MATH. SCAND. 88 (2001), 246–256

UNIVERSAL SPECTRA, UNIVERSAL TILING SETS AND THE SPECTRAL SET CONJECTURE

STEEN PEDERSEN and YANG WANG

Abstract

A subset Ω of R^d with finite positive Lebesgue measure is called a *spectral set* if there exists a subset $\Lambda \subset \mathsf{R}$ such that $\mathscr{E}_\Lambda := \left\{e^{i2\pi\langle\lambda,x\rangle} : \lambda \in \Lambda\right\}$ form an orthogonal basis of $L^2(\Omega)$. The set Λ is called a *spectrum* of the set Ω. The Spectral Set Conjecture states that Ω is a spectral set if and only if Ω tiles R^d by translation. In this paper we prove the Spectral Set Conjecture for a class of sets $\Omega \subset \mathsf{R}$. Specifically we show that a spectral set possessing a spectrum that is a strongly periodic set must tile R by translates of a strongly periodic set depending only on the spectrum, and vice versa.

1. Introduction

Let Ω be a (Lebesgue) measurable subset of R with finite positive measure. For $t \in \mathsf{R}$ let $\Omega + t := \{x + t : x \in \Omega\}$ denote the translate of Ω by t. We say that Ω *tiles* R by translation if there exists a subset $\mathscr{T} \subset \mathsf{R}$ so that $\mathsf{R} \setminus \bigcup_{t\in\mathscr{T}} (\Omega + t)$ is a set of measure zero and $(\Omega + t) \cap \left(\Omega + t'\right)$ is a set of measure zero whenever $t, t' \in \mathscr{T}$ are distinct. In the affirmative case $\mathscr{T}$ is called a *tiling set* for Ω, and $(\Omega, \mathscr{T})$ is called a *tiling pair*. Similarly, we say that Ω tiles the non-negative half line $\mathsf{R}^+ = [0, \infty)$ if there exists a subset $\mathscr{T} \subset \mathsf{R}$ such that $\mathsf{R}^+ \setminus \bigcup_{t\in\mathscr{T}} (\Omega + t)$ is a set of measure zero and $(\Omega + t) \cap \left(\Omega + t'\right)$ is a set of measure zero whenever $t, t' \in \mathscr{T}$ are distinct. Sets that tile the real line by translation have been studied recently, e.g., [9], [8], [7].

For $\lambda \in \mathsf{R}$ we introduce the functions

$$e_\lambda(x) := e^{i2\pi\lambda x}, \qquad x \in \mathsf{R}.$$

We say that Ω is a *spectral set* if there exists a subset $\Lambda \subset \mathsf{R}$ so that the functions $\mathscr{E}_\Lambda := \{e_\lambda : \lambda \in \Lambda\}$ form an orthogonal basis for $L^2(\Omega)$, the Hilbert space of complex valued square integrable functions on Ω with the inner product

$$\langle f, g\rangle := \int_\Omega \overline{f(x)}g(x)\,dx.$$

Received August 25, 1998.

If the functions in $\mathscr{E}_\Lambda$ form an orthogonal basis for $L^2(\Omega)$, then we call (Ω, Λ) a *spectral pair* and Λ a *spectrum* for Ω. Spectral sets have recently been studied in various contexts, e.g., [3], [4], [5], [10], [8], [6].

One of the main open questions concerning spectral sets is the following conjecture, first proposed by Fuglede [3]:

SPECTRAL SET CONJECTURE. *Let Ω be a measurable subset of $\mathbb{R}^d$ with finite positive Lebesgue measure. Then Ω is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation.*

In this paper we study the one dimensional case of the Spectral Set Conjecture. A special class of sets we study consists of tiles that tile the non-negative half line $\mathbb{R}^+$ by translation. We prove:

THEOREM 1.1. *Let Ω be a subset of $\mathbb{R}$ with finite positive Lebesgue measure. Suppose that Ω tiles $\mathbb{R}^+$ by translation. Then Ω tiles $\mathbb{R}$ by translation and is a spectral set.*

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of natural numbers and $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ be the set of non-negative integers. For any $n \in \mathbb{N}$ let $\mathbb{Z}_n^+ := \{0, 1, \ldots, n-1\}$. For any $A, B \subseteq \mathbb{Z}$ we write

$$A + B := \{a + b : a \in A, b \in B\}$$

for the Minkowski sum of A and B. We will write $A \oplus B$ if each element in $A + B$ has a *unique* decomposition of the form $a + b$ with $a \in A$ and $b \in B$.

DEFINITION 1.2. We call $A \subset \mathbb{Z}^+$ a *direct summand of* $\mathbb{Z}_n^+$ if there exists a $B \subset \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}_n^+$. We call a subset $\mathscr{T}$ of $\mathbb{R}$ a *strongly periodic set* if there exist an $n \in \mathbb{N}$ and a direct summand $A \subset \mathbb{Z}^+$ of $\mathbb{Z}_n^+$ such that $\mathscr{T} = \alpha(A \oplus n\mathbb{Z})$ for some non-zero $\alpha \in \mathbb{R}$.

In [8] it was shown that certain tiles that tile $\mathbb{R}$ by translation are spectral sets that possess the so-called *universal spectra*, in the sense that the spectra depend only on the tiling sets, not the tiles. Our main theorem below strengthens this notion by providing a large new class of tiles that possess universal spectra. It shows that a tile that tiles $\mathbb{R}$ by the translates of a strongly periodic set must have a universal spectrum that is also a strongly periodic set. More importantly, the theorem also gives rise to the notion of *universal tiling set*, which can be viewed as the dual of universal spectrum. We show that a spectral set that possesses a spectrum that is a strongly periodic set must have a universal tiling set depending only on the spectrum.

THEOREM 1.3. *Let Ω be a subset of $\mathbb{R}$ with finite positive measure. Suppose that there exists a strongly periodic set $\Lambda \subset \mathbb{R}$ such that (Ω, Λ) is a spectral* *pair. Then there exists a strongly periodic set* $\mathcal{T} \subset \mathbb{R}$ *depending only on* Λ *such that* Ω *tiles* $\mathbb{R}$ *by translates of* $\mathcal{T}$*. Conversely, suppose that there exists a strongly periodic set* $\mathcal{T} \subset \mathbb{R}$ *such that* Ω *tiles* $\mathbb{R}$ *by translates of* $\mathcal{T}$*. Then there exists a strongly periodic set* $\Lambda \subset \mathbb{R}$ *depending only on* $\mathcal{T}$ *such that* (Ω, Λ) *is a spectral pair.*

The strongly periodic sets Λ and $\mathcal{T}$ in Theorem 1.3 are *duals* of each other, and for each given one the other is constructed explicitly in §4. In fact we prove a stronger version of Theorem 1.3 there. For the rest of the paper, in §2 we state a result on the structure of strongly periodic sets, first shown in [2]. In §3 we classify tiles that tile $\mathbb{R}^+$ by translation. The classification is used to prove Theorem 1.1.

2. Structure of Strongly Periodic Sets

In this section we classify subsets A, B of $\mathbb{Z}^+$ satisfying $A \oplus B = \mathbb{Z}_n^+$ for some $n \in \mathbb{N}$. The classification is based on a theorem of de Bruijn [2] establishing the structure of subsets of $\mathbb{Z}^+$ that tile $\mathbb{Z}^+$ by translation. To formulate the result we first introduce some notation regarding divisibility. For $r, s \in \mathbb{Z}$ we use $r \mid s$ to mean that r divides s; for $r \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$ we use $r \mid A$ to mean that r divides every $a \in A$.

PROPOSITION 2.1 (de Bruijn). *Let* $A, B \subseteq \mathbb{Z}^+$ *such that* $A \oplus B = \mathbb{Z}^+$ *and* $A \neq \mathbb{Z}^+$, $B \neq \mathbb{Z}^+$*. Then there exists an integer* $r > 1$ *such that* $r \mid A$ *or* $r \mid B$*. Furthermore, if* $r \mid B$ *and* $B = r\widetilde{B}$ *then there exists an* $\widetilde{A} \subseteq \mathbb{Z}^+$ *such that*

$$A = \mathbb{Z}_r^+ \oplus r\widetilde{A}, \quad \text{and} \quad \widetilde{A} \oplus \widetilde{B} = \mathbb{Z}^+.$$

PROOF. A proof can be found in de Bruijn [2]. For the sake of self-containment we give a short proof here.

Without loss of generality we assume $1 \in A$. Let r be the smallest non-zero member of B. For each $m \in \mathbb{N}$ let $A_m \subseteq A$ and $B_m \subseteq B$ be the minimal subsets so that

$$\mathbb{Z}_{mr}^+ \subseteq A_m + B_m.$$

It follows immediately from the minimality and the uniqueness in $A \oplus B$ that

$$A_m = A \cap \mathbb{Z}_{mr}^+, \qquad B_m = B \cap \mathbb{Z}_{mr}^+.$$

Observe that $\mathbb{Z}_{(m+1)r}^+ \setminus \mathbb{Z}_{mr}^+ = \mathbb{Z}_r^+ + mr$. So

$$A_{m+1} \setminus A_m \subseteq \mathbb{Z}_r^+ + mr, \qquad B_{m+1} \setminus B_m \subseteq \mathbb{Z}_r^+ + mr.$$

We show by induction on m that there are subsets C_m and D_m of $\mathbb{Z}^+$ such that

$$A_m = \mathbb{Z}_r^+ + rC_m, \qquad B_m = rD_m.$$

Let $C_1 := \{0\}$ and $D_1 := \{0\}$. Then $A_1 = \mathbb{Z}_r^+ + rC_1$ and $B_1 = rD_1$ as required. Suppose that C_m, $D_m \subseteq \mathbb{Z}^+$ have been constructed so that $A_m = \mathbb{Z}_r^+ + rC_m$ and $B_m = rD_m$. If $\mathbb{Z}^+_{(m+1)r} \subseteq A_m + B_m$, then $A_{m+1} = A_m$ and $B_{m+1} = B_m$, and so it suffices to set $C_{m+1} := C_m$ and $D_{m+1} := D_m$ to complete the proof.

Now suppose that $\mathbb{Z}^+_{(m+1)r} \not\subseteq A_m + B_m$. Let $j \in \mathbb{Z}_r^+$. If $j + mr \in A_m + B_m = \mathbb{Z}_r^+ + r(C_m + D_m)$ then $m \in C_m + D_m$ and therefore $\mathbb{Z}_r^+ + mr \subseteq A_m + B_m$, contradicting $\mathbb{Z}^+_{(m+1)r} \not\subseteq A_m + B_m$. Hence,

$$(\mathbb{Z}_r^+ + mr) \cap (A_m + B_m) = \emptyset.$$

It follows that $mr \in A_{m+1}$ or $mr \in B_{m+1}$.

If $mr \in B_{m+1}$, then $A_{m+1} = A_m$ and $B_{m+1} = B_m \cup \{rm\}$. Hence we may set $C_{m+1} := C_m$ and $D_{m+1} := D_m \cup \{m\}$.

Assume that $mr \in A_{m+1}$. Let $j \in \mathbb{Z}_r^+$. We have shown above that $j + mr \notin A_m + B_m$, so $j + mr = a + b$ for $a \in A_{m+1} \setminus A_m$, $b \in B_{m+1}$ or $a \in A_m$, $b \in B_{m+1} \setminus B_m$. If $b \in B_{m+1} \setminus B_m$ then $(m+1)r - b \in \mathbb{Z}_r^+$. Thus $mr + r = ((m+1)r - b) + b$ constitute two different decompositions of the same element in $A \oplus B$, a contradiction. This yields $a \in A_{m+1} \setminus A_m$. If $b \neq 0$ then $B_m = rD_m$ and $B_{m+1} \setminus B_m \subseteq \mathbb{Z}_r^+ + mr$ implies that $b \geq r$. So $j + mr = a + b \geq mr + r > j + mr$, again a contradiction. So $b = 0$ and therefore $j + mr = a \in A_{m+1}$. It follows that

$$A_{m+1} = A_m \cup (\mathbb{Z}_r^+ + mr).$$

The inductions steps are now complete by setting $C_{m+1} := C_m \cup \{m\}$ and $D_{m+1} := D_m$.

Finally, the proposition follows by letting $\widetilde{A} := \bigcup_{m=1}^{\infty} C_m$ and $\widetilde{B} = \bigcup_{m=1}^{\infty} D_m$.

Proposition 2.1 immediately leads to the following classification of strongly periodic sets.

COROLLARY 2.2. *Let A, $B \subseteq \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}_n^+$ and $A \neq \mathbb{Z}_n^+$, $B \neq \mathbb{Z}_n^+$. Then there exists an $r > 1$ such that $r \mid n$ and either $r \mid A$ or $r \mid B$. Furthermore, if $r \mid B$ and $B = r\widetilde{B}$ then there exists an $\widetilde{A} \subset \mathbb{Z}^+$ so that*

$$A = \mathbb{Z}_r^+ \oplus r\widetilde{A}, \qquad \text{and} \qquad \widetilde{A} \oplus \widetilde{B} = \mathbb{Z}^+_{\frac{n}{r}}.$$

PROOF. Suppose that $1 \in A$. Applying Proposition 2.1 to $A \oplus (B \oplus n\mathbb{Z}^+) = \mathbb{Z}^+$ yields an $r > 1$ and a set $\widetilde{A}$ so that $A = \mathbb{Z}_r^+ \oplus r\widetilde{A}$ and $r \mid (B \oplus n\mathbb{Z}^+)$. Since

$0 \in B$ and $0 \in \mathbb{Z}^+$ it follows that $r \mid n$ and $r \mid B$. Finally, $\mathbb{Z}_r^+ \oplus r(\widetilde{A} + \widetilde{B}) = A \oplus B = \mathbb{Z}_n^+$ implies $\widetilde{A} \oplus \widetilde{B} = \mathbb{Z}_{\frac{n}{r}}^+$.

Corollary 2.3. *Let A, $B \subseteq \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}_n^+$. Assume that $1 \in A$. Then there exists a unique finite sequence $d_0 = 1, d_1, \dots, d_{k-1}, d_k = n$ in $\mathbb{N}$ with $r_j := d_j/d_{j-1} \in \mathbb{N}$ and $r_j > 1$ for $1 \le j \le k$ such that*

$$A = d_0\mathbb{Z}_{r_1}^+ \oplus d_2\mathbb{Z}_{r_3}^+ \oplus \cdots, \tag{2.1}$$

$$B = d_1\mathbb{Z}_{r_2}^+ \oplus d_3\mathbb{Z}_{r_4}^+ \oplus \cdots. \tag{2.2}$$

Proof. Since $1 \in A$, the proof of Proposition 2.1 yields $A = \mathbb{Z}_{r_1}^+ \oplus r_1\widetilde{A}$ and $B = r_1\widetilde{B}$ where $r_1 = \min\{b : b \in B, b \neq 0\}$, and $\widetilde{A} \oplus \widetilde{B} = \mathbb{Z}_{\frac{n}{r_1}}^+$. The proof is completed by applying Corollary 2.2 iteratively to $\widetilde{A} \oplus \widetilde{B} = \mathbb{Z}_{\frac{n}{r_1}}^+$. Note that the uniqueness follows from the fact that $r_1 = d_1/d_0 = \min\{b : b \in B, b \neq 0\}$, $r_2 = d_2/d_1 = \{a : a \in \widetilde{A}, a \neq 0\}$, etc.

Corollary 2.4. *Suppose that A, $B \subseteq \mathbb{Z}^+$ such that $A \oplus B = \mathbb{Z}^+$, and that B is finite. Then B is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$.*

Proof. By the same argument for Corollary 2.3 B must have the form (2.1) or (2.2), depending on whether $1 \in B$. So B must be a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$.

Call a polynomial a $0-1$ *polynomial* if each of its coefficients is either 0 or 1. We associate each finite $A \subseteq \mathbb{Z}^+$ with the following $0-1$ polynomial

$$A(x) := \sum_{a \in A} x^a,$$

called the *characteristic polynomial* of A. Clearly every $0-1$ polynomial is the characteristic polynomial of the set of exponents corresponding to its non-zero coefficients. If A, B, $C \subseteq \mathbb{Z}^+$ are finite, then $A \oplus B = C$ if and only if $A(x)B(x) = C(x)$. We call a $0-1$ polynomial *c-irreducible* if $A(x) \neq A_1(x)A_2(x)$ for any $0-1$ polynomials $A_1(x) \not\equiv 1$, $A_2(x) \not\equiv 1$. The following result was first stated in [1] (simple examples, however, show that Lemma 1 in [1] is false).

Theorem 2.5. *Let $n > 1$. Then every factorization of $\frac{x^n-1}{x-1}$ into c-irreducible $0-1$ polynomials has the form*

$$\frac{x^n - 1}{x - 1} = F_{p_1}(x)F_{p_2}(x^{p_1})F_{p_3}(x^{p_1p_2}) \dots F_{p_k}(x^{p_1p_2\cdots p_{k-1}}),$$

where $F_m(x) := \frac{x^m-1}{x-1}$, *all* p_j *are primes (not necessarily distinct) and* $n = p_1 p_2 \dots p_k$.

PROOF. This is a direct consequence of Corollary 2.3, by observing that

$$\mathsf{Z}^+_{p_1 p_2 \cdots p_k} = \mathsf{Z}^+_{p_1} \oplus p_1 \mathsf{Z}^+_{p_2} \oplus p_1 \dots p_{k-1} \mathsf{Z}_{p_k}.$$

Note that each term in the factorization is c-irreducible, because it contains a prime number of terms.

3. Tiling the Non-Negative Real Line

Let $\Omega \subset \mathsf{R}$ be a tile with finite and positive Lebesgue measure that tiles R^+ by translates of $\mathscr{T}$. In this case we will write $\Omega \oplus \mathscr{T} = \mathsf{R}^+$. In this section we derive the structure of tiles $\Omega \subset \mathsf{R}$ that tile R^+ by translation.

THEOREM 3.1. *Let* $\Omega \subset \mathsf{R}$ *with finite positive Lebesgue measure. Suppose that* Ω *tiles* R^+ *by translation. Then there exists an affine map* $\varphi(x) = ax + b$ *such that*

$$\varphi(\Omega) = [0, 1] + B$$

for some finite subset $B \subset \mathsf{Z}^+$ *with* $0 \in B$. *Furthermore, B is a direct summand of* Z^+_n *for some* $n \in \mathsf{N}$. *Hence* Ω *tiles* R *by translation.*

PROOF. In this proof, all set relations involving the tile Ω will be interpreted as up to measure zero sets.

Let $\mathscr{T} \subset \mathsf{R}$ such that $\Omega \oplus \mathscr{T} = \mathsf{R}^+$. We first examine the special case $\mathscr{T} = \{0, 1, t_2, t_3, \dots\}$ where $t_j > 1$ for all $j \geq 2$. In this special case we prove that $\Omega = [0, 1] + B$ for some $B \subset \mathsf{Z}^+$ and $0 \in B$. Let $\mathscr{T}_n = \mathscr{T} \cap [0, n-1]$ and $\Omega_n = \Omega \cap [0, n]$. We claim that $\mathscr{T}_n \subset \mathsf{Z}^+$ and $\Omega_n = [0, 1] + B_n$ for some $B_n \subset \mathsf{Z}^+$, by induction on n.

Since $t_j > 1$, we must have $[0, 1] \subseteq \Omega$. So the claim is clearly true for $n = 1$. Assume that the claim is true for all $n < k$. We show that the claim is also true for $n = k$. We divide the proof into two cases: $\Omega_{k-1} \subsetneq \Omega_k$ and $\Omega_{k-1} = \Omega_k$. Suppose that $\Omega_{k-1} \subsetneq \Omega_k$. Then $\Omega \cap (k-1, k] \neq \emptyset$. If $\Omega_k \neq [0, 1] + B_k$ for any $B_k \subset \mathsf{Z}^+$, then $\Omega \cap (k-1, k] \subsetneq (k-1, k]$. Hence there exists a $t \in \mathscr{T}$ such that $(\Omega + t) \cap (k-1, k] \neq \emptyset$. Note that $t \in \mathscr{T}_{k-1}$, so $t \in \mathsf{Z}^+$. It follows that

$$\emptyset \subsetneq \Omega \cap (k-1-t, k-t] \subsetneq (k-1-t, k-t],$$

contradicting the inductive hypothesis. So $\Omega_k = [0, 1] + B_k$ for some $B_k \subset \mathsf{Z}^+$. The assumption that $\Omega_{k-1} \subsetneq \Omega_k$ now implies that $B_k = B_{k-1} \cup \{k-1\}$, so $\mathscr{T}_k = \mathscr{T}_{k-1}$. This proves the claim for $n = k$ in the first case. Suppose that $\Omega_{k-1} = \Omega_k$. Then $\Omega_k = [0, 1] + B_k$ with $B_k = B_{k-1}$. Therefore $\mathscr{T}_k =$

$\mathscr{T}_{k-1} \cup \{k-1\}$. This completes the induction steps and proves the claim. So we have shown that $B, \mathscr{T} \subseteq \mathbb{Z}^+$, and clearly $0 \in B$.

It remains to show that B is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$. Observe that $B \oplus \mathscr{T} = \mathbb{Z}^+$. Therefore B is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$ by Corollary 2.4.

In general, suppose that Ω tiles $\mathbb{R}^+$ by translates of $\mathscr{T}$ where the elements in $\mathscr{T}$ are $t_0 < t_1 < t_2 < \cdots$. Let $\varphi(x) = \frac{1}{t_1 - t_0}(x - t_0)$ and $t_j' = \varphi(t_j)$. Then

$$\varphi(\Omega) \oplus \{0, 1, t_2', t_3', \ldots\} = \mathbb{R}^+.$$

Hence $\varphi(\Omega) = [0, 1] + B$ for some $B \subset \mathbb{Z}^+$ with $0 \in B$.

4. Proofs of Main Theorems

To prove our main theorems we first introduce some notation. For any finite set $A \subset \mathbb{Z}$ we denote $f_A(\xi) := A\left(e^{i2\pi\xi}\right)$ where $A(z)$ is the characteristic (Laurent) polynomial of A. We will use $\mathscr{Z}_A$ to denote the set of zeros of f_A. For a subset $\Omega \subset \mathbb{R}$ with positive and finite measure we will use $\mathscr{Z}_\Omega$ to denote the set of zeros of $\widehat{\chi_\Omega}(\xi)$.

Observe that for any finite $A \subset \mathbb{Z}$, $\xi \in \mathscr{Z}_A$ implies $\xi + m \in \mathscr{Z}_A$ for all $m \in \mathbb{Z}$. So $\mathscr{Z}_A = \mathbb{Z} \oplus X$ for some finite $X \subset \mathbb{R}$. If in addition A is a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$, then $n\mathscr{Z}_A \subseteq \mathbb{Z}$.

Lemma 4.1. *Let $A \subset \mathbb{Z}^+$ be a direct summand of $\mathbb{Z}_n^+$ for some $n \in \mathbb{N}$. Then there exists a direct summand A^* of $\mathbb{Z}_n^+$ with the same cardinality such that*

$$A - A \subseteq n\mathscr{Z}_{A^*} \cup \{0\}, \qquad A^* - A^* \subseteq n\mathscr{Z}_A \cup \{0\}. \tag{4.1}$$

Proof. We procced by induction on n. For $n = 1, 2$ it is easy to check that the lemma holds. Assume that the lemma holds for all $n < k$, where $k \geq 3$. We show that it holds for $n = k$.

Case 1. $1 \notin A$. Then $A = rA_1$ for some $r > 1$, $r \mid k$ and direct summand A_1 of $\mathbb{Z}_{\frac{k}{r}}^+$. By the hypothesis there exists a direct summand A_1^* of $\mathbb{Z}_{\frac{k}{r}}^+$ such that (4.1) holds for A_1, A_1^* and $n = k/r$. Now $f_A(\xi) = f_{A_1}(r\xi)$ yields $\mathscr{Z}_A = \frac{1}{r}\mathscr{Z}_{A_1}$. Set $A^* = A_1^*$. Clearly A^* is a direct summand of $\mathbb{Z}_k^+$ because it is a direct summand of $\mathbb{Z}_{\frac{k}{r}}^+$, and we have

$$A - A = r(A_1 - A_1) \subseteq r \cdot \frac{k}{r}\mathscr{Z}_{A_1^*} \cup \{0\} = k\mathscr{Z}_{A^*} \cup \{0\},$$

and

$$A^* - A^* = A_1^* - A_1^* \subseteq \frac{k}{r}\mathscr{Z}_{A_1} \cup \{0\} = k\mathscr{Z}_A \cup \{0\}.$$

Case 2. $1 \in A$. Then $A = \mathbb{Z}_r^+ \oplus rA_1$ for some $r > 1$, $r \mid k$ and direct summand A_1 of $\mathbb{Z}_{\frac{k}{r}}^+$. By the hypothesis there exists a direct summand A_1^* of $\mathbb{Z}_{\frac{k}{r}}^+$ such that (4.1) holds for A_1, A_1^* and $n = k/r$. Set $A^* = A_1^* \oplus \frac{k}{r}\mathbb{Z}_r^+$. A^* is a direct summand of $\mathbb{Z}_k^+$ because $A^* \oplus B_1^* = \mathbb{Z}_k^+$ where $A_1^* \oplus B_1^* = \mathbb{Z}_{\frac{k}{r}}$. We have

$$f_A(\xi) = f_{\mathbb{Z}_r^+}(\xi) f_{A_1}(r\xi), \qquad f_{A^*}(\xi) = f_{A_1^*}(\xi) f_{\mathbb{Z}_r^+}\left(\tfrac{k}{r}\xi\right).$$

It follows from $\mathscr{Z}_{\mathbb{Z}_r^+} = \frac{1}{r}\mathbb{Z} \setminus \mathbb{Z}$ that

$$\mathscr{Z}_A = \frac{1}{r}(\mathbb{Z} \cup \mathscr{Z}_{A_1}) \setminus \mathbb{Z}, \qquad \mathscr{Z}_{A^*} = \mathscr{Z}_{A_1^*} \cup \frac{r}{k}\left(\tfrac{1}{r}\mathbb{Z} \setminus \mathbb{Z}\right). \tag{4.2}$$

Let $m = a + \frac{k}{r}j$ and $m = a' + \frac{k}{r}j'$ be two distinct elements in A^*, where $a, a' \in A_1^*$ and $j, j' \in \mathbb{Z}_r^+$. If $a = a'$ then

$$m - m' = \frac{k}{r}(j - j') \in k\left(\tfrac{1}{r}\mathbb{Z} \setminus \mathbb{Z}\right) \subseteq k\mathscr{Z}_A.$$

If $a \neq a'$ then $a - a' \in \frac{k}{r}\mathscr{Z}_{A_1}$. Hence $a - a' + \frac{k}{r}l \in \frac{k}{r}\mathscr{Z}_{A_1}$ for all $l \in \mathbb{Z}$. Since $m - m' \notin k\mathbb{Z}$, we have

$$m - m' \in \tfrac{k}{r}\mathscr{Z}_{A_1} \setminus k\mathbb{Z} \subseteq k\mathscr{Z}_A.$$

Hence $A^* - A^* \subseteq k\mathscr{Z}_A \cup \{0\}$.

Now let $m = j + ra$, $m' = j' + ra'$ be two distinct elements in A, where $a, a' \in A_1$ and $j, j' \in \mathbb{Z}_r^+$. If $j = j'$ then $a \neq a'$, and by the hypothesis $a - a' \in \frac{k}{r}\mathscr{Z}_{A_1^*}$. So $m - m' = r(a - a') \in k\mathscr{Z}_{A_1^*}$. If $j \neq j'$ then $j - j' \notin r\mathbb{Z}$, so

$$m - m' = j - j' + r(a - a') \in \mathbb{Z} \setminus r\mathbb{Z} = \frac{r}{k}\left(\tfrac{1}{r}\mathbb{Z} \setminus \mathbb{Z}\right) \subseteq \mathscr{Z}_{A^*}.$$

Hence $A - A \subseteq \mathscr{Z}_{A^*}$.

We have now completed the induction steps and proven the lemma.

We will call two direct summand A and A^* satisfying (4.1) a *conjugate pair*, and A^* a *conjugate* of A. The proof of Lemma 4.1 leads to an explicit construction of conjugate pairs. Let $A \subset \mathbb{Z}^+$ be a direct summand of $\mathbb{Z}_n^+$. Then by Corollary 2.3 there exists a unique sequence $r_0, r_1, \ldots, r_{2k+1}$ in $\mathbb{N}$ with $\prod_{j=0}^{2k+1} r_j = n$, $r_j > 1$ for $0 < j < 2k + 1$ and $r_0, r_{2k+1} \geq 1$, such that

$$A = \bigoplus_{j=0}^{k} d_{2j}\mathbb{Z}_{r_{2j+1}}^+, \qquad \text{where} \quad d_m := \prod_{j=0}^{m} r_j. \tag{4.3}$$

Define the map ϑ_n on the set of direct summand of Z_n^+ by

$$\vartheta_n(A) = \bigoplus_{j=0}^{k} \frac{n}{d_{2j+1}} \mathsf{Z}^+_{r_{2j+1}}. \tag{4.4}$$

Then $\vartheta_n(A)$ is exactly the conjugate set A^* constructed inductively in the proof of Lemma 4.1.

LEMMA 4.2. *Suppose that $A \subset \mathsf{Z}^+$ is a direct summand of Z_n^+. Then A and $\vartheta_n(A)$ form a conjugate pair, and $\vartheta_n(\vartheta_n(A)) = A$. Furthermore, if $A, B \subset \mathsf{Z}^+$ such that $A \oplus B = \mathsf{Z}_n^+$, then $\vartheta_n(A) \oplus \vartheta_n(B) = \mathsf{Z}_n^+$*

PROOF. The proof of Lemma 4.1 already implies that A, $\vartheta_n(A)$ form an conjugate pair. It is easy to see that $\vartheta_n(\vartheta_n(A)) = A$ by directly applying (4.3) and (4.4). Now, suppose that A is given by (4.3) and $B \subset \mathsf{Z}^+$ satisfies $A \oplus B = \mathsf{Z}_n^+$. Then there are several cases: $r_0 = 1$ or $r_0 > 1$, and $r_{2k+1} = 1$ or $r_{2k+1} > 1$. If $r_0 = 1$, $r_{2k+1} > 1$ then

$$B = \bigoplus_{j=1}^{k+1} d_{2j-1} \mathsf{Z}^+_{r_{2j}}, \qquad \text{where} \quad r_{2k+2} := 1. \tag{4.5}$$

So

$$\vartheta_n(B) = \bigoplus_{j=1}^{k+1} \frac{n}{d_{2j}} \mathsf{Z}^+_{r_{2j}}. \tag{4.6}$$

It is now straightforward to check from (4.4) and (4.6) that $\vartheta_n(A) \oplus \vartheta_n(B) = \mathsf{Z}_n^+$. Other cases can be checked similarly.

DEFINITION 4.3. Let $\Lambda, \mathscr{T} \subset \mathsf{R}$ be strongly periodic sets. We say that $\mathscr{T}$ is a *dual* of Λ if there exist a non-zero $\alpha \in \mathsf{R}$ and $A, B \subset \mathsf{Z}^+$ with $A \oplus B = \mathsf{Z}_n^+$ for some $n \in \mathsf{N}$ such that

$$\Lambda = \alpha(A \oplus n\mathsf{Z}), \qquad \mathscr{T} = \frac{1}{n\alpha}\big(\vartheta_n(B) \oplus n\mathsf{Z}\big).$$

By Lemma 4.2 if $\mathscr{T}$ is a dual of Λ then Λ is a dual of $\mathscr{T}$.

LEMMA 4.4. *Let $\Omega \subset \mathsf{R}$ satisfy $\mu(\Omega) = n \in \mathsf{N}$. Suppose that $\Lambda = L \oplus \mathsf{Z}$ where L is a finite subset of R such that $\Lambda - \Lambda \subseteq \mathscr{Z}_\Omega \cup \{0\}$. Then (Ω, Λ) is a spectral pair if and only if $|L| = n$.*

PROOF. See [10], Theorem 1, or [8], Theorem 2.1.

We shall establish the following result, which is a stronger version of our main theorem.

THEOREM 4.5. *Suppose that* $\Omega \subset \mathsf{R}$ *has positive and finite Lebesgue measure. Let* $\Lambda, \mathscr{T} \subset \mathsf{R}$ *be strongly periodic sets such that* $\mathscr{T}$ *is a dual of* Λ. *Then* (Ω, Λ) *is a spectral pair if and only if* Ω *tiles* R *by translates of* $\mathscr{T}$.

PROOF. Without loss of generality we may assume that $\Lambda = \frac{1}{n}(A \oplus n\mathsf{Z})$ and $\mathscr{T} = \vartheta_n(B) \oplus n\mathsf{Z}$ for some $n \in \mathsf{N}$ and $A, B \subset \mathsf{Z}^+$ with $A \oplus B = \mathsf{Z}_n^+$.
($\Leftarrow$) The set $\Omega' = \Omega \oplus \vartheta_n(B)$ tiles R by translates of $n\mathsf{Z}$, so it is a fundamental domain of the lattice $n\mathsf{Z}$. Hence

$$\mathscr{Z}_{\Omega'} = \mathscr{Z}_\Omega \cup \mathscr{Z}_{\vartheta_n(B)} \supseteq \frac{1}{n}\mathsf{Z} \setminus \{0\}.$$

Since $\vartheta_n(A) \oplus \vartheta_n(B) = \mathsf{Z}_n^+$ we have

$$\mathscr{Z}_{\vartheta_n(A)} \cup \mathscr{Z}_{\vartheta_n(B)} = \mathscr{Z}_{\mathsf{Z}_n^+} = \frac{1}{n}\mathsf{Z} \setminus \mathsf{Z}.$$

Furthermore, $\mathscr{Z}_{\vartheta_n(A)} \cap \mathscr{Z}_{\vartheta_n(B)} = \emptyset$ because $f_{\vartheta_n(A)}(\xi) f_{\vartheta_n(B)}(\xi)$ has no multiple roots. Hence

$$\mathscr{Z}_\Omega \supseteq \mathscr{Z}_{\vartheta_n(A)} \cup \mathsf{Z} \setminus \{0\}.$$

Now, for any distinct $\lambda, \lambda' \in \Lambda$ we have $\lambda - \lambda' = \frac{1}{n}k + j$ for some $k \in A - A$, $j \in \mathsf{Z}$. If $k \neq 0$ then $\frac{k}{n} \in \mathscr{Z}_{\vartheta_n(A)}$ by (4.1), which implies that $\lambda - \lambda' = \frac{k}{n} + j \in \mathscr{Z}_{\vartheta_n(A)} \subseteq \mathscr{Z}_\Omega$. Otherwise $\lambda - \lambda' = j \in \mathsf{Z} \setminus \{0\} \subseteq \mathscr{Z}_\Omega$. By Lemma 4.4 (Ω, Λ) is a spectral pair.

($\Rightarrow$) Suppose that (Ω, Λ) is a spectral pair. For any $x \in [0, 1)$ let $D_x := \Omega \cap (\mathsf{Z} + x)$. It follows from [10], Theorem 2, that

$$|D_x| = |A|, \qquad D_x - D_x \subseteq n\mathscr{Z}_A \cup \{0\} \tag{4.7}$$

for almost all $x \in [0, 1)$. We show that $(D_x - x) + \vartheta_n(B)$ is a complete residue system $(\bmod\, n)$ for every D_x satisfying (4.7). Note that $\vartheta_n(B) - \vartheta_n(B) \subseteq n\mathscr{Z}_B \cup \{0\}$, and observe that $k \not\equiv m \bmod n$ for any $k \in n\mathscr{Z}_A$ and $m \in n\mathscr{Z}_B$. Thus for any $k_1, k_2 \in D_x - x$ and $m_1, m_2 \in \vartheta_n(B)$ we must have $k_1 - k_2 \not\equiv m_2 - m_1 \bmod n$ unless $k_1 = k_2$ and $m_1 = m_2$. Hence $k_1 + m_1 \not\equiv k_2 + m_2 \bmod n$. Since $|D_x - x| \cdot |\vartheta_n(B)| = n$ it follows that $(D_x - x) + \vartheta_n(B) = (D_x - x) \oplus \vartheta_n(B)$ contains n distinct residue classesmod n, and hence is a complete residue systemmod n. Therefore

$$D_x + \mathscr{T} = D_x \oplus \mathscr{T} = x + \mathsf{Z}$$

for almost all $x \in [0, 1)$. This implies that Ω tiles R by translates of $\mathscr{T}$.

Theorem 1.1 is a simple consequence of Theorem 3.1 and Theorem 4.5.

ACKNOWLEDGEMENT. The first author would like to thank Palle E. T. Jorgensen for many useful conversations about spectral pairs.

REFERENCES

1. Carlitz and Moser, *On some special factorizations of* $(1-x^n)/(1-x)$, Canad. Math. Bull. 9 (1966), 421–426.
2. de Bruijn, N. G., *On number systems*, Nieuw Arch. Wisk. (4) 4 (1956), 15–17.
3. Fuglede, B., *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal. 16 (1974), 101–121.
4. Jorgensen, P. E. T., and Pedersen, S., *Spectral theory for Borel sets in* R^n *of finite measure*, J. Funct. Anal. 107 (1992), 72–104.
5. Jorgensen, P. E. T., and Pedersen, S., *Harmonic analysis and fractal limit-measures induced by representations of a certain* C^**-algebra*, J. Funct. Anal. 125 (1994), 90–110.
6. Jorgensen, P. E. T., and Pedersen, S., *Orthogonal harmonic analysis of fractal measures*, Electron. Res. Announc. Amer. Math. Soc. 4 (1998), 35–42.
7. Lagarias, J. C. and Wang, Y., *Tiling the line with translates of one tile*, Invent. Math. 124 (1996), 341–365.
8. Lagarias, J. C. and Wang, Y., *Spectral sets and factorizations of finite abelian groups*, J. Funct. Anal. 145 (1997), 73–98.
9. Odlyzko, A. M., *Non-negative digit sets in positional number systems*, Proc. London Math. Soc. 37 (1978), 213–229.
10. Pedersen, S., *Spectral sets whose spectrum is a lattice with a base*, J. Funct. Anal. 141 (1996), 496–509.

STEEN PEDERSEN
DEPARTMENT OF MATHEMATICS
WRIGHT STATE UNIVERSITY
DAYTON OH 45435
USA
E-mail: steen@math.wright.edu

YANG WANG
DEPARTMENT OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA GA 30332
USA
E-mail: wang@math.gatech.edu