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REGULARITY AT INFINITY FOR A MIXED PROBLEM FOR DEGENERATE ELLIPTIC OPERATORS IN A HALF-CYLINDER

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Abstract.

We consider a mixed boundary value problem for the degenerate elliptic equation

$$\operatorname{div}(\mathscr{A}(x)\nabla u(x)) = f(x)$$

in an infinite half-cylinder G_0. The matrix $\mathscr{A}$ satisfies a one-weighted boundedness and ellipticity condition with a weight satisfying a modified Muckenhoupt condition. The right-hand side f is assumed to have a compact support. On a subset F of $\overline{G}_0$, with infinity as a limit point, the zero Dirichlet data are prescribed, while on $\partial G_0 \setminus F$ we consider the Neumann condition. We obtain a necessary and sufficient condition for the Wiener regularity at infinity, generalizing the criterion obtained in Kerimov–Maz'ya–Novruzov [6] for the Laplace operator.

1. Introduction.

Let $G_0 = \omega \times (0,\infty)$ be an infinite half-cylinder in R^n, $n \geq 2$, where $\omega \subset \mathsf{R}^{n-1}$ is a bi-Lipschitzian image of a ball in R^{n-1}. Let F be a closed unbounded subset of $\overline{G}_0$ such that $G_0 \setminus F$ is connected, and assume that F contains the base $\omega \times \{0\}$ of G_0. We shall use the notation $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathsf{R}^{n-1}$ and $x_n \in \mathsf{R}$.

In $G_0 \setminus F$, consider the following differential equation in divergence form

$$\operatorname{div}(\mathscr{A}(x)\nabla u(x)) = \operatorname{div} \mathbf{f}(x) - f_0(x), \quad x \in G_0 \setminus F, \tag{1}$$

with the boundary conditions $u = 0$ on F and $Nu = \langle \mathbf{f}, \nu \rangle$ on $\partial G_0 \setminus F$, where ν is the outer normal of G_0 and $Nu(x) = \langle \mathscr{A}(x)\nabla u(x), \nu(x) \rangle$ is the conormal derivative of u at x. The brackets $\langle \cdot, \cdot \rangle$ denote the scalar product in R^n.

Remark. The problem (1) is sometimes called the Zaremba problem, named after S. Zaremba who studied the mixed boundary value problem for the Laplace operator in [11].

The matrix $\mathscr{A}(x) = (a_{ij}(x))_{i,j=1}^n$ is symmetric with real-valued measurable

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entries a_{ij}, satisfying for a.e. $x \in G_0$ and all $q \in \mathsf{R}^n$ the weighted boundedness and ellipticity condition

$$\alpha_1 w(x)|q|^2 \le \langle \mathscr{A}(x)q, q\rangle \le \alpha_2 w(x)|q|^2.$$

The weight w is a non-negative measurable function on G_0 satisfying the following condition. There exist positive constants r_w and C_w such that

$$w(B \cap G_0) w^{-1}(B \cap G_0) \le C_w r^{2n} \tag{2}$$

holds for all balls B with center in $\overline{G}_0$ and radius $r \le r_w$. Here and in what follows, $w(E)$ stands for the integral $\int_E w(x)\,dx$.

The generality of the coefficient matrix $\mathscr{A}$ requires that the equation (2) be understood in the weak sense, i.e. through the integral identity

$$\int_{G_0} \langle \mathscr{A}(x)\nabla u(x), \nabla v(x)\rangle\,dx = \int_{G_0} \left(\sum_{i=1}^{n} f_i(x)\frac{\partial v(x)}{\partial x_i} + f_0(x)v(x)\right)dx,$$

where the test functions v belong to $L_0^{1,2}(G_0, F, w)$ and the solution u belongs to $L_0^{1,2}(G_0, F, w)$ or to $L^{1,2}(G_0, F, w)$. The spaces $L_0^{1,2}(G_0, F, w)$ and $L^{1,2}(G_0, F, w)$ will be defined later, see Definition 2.3.

The notion of regularity at infinity will be made precise later, but roughly speaking it means that for any right-hand side with compact support, the weak solution of (1) tends to zero, as $x_n \to \infty$.

In the special case when the matrix $\mathscr{A}$ is the unit matrix, i.e. for the equation $\Delta u = \operatorname{div}\mathbf{f} - f_0$, a necessary and sufficient condition for the regularity at infinity was obtained by T. M. Kerimov, V. G. Maz'ya and A. A. Novruzov in [6]. Their result is as follows (cap denotes the Newtonian capacity in R^n).

THEOREM 1.1. *Infinity is regular for the Zaremba problem for the equation* $\Delta u = \operatorname{div}\mathbf{f} - f_0$ *if and only if*

$$\sum_{j=1}^{\infty} j\,\operatorname{cap}(\{x \in F : j \le x_n < j+1\}) = \infty.$$

Unfortunately, some of the methods used in the proof cannot be directly applied to the general operator $\operatorname{div}(\mathscr{A}(x)\nabla u(x))$. However, the connection between the Zaremba problem on $G_0 \setminus F$ and the Dirichlet problem on a bounded domain, together with the Wiener test for degenerate elliptic equations (see Fabes–Jerison–Kenig [2]), makes it possible to obtain a criterion of regularity at infinity for the Zaremba problem for the operator $\operatorname{div}(\mathscr{A}(x)\nabla u(x))$.

We introduce a change of variables which maps the infinite half-cylinder

G_0 onto a bounded domain and preserves the weighted ellipticity of the operator $\operatorname{div}(\mathscr{A}(x)\nabla u(x))$. In order to obtain a Dirichlet problem, the part of the boundary, on which the Neumann data are prescribed, is eliminated.

The equivalence of the regularity for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$ and the regularity for the Dirichlet problem is proved. In this way, the Wiener test for degenerate elliptic equations, see Fabes–Jerison–Kenig [2], provides us with a necessary and sufficient condition for the regularity at infinity for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$, see Theorem 4.5. The regularity in $L^{1,2}(G_0, F, w)$ is investigated separately, see Theorems 4.8 and 6.8.

The reason for considering two different spaces is that infinity may or may not be regarded as an element of $\overline{F}$. In some sense, $u \in L_0^{1,2}(G_0, F, w)$ means that zero Dirichlet data are considered at infinity, whereas for $u \in L^{1,2}(G_0, F, w)$ there is no such condition. We show that in some cases the spaces $L_0^{1,2}(G_0, F, w)$ and $L^{1,2}(G_0, F, w)$ coincide, see Corollaries 4.7 and 6.9. In fact, they differ if and only if $w^{-1}(G_0) < \infty$ and $\operatorname{cap}_{\mathscr{K}}(F) < \infty$, where the capacity $\operatorname{cap}_{\mathscr{K}}$ is generated by the kernel

$$\mathscr{K}(x, y) = \int_{|x-y|}^{R} \frac{r^2}{w(B(x, r) \cap G_0)} \frac{dr}{r}, \quad x, y \in \overline{G}_0, \ |x - y| \le R,$$

with some fixed $R > 0$.

The presence of the weight on G_0 leads to some peculiar cases. If the weight w grows sufficiently fast at infinity, i.e. if $w^{-1}(G_0) < \infty$, then infinity is always regular for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$. This corresponds to the fact that in the weighted potential theory, single points may have positive capacity. In the opposite case and for the regularity in $L^{1,2}(G_0, F, w)$, the regularity criterion reads

$$(3) \qquad \sum_{j=1}^{\infty} w^{-1}(\{x \in G_0 : x_n < j\}) \operatorname{cap}_{\mathscr{K}}(\{x \in F : j \le x_n < j+1\}) = \infty,$$

see Theorem 6.8.

The following table summarizes the possible cases.

$w^{-1}(G_0) < \infty$	(3)	$L_0^{1,2}(G_0, F, w) = L^{1,2}(G_0, F, w)$	Regularity in $L_0^{1,2}(G_0, F, w)$	Regularity in $L^{1,2}(G_0, F, w)$
True	True	True	True	
True	False	False	True	False
False	True	True	True	
False	False	True	False	

In the last two sections we study and compare two different capacities on G_0 associated with the weight w, cap_Γ and $\mathrm{cap}_\mathscr{K}$, see Theorems 5.3, 5.7 and 6.3–6.7.

We also obtain a two-sided estimate for the Neumann function for the operator $-\operatorname{div}(\mathscr{A}(x)\nabla u(x))$ in G_0, see Theorem 5.7.

REMARK. It should be pointed out that in Kerimov–Maz'ya–Novruzov [6] the assumptions on the half-cylinder G_0 are more general then here, viz. they do not assume that ω is homeomorphic to a ball. Thus, in this direction, our result does not cover theirs.

The lemma in Chapter 1.1.8 in Maz'ya [9] shows that if ω is a bounded domain, star-shaped with respect to every point of a ball with center at the origin, then ω is a bi-Lipschitzian image of a ball.

2. Weighted function spaces and weak solutions.

We begin this section by proving some auxiliary results about the weight w.

Let G denote the infinite cylinder $G = \omega \times \mathsf{R}$. Put $G_t = \{x \in G : x_n > t\}$ and $F_t = \{x \in F : x_n \geq t\}$. Let also $B(x, r) = \{y \in \mathsf{R}^n : |y - x| < r\}$ and $B'(x', r) = \{y' \in \mathsf{R}^{n-1} : |y' - x'| < r\}$.

Unless otherwise stated, the letter C will denote a positive constant whose exact value is unimportant and may change even within a line. By $X \simeq Y$ we mean that there exist positive constants C and C' such that $CX \leq Y \leq C'X$.

Recall that w is a weight on G_0 satisfying the condition (2). Note that a restriction to G_0 of a weight belonging to the Muckenhoupt class A_2 satisfies (2). However, the Muckenhoupt class does not allow weights with exponential growth, while the weight $w(x) = e^{ax_n}$, $a \in \mathsf{R}$, satisfies (2). Thus the class of weights considered here is wider than A_2.

Note also that the weight w can be symmetrically extended to $G \setminus G_0$ by $w(x', x_n) = w(x', -x_n)$ and the condition (2) remains valid for all balls B with center in $\overline{G}$ and radius $r \leq r_w$.

LEMMA 2.1. *Let* $t \in \mathsf{R}$. *Then*

(a) $$w(G_{t-1} \setminus G_t) \simeq w(G_t \setminus G_{t+1}),$$

(b) $$w(G_{t-1} \setminus G_t) w^{-1}(G_{t-1} \setminus G_t) \simeq 1.$$

PROOF. Since ω is a bi-Lipschitzian image of a ball, we may assume that $G = (-1, 1)^{n-1} \times \mathsf{R}$. Let $M \geq \sqrt{n}/r_w$ be a fixed integer and put $a = M^{-1}$. Divide $G_{t-1} \setminus G_{t+1}$ by hyperplanes into $2^n M^n$ pairwise disjoint cubes with common sidelength a and edges parallel to the coordinate axes. Let $\mathscr{Q}$ be the collection of all such cubes.

For $Q_1, Q_2 \in \mathscr{Q}$, we can find a chain of cubes from $\mathscr{Q}$ connecting Q_1 and Q_2

in the sense that each cube has at least one vertex in common with its predecessor in the chain. The chain can be chosen so that its length does not exceed $2nM$. Let Q' and Q'' be two neighbouring cubes in the chain and let B be the smallest ball containing $Q' \cup Q''$. If χ is the characteristic function of Q', then by the Hölder inequality,

$$a^{2n} = \left(\int_{B \cap G} \chi(x) w(x)^{1/2} w(x)^{-1/2} dx \right)^2 \leq w(Q') w^{-1}(B \cap G).$$

The condition (2) now yields

$$a^{2n} w(Q'') \leq w(Q') w(B \cap G) w^{-1}(B \cap G) \leq C_w (a\sqrt{n})^{2n} w(Q'),$$

i.e. $w(Q'') \leq C w(Q')$. Repeated application of this inequality to both w and w^{-1} gives $w(Q_1) \simeq w(Q_2)$ and $w^{-1}(Q_1) \simeq w^{-1}(Q_2)$.

To obtain (a), apply $w(Q_1) \simeq w(Q_2)$ to all cubes $Q_1 \subset G_{t-1} \setminus G_t$ and their translations $Q_2 = \{(x', x_n + 1) : (x', x_n) \in Q_1\} \subset G_t \setminus G_{t+1}$.

As for (b), let $Q_1 \subset G_{t-1} \setminus G_t$ be fixed and $Q_2 \subset G_{t-1} \setminus G_t$ arbitrary ($Q_1, Q_2 \in \mathcal{Q}$). Since $G_{t-1} \setminus G_t$ consists of $2^{n-1} M^n$ cubes Q_2, we get using (2),

$$w(G_{t-1} \setminus G_t) w^{-1}(G_{t-1} \setminus G_t) \simeq w(Q_1) w^{-1}(Q_1) \leq C a^{2n}.$$

Conversely, the Hölder inequality yields

$$1 \simeq \left(\int_{G_{t-1} \setminus G_t} w(x)^{1/2} w(x)^{-1/2} dx \right)^2 \leq w(G_{t-1} \setminus G_t) w^{-1}(G_{t-1} \setminus G_t).$$

The following corollary of Lemma 2.1 shows that weights satisfying the condition (2) cannot grow arbitrarily fast at infinity.

COROLLARY 2.2. *There exist positive constants κ and C such that for all $t \in \mathsf{R}$,*

(a) $$\int_{G_t} w(x) e^{-2\kappa x_n} dx \int_{G_t} w(x)^{-1} e^{(2-2n)\kappa x_n} dx \leq C e^{-2n\kappa t},$$

(b) $$\int_{G_t} w(x) e^{-2\kappa x_n} dx \simeq e^{-2\kappa t} (w^{-1}(G_{t-1} \setminus G_t))^{-1}.$$

PROOF. Apply Lemma 2.1, part (a), to the partition $\{G_{t+k} \setminus G_{t+k+1}\}_{k=0}^{\infty}$ of G_t. Then for $\kappa > 0$,

$$\int_{G_t} w(x)e^{-2\kappa x_n}dx \le Ce^{-2\kappa t}w(G_{t-1}\setminus G_t)\sum_{k=0}^{\infty} e^{k(\log C-2\kappa)} \quad \text{and}$$

$$\int_{G_t} w(x)^{-1}e^{(2-2n)\kappa x_n}dx \le Ce^{(2-2n)\kappa t}w^{-1}(G_{t-1}\setminus G_t)\sum_{k=0}^{\infty} e^{k(\log C-(2n-2)\kappa)}.$$

Choose κ so that $\log C < 2\kappa$, then the last two series converge and (a) follows. As for (b), we have by above

$$C'e^{-2\kappa t}w(G_t\setminus G_{t+1}) \le \int_{G_t} w(x)e^{-2\kappa x_n}dx \le Ce^{-2\kappa t}w(G_{t-1}\setminus G_t).$$

Lemma 2.1 finishes the proof.

REMARK. Note that C and the constants in "$\simeq$" depend on the choice of κ. Let therefore, κ be fixed from now on.

We can now define suitable function spaces on G_0 and give a precise meaning to the weak definition of a solution of the equation (1). For an open set Ω, let $\mathscr{C}^\infty(\Omega)$ denote the space of infinitely many times differentiable functions on Ω. Let also

$$\mathscr{C}_0^\infty(\Omega) = \{v \in \mathscr{C}^\infty(\Omega) : \operatorname{spt} v \text{ compact}, \ \operatorname{spt} v \subset \Omega\},$$

$$\mathscr{C}^\infty(\overline{G}_0) = \{v : v \in \mathscr{C}^\infty(\Omega) \text{ for some open } \Omega \supset \overline{G}_0\},$$

$$\mathscr{C}_0^\infty(\overline{G}_0\setminus F) = \{v : v \in \mathscr{C}_0^\infty(\Omega\setminus F) \text{ for some open } \Omega \supset \overline{G}_0\}.$$

DEFINITION 2.3. Let κ be the constant from Corollary 2.2. Let $L_0^{1,2}(G_0, F, w)$ be the closure of $\mathscr{C}_0^\infty(\overline{G}_0\setminus F)$ in the norm

$$\|v\|^2_{L^{1,2}(G_0,F,w)} = \int_{G_0} (|v(x)|^2e^{-2\kappa x_n} + |\nabla v(x)|^2)w(x)\,dx.$$

Similarly, let $L^{1,2}(G_0, F, w)$ be the closure of

$$\{v \in \mathscr{C}^\infty(\overline{G}_0) : \|v\|_{L^{1,2}(G_0,F,w)} < \infty \text{ and } v = 0 \text{ in some neighbourhood of } F\}$$

in the $L^{1,2}(G_0, F, w)$ norm.

REMARK. The definition should be understood in the sense that there exist smooth v_k and a function $u : G_0 \to \mathrm{R}^n$ such that $v_k \to v$ in $L^2(G_0, e^{-2\kappa x_n}w)$ and $\nabla v_k \to u$ in $L^2(G_0, w)$, as $k \to \infty$. However, since the weight w^{-1} is locally integrable, it turns out that u is the distributional gradient ∇v of v, see e.g. Section 1.9 in Heinonen–Kilpeläinen–Martio [5].

DEFINITION 2.4. Let Φ be a bounded linear functional on $L_0^{1,2}(G_0, F, w)$. The function $u \in L^{1,2}(G_0, F, w)$ is called a *weak solution* of the Zaremba problem if

$$(4) \qquad \int_{G_0} \langle \mathscr{A}(x)\nabla u(x), \nabla v(x)\rangle\, dx = \Phi(v) \quad \text{for all } v \in L_0^{1,2}(G_0, F, w).$$

REMARKS. 1. By the usual Banach space isometric embedding argument, see e.g. Section 5.9 in Kufner–John–Fučík [8], every bounded linear functional Φ on $L_0^{1,2}(G_0, F, w)$ can be represented as

$$\Phi(v) = \int_{G_0} \left(\sum_{j=1}^{n} f_j(x) \frac{\partial v(x)}{\partial x_j} + f_0(x)v(x) \right) dx,$$

where the functions $e^{\kappa x_n} f_0/w$ and f_j/w, $j = 1, \dots, n$, belong to $L^2(G_0, w)$. For smooth data, this representation leads to the divergence type equation (1).

2. Theorem 3.6 and the weighted Sobolev embedding theorem, Theorem 1.3 in Fabes–Kenig–Serapioni [3], show that the norm $\|\cdot\|_{L^{1,2}(G_0,F,w)}$ is equivalent to

$$\left(\int_{G_0} |\nabla v(x)|^2 w(x)\, dx \right)^{1/2}.$$

The Lax–Milgram theorem then ensures that for every bounded linear functional on $L_0^{1,2}(G_0, F, w)$ there exists a unique weak solution $u \in L_0^{1,2}(G_0, F, w)$ of the problem (4). On the other hand, there is no uniqueness in $L^{1,2}(G_0, F, w)$, unless $L^{1,2}(G_0, F, w) = L_0^{1,2}(G_0, F, w)$.

DEFINITION 2.5. Infinity is *regular* for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$ (in $L^{1,2}(G_0, F, w)$) if for all bounded linear functionals Φ on $L_0^{1,2}(G_0, F, w)$ with compact support, the weak solution $u \in L_0^{1,2}(G_0, F, w)$ (all weak solutions $u \in L^{1,2}(G_0, F, w)$) of (4) tends to zero as $x_n \to \infty$, $x = (x', x_n) \in G_0 \setminus F$.

3. Change of variables.

Since ω is a bi-Lipschitzian image of a ball, and bi-Lipschitzian mappings preserve the weighted ellipticity of the operator $\operatorname{div}(\mathscr{A}(y)\nabla u(y))$ (with a new weight, also satisfying the condition (2)), we may in the following assume that ω is the unit ball in R^{n-1}. Let κ be the constant from Corollary 2.2 and introduce the following change of variables.

DEFINITION 3.1. Let $T : \mathsf{R}^n \to \mathsf{R}^n \setminus \{(\xi', \xi_n) \in \mathsf{R}^n : \xi' = 0,\ \xi_n \le 0\}$ be defined by $T(x', x_n) = (\xi', \xi_n)$, where

$$\xi' = \frac{2e^{-\kappa x_n} x'}{1 + |x'|^2} \quad \text{and} \quad \xi_n = \frac{e^{-\kappa x_n}(1 - |x'|^2)}{1 + |x'|^2}.$$

LEMMA 3.2. *There exist positive constants C_1 and C_2, such that if $x, y \in \overline{G}$ and $x_n \le y_n$, then*

$$C_1 e^{-\kappa y_n}|x-y| \le |Tx - Ty| \le C_2 e^{-\kappa x_n}|x-y|.$$

PROOF. By direct calculation using the definition of T and the inversion formulas $e^{-\kappa x_n} = |\xi|$ and $x' = (|\xi| + \xi_n)^{-1}\xi'$, where $\xi = Tx$.

Let $\mathrm{d}T(x)$ and $J_T(x)$ denote the differential and the Jacobian of T at x, respectively. By A^* we denote the transpose of a matrix A.

COROLLARY 3.3. *Let $x \in \overline{G}$. Then*

(a) $$|\mathrm{d}T(x)^* q| \simeq e^{-\kappa x_n}|q| \quad \text{for all } q \in \mathsf{R}^n,$$

(b) $$|J_T(x)| \simeq e^{-n\kappa x_n}.$$

The mapping T is a $\mathscr{C}^\infty$-diffeomorphism between the circular half-cylinder G_0 and the unit half-ball $T(G_0) = \{\xi \in B(0,1) : \xi_n > 0\}$. In fact, it can be verified that derivatives of any order of ξ with respect to x can be written as polynomials in ξ_j and $|\xi|$. Similarly, derivatives of x with respect to ξ are polynomials in $e^{-\kappa x_n}$, x_j and $1 + |x'|^2$.

The diffeomorphism T maps the base $B'(0,1) \times \{0\}$ of G_0 onto the half-sphere $\{\xi \in \partial B(0,1) : \xi_n > 0\}$. The lateral surface $\partial B'(0,1) \times (0,\infty)$ of G_0 corresponds to the punctured $(n-1)$-dimensional unit ball $(B'(0,1) \times \{0\}) \setminus \{0\}$. Since $|T(x', x_n)| = e^{-\kappa x_n} \to 0$, as $x_n \to \infty$, the set $T(F)$ has a limit point at the origin. In order to eliminate the part of the boundary, where the Neumann data are prescribed, reflect the domain $T(G_0 \setminus F)$ in the hyperplane $\{\xi \in \mathsf{R}^n : \xi_n = 0\}$ and add $T(\partial G_0 \setminus F)$. The result of this fusion is

$$D = B(0,1) \setminus (T(F) \cup PT(F) \cup \{0\}),$$

where $P : \mathsf{R}^n \to \mathsf{R}^n$ is the reflection $P(\xi', \xi_n) = (\xi', -\xi_n)$. Note that

$$\partial D \subset T(F) \cup PT(F) \cup \{0\}$$

and if the interior of F is empty, then $\partial D = T(F) \cup PT(F) \cup \{0\}$.

The change of variables $x \mapsto \xi = Tx$ shows that in the new coordinates (ξ', ξ_n), the operator $\mathrm{div}_x(\mathscr{A}(x)\nabla_x u(x))$ has the form $\mathrm{div}_\xi(\mathscr{B}(\xi)\nabla_\xi u(\xi))$, where

(5) $$\mathscr{B}(\xi) = \mathscr{B}(Tx) = |J_T(x)|^{-1}\mathrm{d}T(x)\mathscr{A}(x)\mathrm{d}T(x)^* \quad \text{for } \xi \in T(G_0).$$

Extend $\mathscr{B}$ to $B(0,1) \setminus T(G_0)$ by putting $\mathscr{B}(\xi) = P\mathscr{B}(P\xi)P$.

Put for $\xi \in \mathsf{R}^n$, $\xi_n \ge 0$,

$$\widetilde{w}(\xi) = w(T^{-1}\xi)|\xi|^{2-n}$$

and extend $\widetilde{w}$ symmetrically to $\xi_n < 0$ by $\widetilde{w}(\xi) = \widetilde{w}(P\xi)$.

The definition of $\mathscr{B}$, formula (5), together with the weighted ellipticity of $\mathscr{A}$ and Corollary 3.3, now yields that $\mathscr{B}$ satisfies the weighted ellipticity condition

$$\langle \mathscr{B}(\xi)q, q\rangle \simeq \widetilde{w}(\xi)|q|^2 \quad \text{for a.e. } \xi \in B(0,1) \text{ and all } q \in \mathsf{R}^n.$$

THEOREM 3.4 *The weight $\widetilde{w}$ belongs to the Muckenhoupt class A_2, i.e. there exists a constant C such that for all balls $B = B(\xi, \rho) \subset \mathsf{R}^n$,*

$$\widetilde{w}(B)\widetilde{w}^{-1}(B) \le C\rho^{2n}.$$

PROOF. Due to the symmetry of $\widetilde{w}$, we may assume that the centre ξ of B satisfies $\xi_n \ge 0$, i.e. $\xi = Tx$ for some $x \in \overline{G}$. Moreover $\widetilde{w}(B) \simeq \widetilde{w}(B^+)$ and $\widetilde{w}^{-1}(B) \simeq \widetilde{w}^{-1}(B^+)$, where $B^+ = \{\xi \in B : \xi_n \ge 0\}$. Hence by Corollary 3.3,

$$\widetilde{w}(B)\widetilde{w}^{-1}(B) \simeq \int_{T^{-1}(B^+)} w(y)e^{-2\kappa y_n}\,dy \int_{T^{-1}(B^+)} w(y)^{-1}e^{(2-2n)\kappa y_n}\,dy. \tag{6}$$

We shall distinguish two cases.

1. Assume that $\rho \le \delta|\xi|$, where $0 < \delta < 1$ will be fixed later. Lemma 3.2 then yields $T^{-1}(B^+) \subset B(x,r) \cap G$, where $r = (C_1(1-\delta)|\xi|)^{-1}\rho \le (C_1(1-\delta))^{-1}\delta$. Fix δ sufficiently small, so that $r \le r_w$. The estimate (6) and the assumption (2) then yield

$$\widetilde{w}(B)\widetilde{w}^{-1}(B) \le Ce^{-2n\kappa x_n} w(B(x,r) \cap G)w^{-1}(B(x,r) \cap G) \le C\rho^{2n}.$$

2. Let $\rho \ge \delta|\xi|$. Then $T^{-1}(B^+) \subset G_t$ with $t = -\kappa^{-1}\log((1+\delta^{-1})\rho)$. Hence (6) and Corollary 2.2, part (a), yield

$$\widetilde{w}(B)\widetilde{w}^{-1}(B) \le Ce^{-2n\kappa t} \simeq \rho^{2n}.$$

It is well-known that A_2-weights have the doubling property

$$\widetilde{w}(B(\xi, 2\rho)) \le C\widetilde{w}(B(\xi, \rho)),$$

where C is independent of ξ and ρ. For more about A_2-weights see e.g. Chapter IV in Garčía-Cuerva–Rubio de Francia [4] or Section 15.2 in Heinonen–Kilpeläinen–Martio [5].

As a consequence of the doubling property of $\widetilde{w}$, we obtain the local doubling property for w. For fixed $R > 0$, there exists C such that

$$w(B(x,2r) \cap G) \le Cw(B(x,r) \cap G) \quad \text{for all } x \in \overline{G} \text{ and } r \le R.$$

In contrast to the doubling property of $\widetilde{w}$, the constant C depends on R and thus, for w, the doubling property need not hold uniformly for all r.

Weights belonging to the Muckenhoupt class are admissible for the theory of weighted Sobolev spaces, as studied by E. B. Fabes, D. S. Jerison, C. E.

Kenig and R. P. Serapioni in [2] and [3]. See also Heinonen–Kilpeläinen–Martio [5].

DEFINITION 3.5. Let Ω be a bounded domain in R^n. The *weighted Sobolev space* $H_0^{1,2}(\Omega,\widetilde{w})$ is the closure of $\mathscr{C}_0^\infty(\Omega)$ in the norm

$$\|v\|^2_{H^{1,2}(\Omega,\widetilde{w})} = \int_\Omega (|v(\xi)|^2 + |\nabla v(\xi)|^2)\widetilde{w}(\xi)\,d\xi.$$

Similarly, $H^{1,2}(\Omega,\widetilde{w})$ is the closure of $\{v \in \mathscr{C}^\infty(\Omega) : \|v\|_{H^{1,2}(\Omega,\widetilde{w})} < \infty\}$ in the $H^{1,2}(\Omega,\widetilde{w})$-norm.

THEOREM 3.6. *Put for* $v \in L^{1,2}(G_0,F,w)$ *and* $(\xi',\xi_n) \in B(0;1)$,

$$\tilde{v}(\xi',\xi_n) = \begin{cases} (v\circ T^{-1})(\xi',\xi_n) & \text{if } \xi_n \ge 0,\\ (v\circ T^{-1})(\xi',-\xi_n) & \text{if } \xi_n < 0.\end{cases}$$

Then $\tilde{v} \in H_0^{1,2}(D\cup B(0,\rho),\widetilde{w})$ *for all* $\rho > 0$ *and conversely, if* $\tilde{v}$ *belongs to* $H_0^{1,2}(D\cup B(0,\rho),\widetilde{w})$ *for all* $\rho > 0$, *then the function* $v = \tilde{v}|_{T(G_0)} \circ T$, *where* $\tilde{v}|_{T(G_0)}$ *is the restriction of* $\tilde{v}$ *to* $T(G_0)$, *belongs to* $L^{1,2}(G_0,F,w)$. *If moreover* $v \in L_0^{1,2}(G_0,F,w)$, *then* $\tilde{v} \in H_0^{1,2}(D,\widetilde{w})$ *and conversely.*

PROOF. 1. By Corollary 3.3, $\|v\|_{L^{1,2}(G_0,F,w)} \simeq \|\tilde{v}\|^2_{H^{1,2}(B(0,1)\setminus\{0\},\widetilde{w})}$. We can assume that $v \in \mathscr{C}^\infty(\overline{G}_0)$. By Lemma 3.2, $\tilde{v}$ is locally Lipschitz in $B(0,1)\setminus\{0\}$ and consequently (by e.g. Lemmas 1.11 and 1.15 in Heinonen–Kilpeläinen–Martio [5]), $\tilde{v} \in H^{1,2}(B(0,1)\setminus\{0\},\widetilde{w})$. Since the $(n-1)$-dimensional Hausdorff measure of $\{0\}$ is zero, $\tilde{v}$ belongs to $H^{1,2}(B(0,1),\widetilde{w})$ (by Theorem 2.6 in Kilpeläinen [7]).

If v vanishes in some neighbourhood of F, then $\tilde{v}$ has compact support in $D\cup B(0,\rho)$ for all $\rho > 0$, and hence $\tilde{v} \in H_0^{1,2}(D\cup B(0,\rho),\widetilde{w})$ for all $\rho > 0$ (by e.g. Lemma 1.25(i) in Heinonen–Kilpeläinen–Martio [5]). Similarly, if v belongs to $\mathscr{C}_0^\infty(\overline{G}\setminus F)$, then $\operatorname{spt}\tilde{v} \subset D$ and $\tilde{v} \in H_0^{1,2}(D,\widetilde{w})$.

2. Conversely, if $\tilde{v} \in \mathscr{C}_0^\infty(D)$, then $v = \tilde{v}|_{T(G_0)} \circ T \in \mathscr{C}_0^\infty(\overline{G}_0\setminus F)$. Finally, assume that $\tilde{v}$ belongs to $H_0^{1,2}(D\cup B(0,\rho),\widetilde{w})$ for all $\rho > 0$. Fix $\varepsilon > 0$ and choose $\tilde{v}_j \in \mathscr{C}_0^\infty(D\cup B(0,e^{-\kappa j}))$ so that

$$\|\tilde{v}_j - \tilde{v}\|_{H^{1,2}(B(0,1),\widetilde{w})} \le 2^{-j}e^{-\kappa j}\varepsilon. \tag{7}$$

Put $v_j = \tilde{v}_j|_{T(G_0)} \circ T$, then $v_j \in \mathscr{C}^\infty(\overline{G}_0)$ and v_j vanishes in some neighbourhood of $F\setminus G_j$. Let $\varphi_j \in \mathscr{C}_0^\infty(I_j)$, $j \in \mathbf{Z}$, be a partition of unity subordinate to the covering $I_j = (j-2,j)$ of R, such that $|\varphi_j'(t)| \le C$ for all $t \in \mathsf{R}$ and all j. Put $\bar{v} = \sum_{j=1}^\infty v_j\varphi_j$. Then $\bar{v} \in \mathscr{C}^\infty(\overline{G}_0)$ and $\bar{v}$ vanishes in some neighbourhood of F. If $v = \bar{v}|_{T(G_0)} \circ T$, then

$$\|\bar{v} - v\|_{L^{1,2}(G_0,F,w)} \leq \sum_{j=1}^{\infty} \|(v_j - v)\varphi_j\|_{L^{1,2}(G_0,F,w)}.$$

At the same time, using $|\varphi_j(t)| \leq 1$, $|\varphi_j'(t)| \leq C$ and (7),

$$\|(v_j - v)\varphi_j\|^2_{L^{1,2}(G_0,F,w)} \leq \|v_j - v\|^2_{L^{1,2}(G_0,F,w)} + Ce^{2\kappa j}\int_{G_{j-2}\setminus G_j} |v_j(x) - v(x)|^2 e^{-2\kappa x_n} w(x)\,dx$$

$$\leq Ce^{2\kappa j}\|v_j - v\|^2_{L^{1,2}(G_0,F,w)} \leq C2^{-2j}\varepsilon^2.$$

It follows that $\|\bar{v} - v\|_{L^{1,2}(G_0,F,w)} < C\varepsilon$ and letting $\varepsilon \to 0$ shows that $v \in L^{1,2}(G_0, F, w)$.

DEFINITION 3.7. We say that $\operatorname{div}(\mathscr{B}(\xi)\nabla u(\xi)) = 0$ in the $H^{1,2}(\Omega, \widetilde{w})$ sense if $u \in H^{1,2}(\Omega, \widetilde{w})$ and

$$\int_{\Omega} \langle \mathscr{B}(\xi)\nabla u(\xi), \nabla v(\xi)\rangle\,d\xi = 0 \quad \text{for all } v \in H_0^{1,2}(\Omega, \widetilde{w}).$$

By Theorem 2.3.12 in Fabes–Kenig–Serapioni [3], the solution u is locally Hölder continuous in Ω.

THEOREM 3.8. *Let $u \in L_0^{1,2}(G_0, F, w)$ be a weak solution of the Zaremba problem (4) and suppose that the right-hand side Φ has compact support. Then there exists $\rho > 0$ such that (with the notation as in Theorem 3.6) the function $\tilde{u}$ is a solution of* $\operatorname{div}(\mathscr{B}(\xi)\nabla\tilde{u}(\xi)) = 0$ *in the $H^{1,2}(D \cap B(0,\rho), \widetilde{w})$ sense.*

PROOF. Since the support of Φ is compact, there exists t such that

$$\int_{G_0} \langle \mathscr{A}(x)\nabla u(x), \nabla v(x)\rangle\,dx = 0 \tag{8}$$

for all $v \in L_0^{1,2}(G_0, F, w)$ with $\operatorname{spt} v \subset \overline{G}_t$. Due to the symmetry of $\mathscr{B}$ and $\tilde{u}$, it suffices to show

$$\int_{T(G_0)} \langle \mathscr{B}(\xi)\nabla\tilde{u}(\xi), \nabla v(\xi)\rangle\,d\xi = 0$$

for all $v \in H_0^{1,2}(D \cap B(0,\rho), \widetilde{w})$, $\rho \leq e^{-\kappa t}$. This follows directly from (8) using Theorem 3.6 and the definition of $\mathscr{B}$, formula (5).

4. Regularity and the Wiener test.

If $E \subset \overline{\Omega}$, we say that $v \geq c$ on E in the $H^{1,2}(\Omega, \widetilde{w})$ sense, if v can be approximated in $H^{1,2}(\Omega, \widetilde{w})$ by Lipschitz continuous functions v_k on Ω satisfying $v_k \geq c$ on E.

DEFINITION 4.1. Let $K \subset B(0,1)$ be compact. The $\mathscr{B}$-capacity of K in $B(0,1)$ is

$$\operatorname{cap}_{\mathscr{B}}(K) = \inf \int_{B(0,1)} \langle \mathscr{B}(\xi) \nabla v(\xi), \nabla v(\xi) \rangle \, d\xi,$$

where the infimum is taken over all $v \in H_0^{1,2}(B(0,1), \widetilde{w})$ satisfying $v \geq 1$ on K in the $H^{1,2}(B(0,1), \widetilde{w})$ sense.

By Theorem 1.20 in Fabes–Jerison–Kenig [2], there exists a unique minimizing function, called the capacitary potential of K. Moreover, the capacitary potential satisfies the equation $\operatorname{div}(\mathscr{B}(\xi)\nabla u(\xi)) = 0$ in the $H^{1,2}(B(0,1) \setminus K, \widetilde{w})$ sense. By Theorems 4.5 and 4.7 in Fabes–Jerison–Kenig [2], the capacity $\operatorname{cap}_{\mathscr{B}}$ extends to Borel (even analytic) sets $E \subset B(0,1)$ by

$$\operatorname{cap}_{\mathscr{B}}(E) = \sup\{\operatorname{cap}_{\mathscr{B}}(K) : K \subset E \text{ compact}\}.$$

A property is said to hold quasieverywhere (q.e.) if it holds except for a set of $\mathscr{B}$-capacity zero.

We are now ready to prove the equivalence of the regularity for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$ and for the Dirichlet problem. Put $D_0 = D \cap B(0, \frac{1}{4})$. For a detailed definition of the regularity for the Dirichlet problem see Section 5 in Fabes–Jerison–Kenig [2]. For our purposes, it is important, that by Lemma 5.3 in Fabes–Jerison–Kenig [2], $\eta \in \partial D_0$ is a regular point of D_0 if and only if for all $\rho > 0$, the capacitary potentials u_ρ of the sets $\overline{B(\eta, \rho)} \setminus D_0$ satisfy $u_\rho(\xi) \to 1$, as $\xi \to \eta$, $\xi \in D_0$.

THEOREM 4.2. *Infinity is regular for the Zaremba problem in* $L_0^{1,2}(G_0, F, w)$ *if and only if the origin* 0 *is a regular point of* D_0.

PROOF. 1. Let the origin 0 be a regular point of D_0. We shall show that for every bounded linear functional Φ on $L_0^{1,2}(G_0, F, w)$ with compact support, the weak solution $u \in L_0^{1,2}(G_0, F, w)$ of the Zaremba problem (4) tends to zero, as $x_n \to \infty$, $x \in G_0 \setminus F$. Let ρ and $\tilde{u}$ be as in Theorem 3.8. Then $\tilde{u} \in H_0^{1,2}(D, \widetilde{w})$ and $\operatorname{div}(\mathscr{B}(\xi)\nabla \tilde{u}(\xi)) = 0$ in the $H^{1,2}(D \cap B(0, \rho), \widetilde{w})$ sense. By Theorem 2.4.3 in Fabes–Kenig–Serapioni [3] about L^∞ estimates for solutions of $\operatorname{div}(\mathscr{B}(\xi)\nabla \tilde{u}(\xi)) = 0$, $|\tilde{u}(\xi)| \leq M$ for $\xi \in D_0$, $|\xi| = \frac{1}{2}\rho$.

The strong maximum principle (Corollary 2.3.10 in Fabes–Kenig–Serapioni [3]) yields that the capacitary potential of $\overline{B(0,\frac{1}{4}\rho)} \setminus D_0$ satisfies $u_{\rho/4}(\xi) \le c < 1$ for $|\xi| \ge \frac{1}{2}\rho$. Since $u_{\rho/4} = 1$ on $\overline{B(0,\frac{1}{4}\rho)} \setminus D_0$, it follows that

$$|\tilde{u}| \le \frac{M}{1-c}(1 - u_{\rho/4}) \quad \text{on } \partial\big(D_0 \cap B(0,\tfrac{1}{2}\rho)\big)$$

in the $H^{1,2}\big(D_0 \cap B(0,\frac{1}{2}\rho), \widetilde{w}\big)$ sense. The maximum principle implies

$$|\tilde{u}(\xi)| \le \frac{M}{1-c}(1 - u_{\rho/4}(\xi)) \quad \text{for all } \xi \in D_0 \cap B\big(0,\tfrac{1}{2}\rho\big).$$

Since $u_{\rho/4}(\xi) \to 1$, as $\xi \to 0$, $\xi \in D_0$, it follows that $\tilde{u}(\xi) \to 0$, as $\xi \to 0$, $\xi \in D_0$, i.e. $u(x', x_n) \to 0$, as $x_n \to \infty$, $x \in G_0 \setminus F$. Thus, infinity is regular for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$.

2. Assume that the origin is an irregular point of D_0 and find $\rho > 0$ such that the capacitary potential $u_\rho(\xi)$ of the set $\overline{B(0,\rho)} \setminus D_0$ does not tend to 1, as $\xi \to 0$, $\xi \in D_0$. Let φ be a piecewise linear cut-off function on $\overline{G}_0$ such that $\operatorname{spt}\varphi \subset \overline{G}_t$ and $\varphi = 1$ on $\overline{G}_{t+1}$, where $t = -\kappa^{-1}\log\rho$. Put for $x \in \overline{G}_0$,

$$u(x) = (1 - u_\rho(Tx))\varphi(x).$$

Then $u \in L_0^{1,2}(G_0, F, w)$ and $u(x', x_n)$ does not tend to 0, as $x_n \to \infty$. Since $\operatorname{div}(\mathscr{B}(\xi)\nabla u_\rho(\xi)) = 0$ in the $H^{1,2}(D, \widetilde{w})$ sense and $u_\rho = u_\rho \circ P$, we have

$$\int_{T(G_0)} \langle \mathscr{B}(\xi)\nabla u_\rho(\xi), \nabla v(\xi)\rangle \, d\xi = 0$$

for all $v \in H_0^{1,2}(D, \widetilde{w})$. The definition of u and the Hölder inequality now imply that u is a weak solution of the Zaremba problem $\operatorname{div}(\mathscr{A}(x)\nabla u(x)) = \Phi$ for some bounded linear functional on $L_0^{1,2}(G_0, F, w)$ with compact support. Thus, infinity is not regular for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$.

The following criterion of regularity for degenerate elliptic equations was given in Fabes–Jerison–Kenig [2], Theorem 5.1.

THEOREM 4.3 (Wiener test). *Let* $0 < \delta < 1$. *Then the origin* 0 *is a regular point of* D_0 *if and only if one of the following conditions holds*

(a)
$$\int_0^\delta \frac{\rho^2}{\widetilde{w}(B(0,\rho))}\frac{d\rho}{\rho} < \infty,$$

(b)
$$\int_0^\delta \operatorname{cap}_{\mathscr{B}}\big(\overline{B(0,\rho)} \setminus D_0\big)\frac{\rho^2}{\widetilde{w}(B(0,\rho))}\frac{d\rho}{\rho} = \infty.$$

REMARKS. 1. Theorem 4.3 differs slightly from the formulation in Fabes–Jerison–Kenig [2], but the proof is essentially the same.

2. The conditions (a) and (b) in Theorem 4.3 are mutually exclusive and (a) is equivalent to $\operatorname{cap}_{\mathscr{B}}(\{0\}) > 0$ (see the properties (i)–(iv) in the introduction of Fabes–Jerison–Kenig [2]).

The Wiener test, together with Theorem 4.2, provides us with a necessary and sufficient condition for the regularity at infinity for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$. To obtain a criterion of regularity in terms of F and w, we need to estimate $\operatorname{cap}_{\mathscr{B}}(\overline{B(0,\rho)} \setminus D_0)$ and $\widetilde{w}(B(0,\rho))$.

DEFINITION 4.4. Let $E \subset \overline{G}_0$ be a Borel set. Put

$$\operatorname{cap}_\Gamma(E) = \tfrac{1}{2}\operatorname{cap}_{\mathscr{B}}(T(E) \cup PT(E)).$$

REMARK. It follows from the definition of $\operatorname{cap}_{\mathscr{B}}$ and Theorem 3.6 that for a compact subset of $\overline{G}_0$,

$$\operatorname{cap}_\Gamma(K) = \inf \int_{G_0} \langle \mathscr{A}(x)\nabla v(x), \nabla v(x)\rangle \, dx,$$

where the infimum is taken over all $v \in \mathscr{C}^\infty(\overline{G}_0)$, such that $\|v\|_{L^{1,2}(G_0,F,w)} < \infty$, $v \geq 1$ on K and $v = 0$ on $\omega \times \{0\}$.

THEOREM 4.5. *Infinity is regular for the Zaremba problem in* $L_0^{1,2}(G_0, F, w)$ *if and only if one of the following conditions holds*

(a) $$\int_0^\infty w^{-1}(G_{t-1} \setminus G_t)\, dt < \infty,$$

(b) $$\int_1^\infty w^{-1}(G_{t-1} \setminus G_t) \operatorname{cap}_\Gamma(F_t)\, dt = \infty.$$

PROOF. Corollary 2.2, part (b), shows that

$$\widetilde{w}(B(0,\rho)) \simeq \int_{G_t} w(x) e^{-2\kappa x_n} dx \simeq e^{-2\kappa t}(w^{-1}(G_{t-1} \setminus G_t))^{-1},$$

where $t = -\kappa^{-1}\log\rho$. Thus, with $\delta < 1$ fixed, the condition (a) in Theorem 4.3 is equivalent to the condition (a) in Theorem 4.5.

On the other hand, if (a) in Theorem 4.3 fails, i.e. if $\operatorname{cap}_{\mathscr{B}}(\{0\}) = 0$, then $\operatorname{cap}_{\mathscr{B}}(\overline{B(0,\rho)} \setminus D_0) = 2\operatorname{cap}_\Gamma(F_t)$, which inserted into the condition (b) in Theorem 4.3 gives the condition (b) in Theorem 4.5.

REMARK. Note that by Lemma 2.1, the integrals in Theorem 4.5 can be replaced by infinite sums, i.e. (a) is equivalent to $w^{-1}(G_0) < \infty$.

We now turn our attention to the regularity at infinity for the Zaremba problem in $L^{1,2}(G_0, F, w)$. Put for $\rho > 0$, $\widetilde{F}_\rho = (T(F) \cup PT(F)) \cap \overline{B(0,\rho)}$.

THEOREM 4.6. *Assume that there exists $C > 0$ such that*

$$\operatorname{cap}_{\mathscr{B}}(\widetilde{F}_\rho) \geq C \operatorname{cap}_{\mathscr{B}}(\{0\}) \quad \text{for all } \rho > 0.$$

Then $v \in H_0^{1,2}(D, \widetilde{w})$ if and only if $v \in H_0^{1,2}(D \cup B(0,\rho), \widetilde{w})$ for all $\rho > 0$. In particular, this holds whenever $\operatorname{cap}_{\mathscr{B}}(\{0\}) = 0$.

PROOF. Clearly, $v \in H_0^{1,2}(D, \widetilde{w})$ implies $v \in H_0^{1,2}(D \cup B(0,\rho), \widetilde{w})$ for all $\rho > 0$. Conversely, let $v \in H_0^{1,2}(D \cup B(0,\rho), \widetilde{w})$ for all $\rho > 0$. Multiplying v by a smooth compactly supported cut-off function, equal to 1 in some neighbourhood of the origin, we may assume that $\operatorname{spt} v \subset B(0, \frac{1}{2})$. By Theorem 4.14 in Heinonen–Kilpeläinen–Martio [5], we may assume that v is quasicontinuous, i.e. for every $\varepsilon > 0$ there exists an open set U such that $\operatorname{cap}_{\mathscr{B}}(U) < \varepsilon$ and the restriction of v to $\mathsf{R}^n \setminus U$ is continuous. By Theorem 4.5 in Heinonen–Kilpeläinen–Martio [5], a quasicontinuous function belongs to $H_0^{1,2}(\Omega, \widetilde{w})$ if and only if it vanishes q.e. on $\partial\Omega$. It follows that v vanishes q.e. on $T(F) \cup PT(F)$. We shall distinguish two cases.

1. If $\operatorname{cap}_{\mathscr{B}}(\{0\}) = 0$, then v vanishes q.e. on ∂D, and consequently, v belongs to $H_0^{1,2}(D, \widetilde{w})$ (by Theorem 4.5 in Heinonen–Kilpeläinen–Martio [5]).

2. Assume that $\operatorname{cap}_{\mathscr{B}}(\{0\}) > 0$ and find an open set U such that $\operatorname{cap}_{\mathscr{B}}(U) < \operatorname{cap}_{\mathscr{B}}(\widetilde{F}_\rho)$ for all $\rho > 0$, and the restriction of v to $\mathsf{R}^n \setminus U$ is continuous. Then $0 \notin U$ and there exist $\xi^j \in (T(F) \cup PT(F)) \setminus U$ such that $v(\xi^j) = 0$ and $\xi^j \to 0$, as $j \to \infty$. It follows that $v(0) = 0$ and by Theorem 4.5 in Heinonen–Kilpeläinen–Martio [5], $v \in H_0^{1,2}(D, \widetilde{w})$.

REMARK. Theorems 4.5 and 4.14 in Heinonen–Kilpeläinen–Martio [5] are stated in terms of the so called Sobolev capacity. Nevertheless, by Theorem 2.38 in Heinonen–Kilpeläinen–Martio [5], if $E \subset B(0, \frac{1}{2})$, then the Sobolev capacity of E is comparable to $\operatorname{cap}_{\mathscr{B}}(E)$.

COROLLARY 4.7. *If $w^{-1}(G_0) = \infty$ or $\operatorname{cap}_\Gamma(F_t) \geq C > 0$ for all t, then*

$$L_0^{1,2}(G_0, F, w) = L^{1,2}(G_0, F, w).$$

THEOREM 4.8. *Assume that $w^{-1}(G_0) < \infty$ and that $\operatorname{cap}_\Gamma(F_t) \to 0$, as $t \to \infty$. Then infinity is not regular for the Zaremba problem in $L^{1,2}(G_0, F, w)$.*

PROOF. Consider the capacitary potentials $u_{\rho,\varepsilon}$ of the sets $\widetilde{F}_\rho \setminus B(0,\varepsilon)$. With $\rho > 0$ fixed and $0 < \varepsilon < \rho$, the functions $u_{\rho,\varepsilon}$ form a bounded subset of $H_0^{1,2}(B(0,1), \widetilde{w})$ and as $\varepsilon \to 0$, $u_{\rho,\varepsilon}$ converge pointwise to

$$\tilde{u}_\rho = \sup_{0<\varepsilon<\rho} u_{\rho,\varepsilon}.$$

By Theorem 1.32 in Heinonen–Kilpeläinen–Martio [5], $\tilde{u}_\rho \in H_0^{1,2}(B(0,1),\widetilde{w})$ and both $u_{\rho,\varepsilon} \to \tilde{u}_\rho$ and $\nabla u_{\rho,\varepsilon} \to \nabla \tilde{u}_\rho$ weakly in $L^2(B(0,1),\widetilde{w})$.

Using the weak convergence of $\nabla u_{\rho,\varepsilon}$, we obtain that $\tilde{u}_\rho$ is a solution of $\operatorname{div}(\mathscr{B}(\xi)\nabla u(\xi)) = 0$ in the $H^{1,2}(D,\widetilde{w})$ sense. Indeed, if $v \in H_0^{1,2}(D,\widetilde{w})$, then $\mathscr{B}\widetilde{w}^{-1}\nabla v \in L^2(B(0,1),\widetilde{w})$ and consequently,

$$\int_{B(0,1)} \langle \mathscr{B}(\xi)\nabla u_{\rho,\varepsilon}(\xi), \nabla v(\xi)\rangle \, d\xi \to \int_{B(0,1)} \langle \mathscr{B}(\xi)\nabla \tilde{u}_\rho(\xi), \nabla v(\xi)\rangle \, d\xi, \quad \text{as } \varepsilon \to 0.$$

Consider a quasicontinuous representative of $\tilde{u}_\rho$. If $\tilde{u}_\rho(0) \neq 0$, then by Corollary 4.13 in Heinonen–Kilpeläinen–Martio [5], the function $\tilde{u}_\rho/\tilde{u}_\rho(0)$ is admissible for the Sobolev capacity $C_{2,\widetilde{w}}(\{0\})$, see the remark after Theorem 4.6 here. The weighted Sobolev embedding theorem, Theorem 1.3 in Fabes–Kenig–Serapioni [3], now yields

$$C_{2,\widetilde{w}}(\{0\}) \le \|\tilde{u}_\rho/\tilde{u}_\rho(0)\|^2_{H^{1,2}(B(0,1),\widetilde{w})} \le \frac{C}{|\tilde{u}_\rho(0)|^2} \|\nabla \tilde{u}_\rho\|^2_{L^2(B(0,1),\widetilde{w})}. \tag{9}$$

At the same time, $\|\nabla u_{\rho,\varepsilon}\|^2_{L^2(B(0,1),\widetilde{w})} \simeq \operatorname{cap}_{\mathscr{B}}(\widetilde{F}_\rho \setminus B(0,\varepsilon)) \le \operatorname{cap}_{\mathscr{B}}(\widetilde{F}_\rho)$, and the weak convergence of $\nabla u_{\rho,\varepsilon}$ in $L^2(B(0,1),\widetilde{w})$ yields

$$\|\nabla \tilde{u}_\rho\|^2_{L^2(B(0,1),\widetilde{w})} \le C \operatorname{cap}_{\mathscr{B}}(\widetilde{F}_\rho). \tag{10}$$

The assumption $w^{-1}(G_0) < \infty$ implies $C_{2,\widetilde{w}}(\{0\}) \simeq \operatorname{cap}_{\mathscr{B}}(\{0\}) > 0$ and it follows from (9) and (10) that $|\tilde{u}_\rho(0)|^2 \le C \operatorname{cap}_{\mathscr{B}}(\widetilde{F}_\rho)$, which is trivially true if $\tilde{u}_\rho(0) = 0$.

Since $\operatorname{cap}_{\mathscr{B}}(\widetilde{F}_\rho) = 2\operatorname{cap}_\Gamma(F_t)$, where $t = -\kappa^{-1}\log\rho$, there exists $\rho > 0$ such that $\tilde{u}_\rho(0) \le \frac{1}{2}$. Since $\tilde{u}_\rho$ is quasicontinuous and $\operatorname{cap}_{\mathscr{B}}(\{0\}) > 0$, $\tilde{u}_\rho(\xi)$ cannot tend to 1, as $\xi \to 0$. The construction in the second part of the proof of Theorem 4.2, with u_ρ replaced by $\tilde{u}_\rho$, provides us with a weak solution $u \in L^{1,2}(G_0,F,w)$ of the Zaremba problem $\operatorname{div}(\mathscr{A}(x)\nabla u(x)) = \Phi$ for some bounded linear functional on $L_0^{1,2}(G_0,F,w)$ with bounded support. Since $u(x)$ does not tend to zero, as $x_n \to \infty$, infinity is not regular for the Zaremba problem in $L^{1,2}(G_0,F,w)$.

5. The capacity $\operatorname{cap}_\Gamma$.

In this section, we obtain a characterization of $\operatorname{cap}_\Gamma$ in terms of the Neumann function for the operator $-\operatorname{div}(\mathscr{A}(x)\nabla u(x))$ in G_0. We also give a two-sided estimate for the Neumann function.

Let $g(\cdot,\cdot)$ be the Green function for the operator $-\operatorname{div}(\mathscr{B}(\xi)\nabla u(\xi))$ in

B(0,1), as defined in Section 2 in Fabes–Jerison–Kenig [2], i.e. the unique function satisfying

$$\int_{B(0,1)} g(\xi,\eta)\Psi(\xi)\,d\xi = S\Psi(\eta), \tag{11}$$

for all $\eta \in B(0,1)$ and all Ψ such that $\Psi/\widetilde{w}$ is essentially bounded. Here, $S\Psi$ is the unique solution of $-\operatorname{div}(\mathscr{B}(\xi)\nabla u(\xi)) = \Psi(\xi)$ in the $H^{1,2}(B(0,1),\widetilde{w})$ sense, belonging to $H_0^{1,2}(B(0,1),\widetilde{w})$.

LEMMA 5.1. *Let $\xi,\eta \in B(0,1)$. Then $g(P\xi,P\eta) = g(\xi,\eta)$.*

PROOF. We have

$$\int_{B(0,1)} g(P\xi,P\eta)\Psi(\xi)\,d\xi = \int_{B(0,1)} g(\xi,P\eta)(\Psi\circ P)(\xi)\,d\xi = S(\Psi\circ P)(P\eta).$$

At the same time, $P\mathscr{B}(\xi)P = \mathscr{B}(P\xi)$ implies that $S(\Psi\circ P)\circ P$ is a solution of $-\operatorname{div}(\mathscr{B}(\xi)\nabla u(\xi)) = \Psi(\xi)$ in the $H^{1,2}(B(0,1),\widetilde{w})$ sense, and hence by uniqueness, $S(\Psi\circ P)(P\eta) = S\Psi(\eta)$. Thus, $g(P\,\cdot\,,P\eta)$ satisfies the integral identity (11) for all admissible Ψ. The uniqueness of g and its continuity (Proposition 2.6 in Fabes–Jerison–Kenig [2]), imply $g(P\xi,P\eta) = g(\xi,\eta)$ for all $\xi,\eta \in B(0,1)$.

Theorem 4.10 in Fabes–Jerison–Kenig [2] provides us with the following equivalent definition of $\operatorname{cap}_{\mathscr{B}}$. Let $K \subset B(0,1)$ be compact. Then

$$\operatorname{cap}_{\mathscr{B}}(K) = \sup \nu(K),$$

where the supremum is taken over all positive measures ν such that

$$\int g(\xi,\eta)\,d\nu(\eta) \le 1 \quad \text{for all } \xi \in K. \tag{12}$$

DEFINITION 5.2. Let $x,y \in \overline{G}_0$ and put $\Gamma(x,y) = g(Tx,Ty) + g(Tx,PTy)$.

REMARK. Γ is the Neumann function for the operator $-\operatorname{div}(\mathscr{A}(x)\nabla u(x))$, i.e. the solution (in a weak sense) of the equation

$$-\operatorname{div}_x(\mathscr{A}(x)\nabla_x\Gamma(x,y)) = \delta(x-y), \quad x \in G_0,$$

$\Gamma(\cdot,y) = 0$ on $\omega\times\{0\}$, $N\Gamma(\cdot,y) = 0$ on $\partial\omega\times(0,\infty)$ (δ is the Dirac distribution).

THEOREM 5.3. *Let $E \subset \overline{G}_1$ be a Borel set. Then*

$$\operatorname{cap}_\Gamma(E) = \sup \mu(E),$$

where the supremum is taken over all positive measures μ with compact support in E such that

$$\int \Gamma(x,y)\,d\mu(y) \le 1 \quad \text{for all } x \in E.$$

PROOF. Denote the supremum in the statement of the theorem by S.

1. Fix $\varepsilon > 0$ and choose a compact $K \subset T(E) \cup PT(E)$ such that

$$\operatorname{cap}_{\mathscr{B}}(K) \ge \operatorname{cap}_{\mathscr{B}}(T(E) \cup PT(E)) - 2\varepsilon.$$

By Theorem 4.6 in Fabes–Jerison–Kenig [2], there exists a positive measure ν supported on K such that $\nu(K) = \operatorname{cap}_{\mathscr{B}}(K)$ and the inequality in (12) holds for all $\xi \in B(0,1)$. Put for $A \subset \overline{G}_0$,

$$\mu(A) = \tfrac{1}{2}\nu(T(A) \cup PT(A)).$$

Then μ is a positive measure with compact support in E and

$$\int \Gamma(x,y)\,d\mu(y) = \frac{1}{2}\int g(Tx,\eta)\,d\nu(\eta) + \frac{1}{2}\int g(Tx,P\eta)\,d\nu(\eta) \le 1,$$

since by Lemma 5.1, the last integrand is equal to $g(PTx,\eta)$. It follows that

$$S \ge \mu(E) = \tfrac{1}{2}\nu(K) \ge \tfrac{1}{2}\operatorname{cap}_{\mathscr{B}}(T(E) \cup PT(E)) - \varepsilon = \operatorname{cap}_\Gamma(E) - \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\operatorname{cap}_\Gamma(E) \le S$.

2. Conversely, let μ be a measure admissible for the supremum S. Put for $A \subset B(0,1)$,

$$\nu(A) = \mu(T^{-1}(A \cap T(\overline{G}_0))) + \mu(T^{-1}(P(A) \cap T(\overline{G}_0))).$$

Then ν is a positive measure with compact support in $T(E) \cup PT(E)$ and

$$\int g(\xi,\eta)\,d\nu(\eta) = \int (g(\xi,Ty) + g(\xi,PTy))\,d\mu(y) \le 1,$$

for all $\xi \in T(E) \cup PT(E)$. Hence ν is admissible for $\operatorname{cap}_{\mathscr{B}}(\operatorname{spt}\nu)$ and

$$\mu(E) = \tfrac{1}{2}\nu(\operatorname{spt}\nu) \le \tfrac{1}{2}\operatorname{cap}_{\mathscr{B}}(\operatorname{spt}\nu) \le \tfrac{1}{2}\operatorname{cap}_{\mathscr{B}}(T(E) \cup PT(E)) = \operatorname{cap}_\Gamma(E).$$

Taking supremum over all admissible μ yields $S \le \operatorname{cap}_\Gamma(E)$.

The size of the Green function is estimated in Theorem 3.3 in Fabes–Jerison–Kenig [2],

$$g(\xi,\eta) \simeq \int_{|\xi-\eta|}^{1} \frac{\rho^2}{\widetilde{w}(B(\xi,\rho))}\frac{d\rho}{\rho} \quad \text{for } \xi,\eta \in B(0,\tfrac{1}{4}). \tag{13}$$

We shall get a similar formula for the function Γ. Together with Theorem 5.3, it can be used to calculate $\operatorname{cap}_\Gamma$. Fix $R \ge 2\ \operatorname{diam}(G_0 \setminus G_1)$.

DEFINITION 5.4. Put for $x,y \in \overline{G}$,

$$\mathscr{K}(x,y) = \begin{cases} \displaystyle\int_{|x-y|}^{R} \frac{r^2}{w(B(x,r)\cap G)}\frac{dr}{r} & \text{if } |x-y| \le R, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK. Note that if $|x-y| \le r \le R$, then by the local doubling property of w, $w(B(x,r)\cap G) \simeq w(B(y,r)\cap G)$, and hence $\mathscr{K}(x,y) \simeq \mathscr{K}(y,x)$.

DEFINITION 5.5. Put for $t \ge 0$,

$$\Gamma_0(t) = \int_0^t w^{-1}(G_{\tau-1} \setminus G_\tau)\, d\tau.$$

REMARK. Lemma 2.1, part (a), yields $\Gamma_0(t) \simeq w^{-1}(G_0 \setminus G_t)$ for $t \ge 1$, but for the time being, it is more convenient to work with the integral representation of Γ_0.

LEMMA 5.6. *Let* $0 \le \rho_1 \le \rho_2 \le \frac{1}{2}\rho_3$ *and let* γ *be a positive function defined on* $[\rho_1, \rho_3]$ *such that* $\gamma(\rho') \simeq \gamma(\rho'')$ *holds for all* $\rho_1 \le \rho', \rho'' \le 2\rho_2$. *Then*

$$\int_{\rho_1}^{\rho_3} \gamma(\rho)\, d\rho \simeq \int_{\rho_2}^{\rho_3} \gamma(\rho)\, d\rho.$$

PROOF. Elementary.

THEOREM 5.7. There exists t^* such that if $x, y \in \overline{G}_{t^*}$, then

$$\Gamma(x,y) \simeq \mathscr{K}(x,y) + \Gamma_0(\min(x_n, y_n)).$$

PROOF. If we choose t^* sufficiently large, so that $|Tx| \le \frac{1}{4}$ for all $x \in \overline{G}_{t^*}$, then the approximate formula (13) and $|Tx - PTy| \ge |Tx - Ty|$ yield

$$0 \le g(Tx, PTy) \le Cg(Tx, Ty).$$

Since $\Gamma(x,y) = g(Tx,Ty) + g(Tx, PTy)$, it suffices to estimate $g(Tx,Ty)$. By Proposition 2.8 in Fabes–Jerison–Kenig [2], $g(Tx,Ty) = g(Ty,Tx)$, and since $\mathscr{K}(x,y) \simeq \mathscr{K}(y,x)$, we may assume that $x_n \le y_n$. We have from (13),

$$g(Tx,Ty) \simeq \int_{|Tx-Ty|}^{1} \frac{\rho^2}{\widetilde{w}(B(Tx,\rho))}\frac{d\rho}{\rho}.$$

We shall distinguish two cases.

1. Let $|x-y| \ge R$, then $y_n \ge x_n + 2$ and $\delta|Tx| \le |Tx - Ty| \le 2|Tx|$, where $\delta = 1 - e^{-2\kappa} > 0$. Thus if $\rho \ge |Tx - Ty|$, then

$$B(Tx,\rho) \subset B(0, (1+\delta^{-1})\rho) \subset B(Tx, (1+2\delta^{-1})\rho),$$

and the doubling property of $\widetilde{w}$, together with Corollary 2.2, part (b), implies

$$\widetilde{w}(B(Tx,\rho)) \simeq \widetilde{w}(B(0,\rho)) \simeq e^{-2\kappa t}(w^{-1}(G_{t-1}\setminus G_t))^{-1}.$$

Lemma 5.6 then yields

$$g(Tx,Ty) \simeq \int_{|Tx|}^{1} \frac{\rho^2}{\widetilde{w}(B(0,\rho))}\frac{d\rho}{\rho} \simeq \int_0^{x_n} w^{-1}(G_{t-1}\setminus G_t)\,dt = \Gamma_0(x_n).$$

2. Let $|x-y| \le R$. Then by Lemma 3.2, $|Tx-Ty| \simeq e^{-\kappa x_n}|x-y|$ and Lemma 5.6 yields

$$g(Tx,Ty) \simeq \int_{e^{-\kappa x_n}|x-y|}^{1} \frac{\rho^2}{\widetilde{w}(B(Tx,\rho))}\frac{d\rho}{\rho},$$

provided that $\max(|Tx-Ty|, e^{-\kappa x_n}|x-y|) \le \frac{1}{2}$. Choose t^* so that this holds for all $x,y \in \overline{G}_{t^*}$ satisfying $|x-y| \le R$.

The last integral splits into two integrals with limits $e^{-\kappa x_n}|x-y|$, $e^{-\kappa x_n}R$ and $e^{-\kappa x_n}R$, 1, respectively. The latter is estimated in the same way as the integral in the first part of the proof and is comparable to $\Gamma_0(x_n)$.

As for the former, consider $\rho \le e^{-\kappa x_n}R$ and put $r = e^{\kappa x_n}\rho$. Then

$$B^+(Tx, C\rho) \subset T(B(x,r)\cap G) \subset B^+(Tx, C'\rho),$$

where $B^+(\cdot\,,\cdot) = \{\xi \in B(\cdot\,,\cdot) : \xi_n \ge 0\}$. The doubling property of $\widetilde{w}$ now yields

$$\widetilde{w}(B(Tx,\rho)) \simeq \widetilde{w}(T(B(x,r)\cap G)) \simeq e^{-2\kappa x_n} w(B(x,r)\cap G),$$

which results in

$$\int_{e^{-\kappa x_n}|x-y|}^{e^{-\kappa x_n}R} \frac{\rho^2}{\widetilde{w}(B(Tx,\rho))}\frac{d\rho}{\rho} \simeq \int_{|x-y|}^{R} \frac{r^2}{w(B(x,r)\cap G)}\frac{dr}{r} = \mathscr{K}(x,y).$$

6. The capacity $\mathrm{cap}_{\mathscr{K}}$.

In this section, we estimate cap_Γ by means of a new capacity on $\overline{G}$, $\mathrm{cap}_{\mathscr{K}}$. Compared with cap_Γ, the capacity $\mathrm{cap}_{\mathscr{K}}$ has the advantage that it is quasiadditive with respect to the partition $\{F_j \setminus F_{j+1}\}_{j=0}^{\infty}$ of F and hence, the Wiener test can be rewritten in terms of the relatively compact sets $F_j \setminus F_{j+1}$, rather than the unbounded sets F_j. As a corollary, we obtain a new proof of Theorem 1.1. Also, the criterion for $L_0^{1,2}(G_0,F,w) = L^{1,2}(G_0,F,w)$ from Corollary 4.7 can be simplified using $\mathrm{cap}_{\mathscr{K}}$.

Definition 6.1. Put for a Borel set $E \subset \overline{G}$,

$$\mathrm{cap}_{\mathscr{K}}(E) = \sup \mu(E),$$

where the supremum is taken over all positive measures μ with compact support in E, satisfying

$$\int \mathcal{K}(x,y)\,d\mu(y) \le 1 \quad \text{for all } x \in \overline{G}.$$

LEMMA 6.2. (Generalized maximum principle for $\mathcal{K}$-potentials). *There exists C such that if μ is a positive measure with support in $\overline{G}$ and*

$$\int \mathcal{K}(x,y)\,d\mu(y) \le M \quad \text{for all } x \in \operatorname{spt}\mu,$$

then

$$\int \mathcal{K}(x,y)\,d\mu(y) \le CM \quad \text{for all } x \in \overline{G}.$$

PROOF. If $\operatorname{dist}(x,\operatorname{spt}\mu) \ge R$, then $\mathcal{K}(x,y) = 0$ for all $y \in \operatorname{spt}\mu$ and the claim follows.

Assume that $\operatorname{dist}(x,\operatorname{spt}\mu) < R$. The ball $B(x,R)$ can be covered by N balls B_j with radii $\frac{1}{4}R$. Let x^j be the point of $\overline{B}_j \cap \operatorname{spt}\mu$, which is closest to x, and $y \in B_j \cap \operatorname{spt}\mu$. Then $|y-x^j| \le |y-x| + |x-x^j| \le 2|y-x|$ and hence

$$\mathcal{K}(x,y) \simeq \mathcal{K}(y,x) = \int_{|y-x|}^{R} \frac{r^2}{w(B(y,r)\cap G)}\frac{dr}{r} \le \int_{|y-x^j|/2}^{R} \frac{r^2}{w(B(y,r)\cap G)}\frac{dr}{r}.$$

The local doubling property of w and Lemma 5.6 imply

$$\int_{|y-x^j|/2}^{R} \frac{r^2}{w(B(y,r)\cap G)}\frac{dr}{r} \simeq \int_{|y-x^j|}^{R} \frac{r^2}{w(B(y,r)\cap G)}\frac{dr}{r} = \mathcal{K}(y,x^j) \simeq \mathcal{K}(x^j,y).$$

This yields

$$\int \mathcal{K}(x,y)\,d\mu(y) \le C\sum_{j=1}^{N}\int_{B_j} \mathcal{K}(x^j,y)\,d\mu(y) \le CNM.$$

LEMMA 6.3. *For $t \ge 0$,*

$$\operatorname{cap}_{\mathcal{K}}(F_t) \simeq \sum_{j=0}^{\infty} \operatorname{cap}_{\mathcal{K}}(F_{t+j} \setminus F_{t+j+1}).$$

PROOF. The inequality "$\le$" is easy. To prove the other inequality, fix $t \ge 0$ and $\varepsilon > 0$. We may, for simplicity, assume that t is an integer. Let μ_j, $j = 0,1,\dots$, be measures admissible for $\operatorname{cap}_{\mathcal{K}}(F_j \setminus F_{j+1})$, such that

$$\mu_j(F_j \setminus F_{j+1}) \ge \operatorname{cap}_{\mathcal{K}}(F_j \setminus F_{j+1}) - 2^{-j}\varepsilon.$$

Let $M \geq R+1$ be a fixed integer. For $m = 1, \ldots, M$, let I_m denote the set of all $j \geq t$ such that $j \equiv m \mod M$. Let $x \in F_k \setminus F_{k+1}$, $k \in I_m$. Then

$$\sum_{j \in I_m} \int \mathscr{K}(x,y)\, d\mu_j(y) = \int \mathscr{K}(x,y)\, d\mu_k(y) \leq 1.$$

By Lemma 6.2, restrictions of the measure $C^{-1} \sum_{j \in I_m} \mu_j$ to compact sets are admissible in the definition of $\operatorname{cap}_{\mathscr{K}}(F_t)$ and hence,

$$\sum_{j \in I_m} \mu_j(F_j \setminus F_{j+1}) \leq C \operatorname{cap}_{\mathscr{K}}(F_t).$$

Since $\mu_j(F_j \setminus F_{j+1}) \geqq \operatorname{cap}_{\mathscr{K}}(F_j \setminus F_{j+1}) - 2^{-j}\varepsilon$, we get summing up over all m,

$$\sum_{j=t}^{\infty} \operatorname{cap}_{\mathscr{K}}(F_j \setminus F_{j+1}) - 2\varepsilon \leq CM \operatorname{cap}_{\mathscr{K}}(F_t).$$

Letting $\varepsilon \to 0$ finishes the proof.

Let for $s \geq 0$, $G^s = \{x \in G : \Gamma_0(x_n) > s\}$ and $F^s = \{x \in F : \Gamma_0(x_n) \geq s\}$.

THEOREM 6.4. *Let $E \subset \overline{G^s}$, $s \geq \Gamma_0(t^*)$, be a Borel set. Then*

$$\operatorname{cap}_\Gamma(E) \leq C \min(s^{-1}, \operatorname{cap}_{\mathscr{K}}(E)).$$

If moreover $E \subset \overline{G^s \setminus G^{2s}}$, then $\operatorname{cap}_\Gamma(E) \simeq \min(s^{-1}, \operatorname{cap}_{\mathscr{K}}(E))$.

PROOF. 1. Let μ be a measure admissible for $\operatorname{cap}_\Gamma(E)$ and $x \in E \subset \overline{G^s}$. By Theorem 5.7,

$$\int \mathscr{K}(x,y)\, d\mu(y) \leq \int (C\Gamma(x,y) - \Gamma_0(\min(x_n, y_n)))\, d\mu(y) \leq C - s\mu(E).$$

By Lemma 6.2, the measure $C'(C - s\mu(E))^{-1}\mu$ is admissible in the definition of $\operatorname{cap}_{\mathscr{K}}(E)$, which leads to

$$\mu(E) \leq \frac{C \operatorname{cap}_{\mathscr{K}}(E)}{C' + s \operatorname{cap}_{\mathscr{K}}(E)} \leq C \min(s^{-1}, \operatorname{cap}_{\mathscr{K}}(E)).$$

Taking supremum over all μ admissible in the definition of $\operatorname{cap}_\Gamma(E)$ finishes the first part of the proof.

2. Conversely, for $E \subset \overline{G^s \setminus G^{2s}}$, let μ be a measure admissible in the definition of $\operatorname{cap}_{\mathscr{K}}(E)$, and $x \in E$. By Theorem 5.7,

$$\int \Gamma(x,y)\, d\mu(y) \simeq \int (\mathscr{K}(x,y) + \Gamma_0(\min(x_n, y_n)))\, d\mu(y) \leq 1 + 2s \operatorname{cap}_{\mathscr{K}}(E),$$

i.e. the measure $C(1 + 2s \operatorname{cap}_{\mathscr{K}}(E))^{-1}\mu$ is admissible for $\operatorname{cap}_\Gamma(E)$. Hence,

$$\operatorname{cap}_\Gamma(E) \geq \frac{C\mu(E)}{1 + 2s \operatorname{cap}_{\mathscr{K}}(E)} \geq \frac{C\mu(E)}{\max(1, s \operatorname{cap}_{\mathscr{K}}(E))}.$$

Taking supremum over all μ admissible in the definition of $\operatorname{cap}_{\mathscr{K}}(E)$ yields

$$\operatorname{cap}_\Gamma(E) \geq C \min(s^{-1}, \operatorname{cap}_{\mathscr{K}}(E)).$$

EXAMPLE 6.5. We show that at least in the unweighted case $w = 1$, the assumption $E \subset G^s \setminus G^{2s}$ is necessary. We have $\Gamma_0(t) \simeq t$ and the proof of Corollary 6.10 below reveals that $\operatorname{cap}_{\mathscr{K}}(E) \simeq \operatorname{cap}(E)$, provided $\operatorname{diam}(E) \leq \frac{1}{2}R$ (cap is the Newtonian capacity in $\mathbf{R}^n$). Fix $0 < r_2 < \frac{1}{4}R$ and $t_1 \geq t^*$. By Theorem 6.4,

$$\operatorname{cap}_\Gamma(B((0,t_1),r_1) \cup B((0,t_2),r_2)) \leq C\big(\operatorname{cap}(B((0,t_1),r_1)) + t_2^{-1}\big).$$

Letting $r_1 \to 0$ and $t_2 \to \infty$, the right-hand side can be made arbitrarily small, while t_1 and $\operatorname{cap}(B((0,t_1),r_1) \cup B((0,t_2),r_2)) \geq \operatorname{cap}(B((0,t_2),r_2)) > 0$ are fixed.

We conclude this section by further simplifying the criterion of regularity for the operator $\operatorname{div}(\mathscr{A}(x)\nabla u(x))$. We need two lemmas, in which we assume

$$\text{(14)} \qquad \Gamma_0(t) \to \infty, \text{ as } t \to \infty \quad \text{and} \quad \int_{\Gamma_0(1)}^{\infty} \operatorname{cap}_\Gamma(F^s)\, ds < \infty.$$

LEMMA 6.6. *Let* (14) *hold. Then there exists s_0 such that for all $s \geq s_0$,*

$$\operatorname{cap}_{\mathscr{K}}(F^s \setminus F^{2s}) < s^{-1},$$

i.e. $\operatorname{cap}_\Gamma(F^s \setminus F^{2s}) \simeq \operatorname{cap}_{\mathscr{K}}(F^s \setminus F^{2s})$.

PROOF. Suppose that there exists an infinite sequence $s_j \to \infty$ of indices such that $\operatorname{cap}_{\mathscr{K}}(F^{s_j} \setminus F^{2s_j}) \geq s_j^{-1}$ for all $j \geqq 0$. By throwing away some s_j, we may assume that $s_j \geq 2s_{j-1}$ and $s_j \geq \Gamma_0(t^*)$. Then,

$$\int_{\Gamma_0(1)}^{\infty} \operatorname{cap}_\Gamma(F^s)\, ds \geq \sum_{j=1}^{\infty} (s_j - s_{j-1}) \operatorname{cap}_\Gamma(F^{s_j} \setminus F^{2s_j}).$$

By Theorem 6.4, $\operatorname{cap}_\Gamma(F^{s_j} \setminus F^{2s_j}) \simeq s_j^{-1}$, and since $s_j - s_{j-1} \geq \frac{1}{2}s_j$, the series diverges, which contradicts the assumption (14).

LEMMA 6.7. *Let* (14) *hold. Then there exists t_0 such that for all $t \geq t_0$,*

$$\operatorname{cap}_\Gamma(F_t) \simeq \operatorname{cap}_{\mathscr{K}}(F_t).$$

PROOF. The inequality "$\leq$" follows from Theorem 6.4. Conversely, let s_0 be as in Lemma 6.6 and define t_0 by $\Gamma_0(t_0) = s_0$. Let $t \geqq t_0$ be arbitrary and put $s = \Gamma_0(t)$. Let μ be a measure admissible in the definition of $\operatorname{cap}_{\mathscr{K}}(F_t)$. Estimate the Γ-potential of μ at $x \in F_t = F^s$. By Theorem 5.7,

$$\int \Gamma(x,y)\,d\mu(y) \simeq \int (\mathscr{K}(x,y) + \Gamma_0(\min(x_n,y_n)))\,d\mu(y) \leq 1 + \sum_{j=0}^{\infty} 2^{j+1} s\mu(F^{2^j s} \setminus F^{2^{j+1}s}). \tag{15}$$

By Lemma 6.6,

$$\mu(F^{2^j s} \setminus F^{2^{j+1}s}) \leq \operatorname{cap}_{\mathscr{K}}(F^{2^j s} \setminus F^{2^{j+1}s}) \simeq \operatorname{cap}_\Gamma(F^{2^j s} \setminus F^{2^{j+1}s}).$$

Next, for $j \geq 1$,

$$2^{j+1} s \operatorname{cap}_\Gamma(F^{2^j s} \setminus F^{2^{j+1}s}) \leq 2^{j+1} s \operatorname{cap}_\Gamma(F^{2^j s}) \leq 4 \int_{2^{j-1}s}^{2^j s} \operatorname{cap}_\Gamma(F^\sigma)\,d\sigma$$

and by (14), the series in (15) converges. It follows that the measure $C^{-1}\mu$ is admissible for $\operatorname{cap}_\Gamma(F_t)$ and taking supremum over all μ admissible for $\operatorname{cap}_{\mathscr{K}}(F_t)$ yields $\operatorname{cap}_{\mathscr{K}}(F_t) \leq C \operatorname{cap}_\Gamma(F_t)$.

Theorem 6.8. *Infinity is regular for the Zaremba problem in $L^{1,2}(G_0, F, w)$ if and only if*

$$\sum_{j=1}^{\infty} w^{-1}(G_0 \setminus G_j) \operatorname{cap}_{\mathscr{K}}(F_j \setminus F_{j+1}) = \infty. \tag{16}$$

Proof. 1. Assume first that $w^{-1}(G_0) = \infty$. By Corollary 4.7, the spaces $L_0^{1,2}(G_0, F, w)$ and $L^{1,2}(G_0, F, w)$ coincide. Theorem 4.5 now implies that infinity is regular for the Zaremba problem in $L^{1,2}(G_0, F, w)$ if and only if

$$\sum_{j=1}^{\infty} w^{-1}(G_{j-1} \setminus G_j) \operatorname{cap}_\Gamma(F_j) = \infty. \tag{17}$$

By Theorem 6.4, $\operatorname{cap}_\Gamma$ in (23) can be replaced by $\operatorname{cap}_{\mathscr{K}}$ and the series still diverges. Lemma 6.3 then shows that (17) implies (16).

Conversely, if (17) fails, then (14) holds. Lemma 6.7 implies $\operatorname{cap}_\Gamma(F_t) \simeq \operatorname{cap}_{\mathscr{K}}(F_t)$ for large t. Hence

$$\sum_{j=1}^{\infty} w^{-1}(G_{j-1} \setminus G_j) \operatorname{cap}_{\mathscr{K}}(F_j) < \infty,$$

and by Lemma 6.3, (16) fails.

2. Assume that $w^{-1}(G_0) < \infty$. Then by Theorem 4.5, infinity is regular for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$. As for the regularity in $L^{1,2}(G_0, F, w)$,

the condition (16) is equivalent to

$$\sum_{j=1}^{\infty} \operatorname{cap}_{\mathscr{K}}(F_j \setminus F_{j+1}) = \infty. \tag{18}$$

By Lemma 6.3, $\operatorname{cap}_{\mathscr{K}}(F_t) = \infty$ for all $t \geq 0$. Since $F_t \subset \overline{G^{\Gamma_0(t)} \setminus G^{2\Gamma_0(t)}}$ for large t, Theorem 6.4 yields $\operatorname{cap}_{\Gamma}(F_t) \simeq \Gamma_0(t)^{-1} \geq C(w^{-1}(G_0))^{-1} > 0$ for large t. It then follows from Corollary 4.7 that $L^{1,2}(G_0, F, w) = L_0^{1,2}(G_0, F, w)$ and hence, infinity is regular for the Zaremba problem in $L^{1,2}(G_0, F, w)$.

On the other hand, if (18) fails, then $\operatorname{cap}_{\mathscr{K}}(F_t) \to 0$, as $t \to \infty$, and Theorem 6.4 yields $\operatorname{cap}_{\Gamma}(F_t) \to 0$, as $t \to \infty$. By Theorem 4.8, infinity is not regular for the Zaremba problem in $L^{1,2}(G_0, F, w)$.

Corollary 6.9. *The spaces $L_0^{1,2}(G_0, F, w)$ and $L^{1,2}(G_0, F, w)$ differ if and only if $w^{-1}(G_0) < \infty$ and $\operatorname{cap}_{\mathscr{K}}(F) < \infty$. In this case, infinity is regular for the Zaremba problem in $L_0^{1,2}(G_0, F, w)$ but not in $L^{1,2}(G_0, F, w)$.*

Proof. See Corollary 4.7 and the proof of Theorem 6.8.

Corollary 6.10 (Theorem 1.1). *For $w = 1$, infinity is regular for the Zaremba problem in $L_0^{1,2}(G_0, F, w) = L^{1,2}(G_0, F, w)$ if and only if*

$$\sum_{j=1}^{\infty} j \operatorname{cap}(F_j \setminus F_{j+1}) = \infty,$$

where cap *is the Newtonian capacity in* R^n.

Proof. Insert $w = 1$ in Theorem 6.8, then $w^{-1}(G_0 \setminus G_j) \simeq j$. For $x, y \in \overline{G}$ satisfying $|x - y| \leq \frac{1}{2}R$, the kernel $\mathscr{K}(x, y)$ is comparable to

$$k(x,y) = \begin{cases} |x-y|^{2-n} & \text{if } n \geq 3, \\ \log\left(\dfrac{R}{|x-y|}\right) & \text{if } n = 2. \end{cases}$$

The kernel $k(x, y)$ generates the Newtonian capacity in R^n, see e.g. Section III in Carleson [1] or Definition 7.1 in Wermer [10]. Since $\operatorname{diam}(F_j \setminus F_{j+1}) \leq \frac{1}{2}R$, we have $\operatorname{cap}_{\mathscr{K}}(F_j \setminus F_{j+1}) \simeq \operatorname{cap}(F_j \setminus F_{j+1})$.

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