f$. Since

$$K(u) \leq C_\alpha \frac{1}{|u|^{1-\alpha}}$$

for every α with $0 < \alpha < 1$, we have

$$\| Rf \|_q \leq C \| f \|_p .$$

We have proved that $|I(u,v)| \leq CK(u-v)$ and it follows that

$$\text{(21)} \qquad \| A_0 \|_2^2 \leq C \int_0^\infty \int_0^\infty K(u-v)|g(u)|\,|g(v)|du\,dv$$
$$= C \int R(|g|)(u)|g(u)|du \leq C \| R(|g|) \|_q \| g \|_p \leq C \| g \|_p^2 .$$

It now remains to prove that

$$\text{(22)} \qquad \| A_1 \|_2 \leq C \| g \|_p,$$

where

$$A_1(\xi) = \psi(\xi)(1+\xi^2)^{-s/2} \int_0^\infty \psi(\xi u) e^{i\xi u} e^{-it(u)|\xi|^a} u^{-\gamma} g(u)du.$$

We have

$$|A_1(\xi)| \leq C\left(\sum_1^\infty \varphi(2^{-k}\xi)2^{-ks}\right)\left|\int_0^\infty \psi(\xi u) e^{i\xi u} e^{-it(u)|\xi|^a} u^{-\gamma} g(u)du\right|$$

and set

$$A_{1,N}(\xi) = C\left(\sum_{1}^{N} \varphi(2^{-k}\xi)2^{-ks}\right)\left|\int_0^\infty \psi(\xi u)e^{i\xi u}e^{-it(u)|\xi|^a}u^{-\gamma}g(u)du\right|,$$

$N = 1, 2, 3, \ldots$. It follows that

$$|A_{1,N}(\xi)|^2 \leq C\left(\sum_{1}^{N} \varphi(2^{-k}\xi)2^{-2ks}\right)\left|\int_0^\infty \psi(\xi u)e^{i\xi u}e^{-it(u)|\xi|^a}u^{-\gamma}g(u)du\right|^2.$$

Changing the order of integration and performing a change of variable we obtain

$$\begin{aligned}
&\int_{\mathsf{R}} |A_{1,N}(\xi)|^2 d\xi \\
&\leq C\int_{\mathsf{R}}\left(\sum_{1}^{N} \varphi(2^{-k}\xi)2^{-2ks}\right)\left(\int_0^\infty \psi(\xi u)e^{i\xi u}e^{-it(u)|\xi|^a}u^{-\gamma}g(u)du\right) \\
&\quad\cdot\left(\int_0^\infty \psi(\xi v)e^{-i\xi v}e^{it(v)|\xi|^a}v^{-\gamma}\,\overline{g(v)}\,dv\right)d\xi \\
&= C\int_0^\infty\int_0^\infty\left(\sum_{1}^{N} 2^{-2ks}\int_{\mathsf{R}} e^{i[(u-v)\xi-(t(u)-t(v))|\xi|^a]}\varphi(2^{-k}\xi)\psi(\xi u)\psi(\xi v)d\xi\right) \\
&\quad\cdot u^{-\gamma}g(u)v^{-\gamma}\,\overline{g(v)}\,du\,dv \\
&= C\sum_{1}^{N} 2^{-2ks}2^k\int_0^\infty\int_0^\infty\left(\int_{\mathsf{R}} e^{i[(u-v)2^k\eta-(t(u)-t(v))2^{ka}|\eta|^a]}\varphi(\eta)\psi(2^k\eta u)\psi(2^k\eta v)d\eta\right) \\
&\quad\cdot u^{s-\gamma-\alpha}g(u)v^{s-\gamma-\alpha}\,\overline{g(v)}\,u^{\alpha-s}v^{\alpha-s}du\,dv,
\end{aligned}$$

where we let $\alpha = \max(1/4, s/n)$.

The inner integral vanishes if $u \leq 2^{-k-1}$ or $v \leq 2^{-k-1}$ and since $\alpha \leq s$ we may therefore use the inequalities

$$u^{\alpha-s} \leq C(2^{-k})^{\alpha-s} = C2^{ks-k\alpha}$$

and

$$v^{\alpha-s} \leq C2^{ks-k\alpha}$$

in the above integral. We obtain

$$\int_{\mathbf{R}} |A_{1,N}(\xi)|^2 d\xi$$

$$\leq C \sum_1^N 2^{k(1-2\alpha)} \int_0^\infty \int_0^\infty \left| \int_{\mathbf{R}} e^{i[(u-v)2^k\eta - d2^{ka}|\eta|^a]} \varphi(\eta)\psi(2^k\eta u)\psi(2^k\eta v)\, d\eta \right|$$

$$\cdot\, u^{s-\gamma-\alpha}|g(u)|v^{s-\gamma-\alpha}|g(v)|\,du\,dv,$$

where we have set $d = t(u) - t(v)$.

Letting $N \to \infty$ we obtain

$$\|A_1\|_2^2 \leq C \int_0^\infty \int_0^\infty \left(\sum_1^\infty J_k \right) u^{s-\gamma-\alpha}|g(u)|v^{s-\gamma-\alpha}|g(v)|\,du\,dv,$$

where

$$J_k = 2^{k(1-2\alpha)} \left| \int_{\mathbf{R}} e^{i[(u-v)2^k\eta - d2^{ka}|\eta|^a]} \varphi(\eta)\psi(2^k\eta u)\psi(2^k\eta v)\, d\eta \right|.$$

We claim that

$$\sum_1^\infty J_k \leq C \frac{1}{|u-v|^{1-2\alpha}}. \tag{23}$$

We first assume that (23) holds and complete the estimate of $\|A_1\|_2$. One has

$$\begin{aligned}
\|A_1\|_2^2 &\leq C \int_0^\infty \int_0^\infty \frac{1}{|u-v|^{1-2\alpha}}\, u^{s-\gamma-\alpha}|g(u)|v^{s-\gamma-\alpha}|g(v)|\,du\,dv \\
&= C \int I_{2\alpha}(v^{s-\gamma-\alpha}|g|)(u) u^{s-\gamma-\alpha}|g(u)|\,du \\
&= C \int |\xi|^{-2\alpha} |(u^{s-\gamma-\alpha}|g|)^\wedge(\xi)|^2 d\xi.
\end{aligned}$$

We now invoke Pitt's inequality (15) again. We obtain

$$\begin{aligned}
\|A_1\|_2 &\leq C \left(\int_{\mathbf{R}} (u^{s-\gamma-\alpha}|g|)^p u^{\alpha_1 p}\, du \right)^{1/p} \\
&= C \left(\int_0^\infty |g(u)|^p u^{sp-\gamma p-1+p/2}\, du \right)^{1/p} = C\, \|g\|_p,
\end{aligned} \tag{24}$$

where $\alpha_1 = 1/2 - 1/p + \alpha$, provided we can show that

(25) $$0 \leq \alpha < 1/2$$

and

(26) $$0 \leq \alpha_1 < 1 - 1/p.$$

The inequality (25) holds by the definition of α. Since $\alpha < 1/2$ it is clear that $\alpha_1 < 1 - 1/p$ and using the fact that $1/p = 1/2 + s/n$ we also obtain

$$\alpha_1 = \alpha - \frac{1}{p} + \frac{1}{2} = \alpha - \frac{s}{n} \geq 0,$$

which completes the proof of (26).

Hence (22) is proved and combining this with (18), (19) and (21) we obtain (14).

It now remains to prove the claim (23). To do this we shall invoke the following two lemmas (see P. Sjölin [14] and the references in that paper).

LEMMA 1. *Let Ω denote an open set in $\mathbb{R}^n$ and let $\varphi \in C_0^\infty(\Omega)$. Assume that $\psi \in C^\infty(\Omega)$, ψ is real-valued and that* $|\det(\partial^2\psi/\partial x_i \partial x_k)| \geq c > 0$ *in Ω. Then*

$$\left| \int_\Omega e^{i(\xi \cdot x + \zeta\psi(x))} \varphi(x) dx \right| \leq C(1 + |\zeta|)^{-n/2}, \ \xi \in \mathbb{R}^n, \ \zeta \in \mathbb{R}.$$

LEMMA 2. *Let I denote an open interval in $\mathbb{R}$, let $g \in C_0^\infty(I)$, $F \in C^\infty(I)$ and assume that F is real-valued and $F' = 0$. If k is a positive integer then*

$$\int_I e^{iF(x)} g(x) dx = \int_I e^{iF(x)} h_k(x) dx,$$

where h_k is a linear combination of functions of the form

$$g^{(s)}(F')^{-k-r} \prod_{q=1}^{r} F^{(j_q)}$$

with $0 \leq s \leq k$, $0 \leq r \leq k$ *and* $2 \leq j_q \leq k + 1$.

To prove (23) we may assume $0 < d < 1$. We shall study three cases.

CASE 1. $|u - v| < Cd^{1/a}$.

First assume $2^k \leq d^{-1/a}$. Then we use the estimate $J_k \leq C2^{k(1-2\alpha)}$ and obtain

$$\sum_{2^k \le d^{-1/a}} J_k \le C d^{-(1-2\alpha)/a} \le C|u-v|^{-(1-2\alpha)}.$$

In the case $2^k > d^{-1/a}$ we have $d2^{ka} > 1$ and we use Lemma 1 to conclude that

$$J_k \le C\,2^{k(1-2\alpha)}(d\,2^{ka})^{-1/2}.$$

Since $\alpha \ge 1/4$ we then obtain

$$\begin{aligned}\sum_{2^k > d^{-1/a}} J_k &\le C\,d^{-1/2} \sum_{2^k > d^{-1/a}} 2^{k(1-2\alpha-a/2)}\\ &\le C\,d^{-1/2} d^{-(1-2\alpha-a/2)/a} = C\,d^{-(1-2\alpha)/a}\\ &\le C|u-v|^{-(1-2\alpha)}.\end{aligned}$$

CASE 2. $C\,d^{1/a} \le |u-v| < 1$.

It follows that $1/|u-v| \le c\,d^{-1/a} \le c(|u-v|/d)^{1/(a-1)}$. First assume $2^k \le 1/|u-v|$. One obtains for these k

$$\sum J_k \le C \sum 2^{k(1-2\alpha)} \le C|u-v|^{-(1-2\alpha)}.$$

Then assume $1/|u-v| < 2^k \le c(|u-v|/d)^{1/(a-1)}$. One has $d\,2^{ka} \le c\,2^k|u-v|$ and using Lemma 2 one obtains

$$\begin{aligned}\sum J_k &\le C \sum 2^{k(1-2\alpha)}|u-v|^{-N} 2^{-kN}\\ &= C|u-v|^{-N} \sum 2^{k(-N+1-2\alpha)}\\ &\le C|u-v|^{-N}|u-v|^{N-1+2\alpha} = C|u-v|^{-(1-2\alpha)},\end{aligned}$$

where N denotes a large integer.

We then assume $2^k > c(|u-v|/d)^{1/(a-1)}$. Using Lemma 1 we obtain

$$\begin{aligned}\sum J_k &\le C d^{-1/2} \sum 2^{k(1-2\alpha-a/2)}\\ &\le C d^{-1/2}(|u-v|/d)^{(1-2\alpha-a/2)/(a-1)}\\ &= C|u-v|^{(1-2\alpha-a/2)/(a-1)} d^{-1/2+(2\alpha+a/2-1)/(a-1)}\\ &= C|u-v|^{(1-2\alpha-a/2)/(a-1)} d^{(2\alpha-1/2)/(a-1)}\\ &\le C|u-v|^{(1-2\alpha-a/2)/(a-1)} |u-v|^{a(2\alpha-1/2)/(a-1)}\\ &= C|u-v|^{-(1-2\alpha)}.\end{aligned}$$

CASE 3. $|u-v| \ge 1$.

First assume $2^k \le c(|u-v|/d)^{1/(a-1)}$. Then $d\,2^{ka} \le c\,2^k|u-v|$ and using Lemma 2 one obtains

$$\sum J_k \le C \sum 2^{k(1-2\alpha)} |u-v|^{-N} 2^{-kN} \le C|u-v|^{-N} \le C|u-v|^{-(1-2\alpha)}.$$

Finally we assume $2^k \ge c(|u-v|/d)^{1/(a-1)}$. We then have

$$J_k \le C\, 2^{k(1-2\alpha)} d^{-1/2} 2^{-ka/2}$$

and

$$\begin{aligned}\sum J_k &\le C\, d^{-1/2} \sum 2^{k(1-2\alpha-a/2)} \\ &\le C\, d^{-1/2}(|u-v|/d)^{(1-2\alpha-a/2)/(a-1)}.\end{aligned}$$

Arguing as in Case 2 we get

$$\sum J_k \le C|u-v|^{(1-2\alpha-a/2)/(a-1)} d^{(2\alpha-1/2)/(a-1)}.$$

We have $d < |u-v|^a$ and as in Case 2 we obtain

$$\sum J_k \le C|u-v|^{-(1-2\alpha)}.$$

The claim (23) has now been proved.

We remark that in the above applications of Lemmas 1 and 2 we have used the fact that the derivative of every order of the function $\varphi(\xi)\psi(2^k u\xi)\psi(2^k v\xi)$ is uniformly bounded in k, u and v. This follows from the definition of the functions φ and ψ. This property of the derivatives gives uniformly bounded constants in the estimates.

We have now proved that the global estimate (2) holds for radial functions if $1/4 \le s < n/2$ and $q = 2n/(n-2s)$.

We shall then prove that there is no local estimate for S^*f if $s < 1/4$. It suffices to prove that there is no inequality

$$\int_0^1 |Tf(u)|u^{n-1}du \le C\left(\int_0^\infty |\hat{f}(r)|^2(1+r^2)^s r^{n-1}dr\right)^{1/2}, \tag{27}$$

for $s < 1/4$, where

$$Tf(u) = c_n u^{1-n/2} \int_0^\infty J_{n/2-1}(ru)e^{it(u)r^a}\hat{f}(r)r^{n/2}dr.$$

We let $\varphi \in C_0^\infty(\mathrm{R})$ with $\operatorname{supp}\varphi \subset (-1,1)$ and choose f such that

$$\hat{f}(r) = N^{-1/2}\varphi(-N^{-1/2}r + N^{1/2})r^{1/2-n/2}.$$

It follows that $\hat{f}(r)$ vanishes outside the interval $[N - N^{1/2}, N + N^{1/2}]$ and

$$\|f\|_{H_s}^2 \leq C \int_{N-N^{1/2}}^{N+N^{1/2}} N^{-1} N^{2s} dr = C N^{2s-1/2}.$$

Hence

$$\|f\|_{H_s} \leq C N^{s-1/4}$$

and $\|f\|_{H_s} \to 0$ as $N \to \infty$ if $s < 1/4$.

We shall study $Tf(u)$ for $\varepsilon \leq u \leq 2\varepsilon$, where $\varepsilon > 0$ is small. Using the estimate

$$t^{1/2} J_{n/2-1}(t) = b_1 e^{it} + b_2 e^{-it} + \mathcal{O}(\min(1, 1/t)) \tag{28}$$

we obtain

$$\begin{aligned} Tf(u) &= c_n u^{1/2-n/2} \int_0^\infty J_{n/2-1}(ru)\,(ru)^{1/2} e^{it(u)r^a} \hat{f}(r) r^{n/2-1/2} dr \\ &= c_n u^{1/2-n/2} \int_0^\infty (b_1 e^{iru} + b_2 e^{-iru} + \mathcal{O}(1/ru)) e^{it(u)r^a} N^{-1/2} \\ &\quad \cdot \varphi(-N^{-1/2} r + N^{1/2}) dr = b_1 B_1(u) + b_2 B_2(u) + C(u), \end{aligned}$$

where

$$B_1(u) = c_n u^{1/2-n/2} \int_0^\infty e^{iru} e^{it(u)r^a} N^{-1/2} \varphi(-N^{-1/2} r + N^{1/2}) dr,$$

$$B_2(u) = c_n u^{1/2-n/2} \int_0^\infty e^{-iru} e^{it(u)r^a} N^{-1/2} \varphi(-N^{-1/2} r + N^{1/2}) dr$$

and

$$C(u) = c_n u^{1/2-n/2} \int_0^\infty \mathcal{O}(1/ru) N^{-1/2} \varphi(-N^{-1/2} r + N^{1/2}) dr.$$

We have

$$B_2(u) = c_n u^{1/2-n/2} \int e^{iu\eta} e^{it(u)|\eta|^a} N^{-1/2} \varphi(N^{-1/2} \eta + N^{1/2}) d\eta$$

and setting $\xi = N^{-1/2}\eta + N^{1/2}$ we obtain

$$B_2(u) = c_n u^{1/2-n/2} \int e^{iu(N^{1/2}\xi - N)} e^{it(u)(N-N^{1/2}\xi)^a} \varphi(\xi) d\xi$$
$$= c_n u^{1/2-n/2} \int e^{iF} \varphi d\xi,$$

where

$$\begin{aligned} F(\xi) &= N^{1/2}u\xi - Nu + t(u)N^a(1 - N^{-1/2}\xi)^a \\ &= N^{1/2}u\xi - Nu + t(u)N^a\left(1 - aN^{-1/2}\xi + \frac{a(a-1)}{2}N^{-1}\xi^2 + \mathcal{O}(N^{-3/2})\right) \\ &= t(u)N^a - Nu + N^{1/2}u\xi - t(u)aN^{a-1/2}\xi \\ &\quad + \frac{a(a-1)}{2}t(u)N^{a-1}\xi^2 + \mathcal{O}(t(u)N^{a-3/2}). \end{aligned}$$

We now choose

$$t(u) = \frac{u}{aN^{a-1}}$$

and it follows that

$$|B_2(u)| = c_n u^{1/2-n/2} \left| \int e^{iG} \varphi d\xi \right|,$$

where

$$G(\xi) = \frac{a-1}{2} u\xi^2 + \mathcal{O}(N^{-1/2}).$$

It is then easy to prove that

$$|B_2(u)| \geq c\, u^{1/2-n/2} \geq c, \qquad \varepsilon \leq u \leq 2\varepsilon, \tag{29}$$

if φ is suitably chosen.

We shall prove that $B_1(u)$ and $C(u)$ are small compared to $B_2(u)$. It then follows that the left hand side of (27) is bounded from below and therefore (27) can not hold for $s < 1/4$.

We shall first estimate $C(u)$. We obtain

$$|C(u)| \leq C \int_{N-N^{1/2}}^{N+N^{1/2}} r^{-1} N^{-1/2} dr \leq C \frac{1}{N}$$

and it remains to study $B_1(u)$. Arguing as above with B_2 we obtain

$$B_1(u) = c_n u^{1/2-n/2} \int e^{iF} \varphi d\xi$$

where

$$F(\xi) = -N^{1/2}u\xi + Nu + t(u)(N - N^{1/2}\xi)^a.$$

Integrating by parts we have

$$\int e^{iF}\varphi d\xi = \int e^{iF} i F' \frac{1}{i} \frac{\varphi}{F'} d\xi = i \int e^{iF} \left(\frac{\varphi'}{F'} - \frac{\varphi F''}{(F')^2} \right) d\xi. \tag{30}$$

Now

$$F'(\xi) = -N^{1/2}u - t(u)a(N - N^{1/2}\xi)^{a-1}N^{1/2}$$

and it follows that

$$|F'| \geq cN^{1/2}.$$

Also

$$F''(\xi) = t(u)a(a-1)\,(N - N^{1/2}\xi)^{a-2}N$$

and

$$|F''| \leq CN^{1-a}N^{a-2}N = C.$$

We therefore conclude from (30) that

$$\left| \int e^{iF}\varphi d\xi \right| \leq CN^{-1/2}.$$

Hence

$$|B_1(u)| \leq CN^{-1/2}$$

and we have proved that also B_1 is small compared to B_2.

Thus we have proved that there is no local estimate for S^*f if $s < 1/4$. We shall then assume $1/4 \leq s < n/2$ and prove that $q \leq 2n/(n-2s)$ is a necessary condition for the local estimate (1).

First let $\varphi \in C_0^\infty \mathbf{R}^n$ be radial and non-negative. Assume that $\operatorname{supp} \varphi \subset \{\xi; 1 < |\xi| < 2\}$ and that $\varphi(\xi) = 1$ for $5/4 \leq |\xi| \leq 7/4$. We then choose f so that $\hat{f}(\xi) = \varphi(\xi/N)$, where $N > 1$. It is then easy to see that

$$\| f \|_{H_s} \leq CN^{n/2+s}.$$

One also has

$$S_0 f(x) = \pi_n N^n \hat{\varphi}(Nx)$$

and it follows that

$$S^*f(x) \geq cN^n$$

for $|x| \leq \delta/N$, where δ is a small constant. If (1) holds we obtain

$$\left(\int_{|x| \leq \delta/N} N^{nq} dx \right)^{1/q} \leq CN^{n/2+s}$$

and hence

$$N^{n-n/q} \leq CN^{n/2+s}.$$

Letting $N \to \infty$ we then conclude that

$$n - \frac{n}{q} \leq \frac{n}{2} + s$$

and it follows that $q \leq 2n/(n-2s)$.

To complete the proof of Theorem 3 it now only remains to prove that (1) does not hold in the case $s = n/2$, $q = \infty$, and this can be proved as in the case $n = 1$.

The proof of Theorem 3 is complete.

PROOF OF THEOREM 4. We shall first prove that the global inequality (2) holds for radial functions in the case $q = 2$, $s > a/4$. We set

$$Pg(u) = u^{1/2} \int_0^\infty J_{n/2-1}(ru) e^{it(u)r^a} g(r)(1+r^2)^{-s/2} r^{1/2} dr$$

and have to prove that

$$\left(\int_0^\infty |Pg(u)|^2 du \right)^{1/2} \leq C \left(\int_0^\infty |g(r)|^2 dr \right)^{1/2} \tag{31}$$

if $s > a/4$ (compare the proof of Theorem 3).

Invoking (28) one has

$$\begin{aligned} Pg(u) &= b_1 \int_0^\infty e^{iru} e^{it(u)r^a} g(r)(1+r^2)^{-s/2} dr \\ &\quad + b_2 \int_0^\infty e^{-iru} e^{it(u)r^a} g(r)(1+r^2)^{-s/2} dr + A(u) + B(u) \\ &= b_1 D_1(u) + b_2 D_2(u) + A(u) + B(u), \end{aligned}$$

where

$$|A(u)| \leq C \int_0^{1/u} |g(r)| dr$$

and

$$|B(u)| \leq C \int_{1/u}^{\infty} \frac{1}{ru} |g(r)| dr = C \frac{1}{u} \int_{1/u}^{\infty} \frac{1}{r} |g(r)| dr$$

for $u > 0$.

We set

$$M(t) = \frac{1}{t} A\left(\frac{1}{t}\right)$$

and then have

$$M(t) \leq \frac{1}{t} \int_0^t |g| dr \leq C Mg(t),$$

where Mg denotes Hardy-Littlewood's maximal function for g. It follows that

$$\| A \|_2 \leq C \| g \|_2$$

for $s \geq 0$.

We have $|B(u)| \leq CQ(u)$, where

$$Q(u) = \frac{1}{u} \int_{1/u}^{\infty} \frac{1}{r} |g(r)| dr, \qquad u > 0.$$

We set

$$L(t) = \frac{1}{t} Q\left(\frac{1}{t}\right) = \int_t^{\infty} \frac{1}{r} |g(r)| dr$$

and then have

$$\|L\|_2 = \sup_{\substack{\|f\|_2=1 \\ f\geq 0}} \int_0^\infty L(t)f(t)dt$$
$$= \sup \int_0^\infty \left(\int_t^\infty \frac{1}{r}|g(r)|dr \right) f(t)dt$$
$$= \sup \int_0^\infty |g(r)| \left(\frac{1}{r} \int_0^r f(t)dt \right) dr$$
$$\leq \sup \int_0^\infty |g(r)| Mf(r)dr \leq \sup \|g\|_2 \|Mf\|_2 \leq C \|g\|_2 .$$

It follows that

$$\|B\|_2 \leq C \|g\|_2$$

for $s \geq 0$.

To estimate D_1 and D_2 we use the global estimate in P. Sjölin [17], which was mentioned in the proof of Theorem 2. One obtains

$$\|D_i\|_2 \leq C \left(\int_0^\infty |g(r)|^2 (1+r^2)^{-s} (1+r^2)^s dr \right)^{1/2} = C \|g\|_2$$

if $s > a/4$. Thus (31) is proved for $s > a/4$.

For

$$Tf(u) = c_n u^{1-n/2} \int_0^\infty J_{n/2-1}(ru) e^{it(u)r^a} \hat{f}(r) r^{n/2} dr$$

we have proved the estimate

$$\left(\int_0^\infty |Tf(u)|^q u^{n-1} du \right)^{1/q} \leq C \left(\int_0^\infty |\hat{f}(r)|^2 (1+r^2)^s r^{n-1} dr \right)^{1/2} \tag{32}$$

in the two cases $q = 4n/(2n-1)$, $s = 1/4$, and $q = 2$, $s = b/4$, where $b > a$. Interpolating between these results (J. Bergh and J. Löfström [1], pp. 120--121) we obtain (32) with

$$\frac{1}{q} = (1-\theta)\frac{2n-1}{4n} + \theta\frac{1}{2}$$

and

$$s = \frac{1}{4}(1-\theta) + \frac{b}{4}\theta,$$

where $0 < \theta < 1$. Hence $\theta = (4s-1)/(b-1)$ and

$$\frac{1}{q} = \frac{2n-1}{4n} + \theta\frac{1}{n}$$

and it follows that

$$\frac{1}{q} = \frac{2n-1}{4n} + \frac{4s-1}{(b-1)4n} = \frac{4s + b(2n-1) - 2n}{4(b-1)n}.$$

We conclude that the global estimate (2) holds for radial functions if $(2n-1)/4n < 1/q \leq 1/2$ and

$$\frac{1}{q} < \frac{4s + a(2n-1) - 2n}{4(a-1)n}.$$

The sufficiency part of Theorem 4 now follows.

To prove the necessity part we first assume that $1/4 \leq s \leq a/4$ and that (32) holds. We shall prove that then

$$q \geq \frac{4(a-1)n}{4s + a(2n-1) - 2n}. \tag{33}$$

First let $\varphi \in C_0^\infty(\mathrm{R})$ with supp $\varphi \subset (-1, 1)$ and choose f such that

$$\hat{f}(r) = \varphi(-N^{a/2-1}r + N^{a/2})r^{1/2-n/2}.$$

It is then easy to see that $\hat{f}(r)$ vanishes outside the interval $[N - N^{1-a/2}, N + N^{1-a/2}]$ and it follows that

$$\| f \|_{H_s} \leq CN^{s+1/2-a/4}. \tag{34}$$

We choose u so that $(1-\varepsilon)N^{a-1} \leq u \leq N^{a-1}$, where $\varepsilon > 0$ is a small number. We then have

$$Tf(u) = c_n u^{1/2-n/2} \int_0^\infty J_{n/2-1}(ru)\,(ru)^{1/2} e^{it(u)r^a} \hat{f}(r) r^{n/2-1/2} dr$$

and invoking (28) we obtain

$$Tf(u) = c_n u^{1/2-n/2} \int_0^\infty \left(b_1 e^{iru} + b_2 e^{-iru} + \mathcal{O}\left(\frac{1}{ru}\right) \right) e^{it(u)r^a} \varphi(-N^{a/2-1} r + N^{a/2}) dr$$
$$= b_1 B_1(u) + b_2 B_2(u) + C(u),$$

where

$$B_1(u) = c_n u^{1/2-n/2} \int_0^\infty e^{iru} e^{it(u)r^a} \varphi(-N^{a/2-1} r + N^{a/2}) dr,$$

$$B_2(u) = c_n u^{1/2-n/2} \int_0^\infty e^{-iru} e^{it(u)r^a} \varphi(-N^{a/2-1} r + N^{a/2}) dr$$

and

$$C(u) = c_n u^{1/2-n/2} \int_0^\infty \mathcal{O}\left(\frac{1}{ru}\right) \varphi(-N^{a/2-1} r + N^{a/2}) dr.$$

We have

$$B_2(u) = c_n u^{1/2-n/2} \int e^{iru} e^{it(u)|r|^a} \varphi(N^{a/2-1} r + N^{a/2}) dr$$

and it is proved in [17] that if we choose

$$t(u) = \frac{u}{aN^{a-1}}$$

then

$$|B_2(u)| \geq c\, u^{1/2-n/2} N^{1-a/2} \geq c\, N^{(a-1)(1/2-n/2)} N^{1-a/2} = c\, N^{n/2-an/2+1/2} \tag{35}$$

(with a suitable choice of φ).

We shall prove that $B_1(u)$ and $C(u)$ are small compared to $B_2(u)$. We have

$$|C(u)| \leq C u^{1/2-n/2} \int_{N-N^{1-a/2}}^{N+N^{1-a/2}} \frac{1}{rN^{a-1}} dr \leq C u^{1/2-n/2} N^{1-a/2} N^{-a}$$

and therefore $C(u)$ is much smaller than $B_2(u)$.

We shall then study B_1. We have

$$B_1(u) = c_n u^{1/2-n/2} \int e^{-iu\eta} e^{it(u)|\eta|^a} \varphi(N^{a/2-1}\eta + N^{a/2}) d\eta$$

and setting $\xi = N^{a/2-1}\eta + N^{a/2}$ we obtain

$$B_1(u) = c_n u^{1/2-n/2} N^{1-a/2} \int e^{-iu(N^{1-a/2}\xi - N)} e^{it(u)(N-N^{1-a/2}\xi)^a} \varphi(\xi) d\xi$$
$$= c_n u^{1/2-n/2} N^{1-a/2} \int e^{iF} \varphi d\xi e^{iNu},$$

where

$$F(\xi) = -N^{1-a/2} u\xi + t(u)(N - N^{1-a/2}\xi)^a.$$

Integrating by parts we have

$$\int e^{iF} \varphi d\xi = \int e^{iF} iF' \frac{1}{i} \frac{\varphi}{F'} d\xi = i \int e^{iF} \left(\frac{\varphi'}{F'} - \frac{\varphi F''}{(F')^2} \right) d\xi. \tag{36}$$

Now

$$F'(\xi) = -N^{1-a/2} u - t(u) a (N - N^{1-a/2}\xi)^{a-1} N^{1-a/2}$$

and it follows that

$$|F'| \geq c N^{1-a/2} N^{a-1} + c N^{a-1} N^{1-a/2} = c N^{a/2}.$$

Also

$$|F''(\xi)| = t(u) a(a-1) (N - N^{1-a/2}\xi)^{a-2} N^{2-a}$$

and

$$|F''| \leq C.$$

We therefore conclude from (36) that

$$\left| \int e^{iF} \varphi d\xi \right| \leq C N^{-a/2}.$$

Hence

$$|B_1(u)| \leq C u^{1/2-n/2} N^{1-a/2} N^{-a/2}$$

and comparing this with the lower bound (35) for $B_2(u)$ we obtain

$$|Tf(u)| \geq c|B_2(u)| \geq c N^{n/2-an/2+1/2}$$

for $(1-\varepsilon)N^{a-1} \leq u \leq N^{a-1}$. If the estimate (32) holds we therefore get

$$N^{n/2-an/2+1/2} N^{(a-1)(n-1)/q} N^{(a-1)/q} \leq C N^{s+1/2-a/4}$$

and it follows that

$$\frac{n}{2} - \frac{an}{2} + \frac{1}{2} + \frac{(a-1)n}{q} \leq s + \frac{1}{2} - \frac{q}{4}$$

and

$$\frac{(a-1)n}{q} \le s - \frac{a}{4} + \frac{an}{2} - \frac{n}{2} = \frac{4s - a + 2an - 2n}{4}.$$

The inequality (33) now follows.

To complete the proof of Theorem 4 it is now sufficient to observe that the global estimate (2) can not hold for radial functions if $q < 2$. This can be proved as in the case $n = 1$ in Theorem 2.

PROOF OF THEOREM 5. We choose $\varphi \in C_0^\infty(\mathbb{R})$ so that $\int \varphi(\xi)d\xi \neq 0$ and $\operatorname{supp} \varphi \subset (-1, 1)$. We shall first carry out the proof for $n = 2$. We choose f_N, $N > 1$, so that

$$\hat{f}_N(\xi) = N^{-1/2}\varphi(N^{-1/2}\xi_1 + N^{1/2})N^{-1/2}\varphi(N^{-1/2}\xi_2 + N^{1/2}).$$

then

$$\| f_N \|_{H_s}^2 = \int_{\substack{-N-N^{1/2} \le \xi_i \\ \le -N+N^{1/2}}} (1 + |\xi|^2)^s |\hat{f}_N(\xi)|^2 d\xi \le C N N^{2s} N^{-2} = C N^{2s-1}$$

and

$$\| f_N \|_{H_s} \le C N^{s-1/2}. \tag{37}$$

We also have

$$\begin{aligned} S_t f_N(x) &= \pi_2 \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it|\xi|^2} N^{-1} \varphi(N^{-1/2}\xi_1 + N^{1/2})\varphi(N^{-1/2}\xi_2 + N^{1/2}) d\xi \\ &= \pi_2 \int_{\mathbb{R}} e^{ix_1\xi_1} e^{it\xi_1^2} N^{-1/2}\varphi(N^{-1/2}\xi_1 + N^{1/2}) d\xi \\ &\quad \cdot \int_{\mathbb{R}} e^{ix_2\xi_2} e^{it\xi_2^2} N^{-1/2}\varphi(N^{-1/2}\xi_2 + N^{1/2}) d\xi_2 \\ &= \pi_2 I_t(x_1) I_t(x_2). \end{aligned}$$

Setting $\eta = N^{-1/2}\xi_1 + N^{1/2}$ we obtain

$$I_t(x_1) = \int e^{ix_1(N^{1/2}\eta - N)} e^{it(N^{1/2}\eta - N)^2} \varphi(\eta) d\eta = \int e^{iF} \varphi d\eta,$$

where

$$F(\eta) = N^{1/2} x_1 \eta - N x_1 + tN\eta^2 + tN^2 - 2tN^{3/2}\eta.$$

It follows that

$$|I_t(x_1)| = \left|\int e^{iG}\varphi d\eta\right|,$$

where

$$G(\eta) = N^{1/2}x_1\eta - 2tN^{3/2}\eta + Nt\eta^2.$$

We now choose $t = t(x_1) = x_1/2N$ and then obtain

$$G(\eta) = \frac{1}{2}x_1\eta^2.$$

Hence

$$|I_t(x_1)| = \left|\int e^{ix_1\eta^2/2}\varphi(\eta)d\eta\right|$$

and we conclude that $|I_t(x_1)| \geq c > 0$ for $x_1 \in I$, where I is a small interval around the origin.

With $t = t(x_1) = x_1/2N$ we also obtain

$$|I_t(x_2)| = \left|\int e^{iH}\varphi d\eta\right|,$$

where

$$H(\eta) = N^{1/2}x_2\eta - 2t\,N^{3/2}\eta + Nt\eta^2 = N^{1/2}x_2\eta - N^{1/2}x_1\eta + \frac{1}{2}x_1\eta^2$$
$$= N^{1/2}(x_2 - x_1)\eta + \frac{1}{2}x_1\eta^2.$$

It follows that $|I_t(x_2)| \geq c > 0$ if $x_1 \in I$ and $|x_2 - x_1| \leq \delta N^{-1/2}$, where δ is a small positive number. Hence $|S_tf_N(x)| \geq c$ for $x_1 \in I$ and $|x_2 - x_1| \leq \delta N^{-1/2}$. Thus $|S_tf_N(x)|$ is bounded from below on a set of measure $\geq cN^{-1/2}$. If $B = \{x \in \mathrm{R}^2; |x| \leq 1\}$ we then obtain

$$\left(\int_B |S^*f_N|^q dx\right)^{1/q} \geq c\,N^{-1/2q}.$$

Assuming that the local estimate (1) holds and using (37) we then obtain

$$N^{-1/2q} \leq C\,N^{s-1/2}$$

and

$$1 \leq C\,N^{s+1/(2q)-1/2}.$$

We conclude that

$$s + \frac{1}{2q} \geq \frac{1}{2}$$

in the case $n = 2$.

In the case $n \geq 3$ we set

$$\hat{f}_N(\xi) = N^{-1/2}\varphi(N^{-1/2}\xi_1 + N^{1/2})N^{-1/2}\varphi(N^{-1/2}\xi_2 + N^{1/2})\dots \cdot N^{-1/2}\varphi(N^{-1/2}\xi_n + N^{1/2}).$$

Then

$$\| f_N \|_{H_s} \leq C(N^{n/2}N^{2s}N^{-n})^{1/2} = C\,N^{s-n/4}$$

and

$$S_t f_N(x) = \pi_n I_t(x_1)I_t(x_2)\dots I_t(x_n).$$

We choose $t = t(x_1) = x_1/2N$ as above and then $|I_t(x_1)| \geq c$ for $x_1 \in I$. Also $|I_t(x_i)| \geq c$ for $|x_i - x_1| \leq \delta N^{-1/2}, i = 2, 3, \dots, n$, and hence $|S_t f_N(x)|$ is bounded from below on a set of measure $\geq N^{-(n-1)/2}$. It follows that

$$\left(\int_B |S^* f|^q dx\right)^{1/q} \geq c\,N^{-(n-1)/2q}$$

(where we have chosen B as above) and if (1) holds then we obtain

$$N^{-(n-1)/2q} \leq C\,N^{s-n/4}$$

and

$$s + \frac{n-1}{2q} \geq \frac{n}{4}.$$

The proof of Theorem 5 is complete.

REFERENCES

1. J. Bergh and J. Löfström, Interpolation Spaces, Grundlehren Math. Wiss. 223. 1976.
2. J. Bourgain, *A remark on Schrödinger operators,* Israel J. Math. 77 (1992), 1–16.
3. A. Carbery, *Radial fourier multipliers and associated maximal functions.Recent Progress in Fourier Analysis,* North-Holland Mathematics Studies 111, 49–56.
4. L. Carleson, *Some analytical problems related to statistical mechanics,* Euclidean Harmonic Analysis, Lecture Notes in Math. 779 (1979), 5–45.
5. M. Cowling, *Pointwise behaviour of solutions to Schrödinger equations,* Harmonic Analysis, Lecture Notes in Math. 992 (1983), 83–90.

6. B.E.J. Dahlberg and C.E. Kenig, *A note on the almost everywhere behaviour of solutions to the Schrödinger equation,* Harmonic Analysis, Lecture Notes in Math. 908 (1982), 205–209.
7. C.E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations,* Indiana Univ. Math. J. 40 (1991), 33–69.
8. C.E. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial value problem for the Korteweg-deVries equation,* J. Amer. Math. Soc 4 (1991), 323–347.
9. C.E. Kenig and A. Ruiz, *A strong type* $(2,2)$ *estimate for a maximal operator associated to the Schrödinger equation,* Trans. Amer. Math. Soc. 280 (1983), 239–246.
10. A. Miyachi, *On some singular Fourier multipliers,* J. Fac. Sci. Univ. Tokyo, Sect. IA .28, (1981), 267–315.
11. B. Muckenhoupt, *Weighted norm inequalities for the Fourier transform,* Trans. Amer. Math. Soc 276 (1983), 729–742.
12. E. Prestini, *Radial functions and regularity of solutions to the Schrödinger equation,* Monatsh. Math. 109 (1990), 135–143.
13. P. Sjögren and P. Sjölin, *Convergence properties for the time-dependent Schrödinger equation,* Ann. Acad. Sci. Fenn. Ser. A.I. Math. 14 (1989), 13–25.
14. P. Sjölin, *Convolution with oscillating kernels,* Indiana Univ. Math. J. 30 (1981), 47–55.
15. P. Sjölin, *Regularity of solutions to the Schrödinger equation,* Duke Math. J. 55 (1987), 699–715.
16. P. Sjölin, *Radial functions and maximal estimates for solutions to the Schrödinger equation,* J. Austral. Math. Soc., (Series A) 59 (1995), 134–142.
17. P. Sjölin, *Global maximal estimates for solutions to the Schrödinger equation,* Studia Math. 110 (1994), 105–114.
18. E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton Univ. Press 1971.
19. L. Vega, *El multiplicador de Schrödinger: la function maximal y los operadores de restriccion,* Ph.D. Thesis, Universidad Autónoma de Madrid, 1988.
20. L. Vega, *Schrödinger equations: pointwise convergence to the initial data,* Proc. Amer. Math. Soc. 102 (1988), 874–878.

DEPARTMENT OF MATHEMATICS
ROYAL INSTITUTE OF TECHNOLOGY
S-100 44 STOCKHOLM
SWEDEN