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ON NETS AND FILTERS

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1. Introduction.

In the theory of convergence classes as described in [3] four axioms are needed, while the corresponding filter version only needs three. Yet there is a natural correspondance between the filter axioms and three of the net axioms. On the basis of this E. M. Alfsen raised the question whether the fourth net axiom could be deduced from the three other ones. After a suitable adjustment of the definition of subnet this is indeed the case. In this paper we give a definition of subnet which meets this purpose. In addition we shall take time to remove some irregularities concerning nets and subnets.

Unless explicity stated we use the notation of [3]. A *net* is a function S defined on a directed set D. We shall write (S,D) for nets instead of $(S,\geqq)$ or $\{S_n, n\in D, \geqq\}$. If the specification of the domain of a net is not needed expressions like "the net S", "T is a subnet of S" etc. will be used. ω denotes the set of non-negative integers with the usual order. If D is a directed set and $n\in D$, we write $D_n=\{p\in D : p\geqq n\}$. By $\mathscr{F}_S$ we mean the filter generated by the net (S,D), that is the filter with base $\{S[D_n]\}_{n\in D}$. Finally, we occasionally write $S(n)$ instead of S_n for the value of S at $n\in D$.

In standard textbooks on general topology (cf. f.ex. [3]) the concept of a subnet is defined in the following way: Let (S,D) be a net in some set X. The net (T,E) is called a *subnet* of (S,D) if there exsists a function $N:E\to D$ such that

(i) $T=S\circ N$,

(ii) for each $n\in D$ there exists $m\in E$ such that $N(p)\geqq n$ for each $p\in E$, $p\geqq m$.

This definition is due to E. H. Moore [4], and we shall refer to a subnet of this type as an M-*subnet*.

The main property of any reasonable definition of a subnet is that it entails that if a net is eventually in a set A, then any subnet is also eventually in A. This is of course true for M-subnets. There are, however, some irregularities in the correspondence between nets/M-subnets and

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filters/subfilters (i.e. finer filters). We have already mentioned the problem concerning convergence classes. We point out two other inconveniences:

(a) Let S and T be two nets in a set X. It seems reasonable to call S and T equivalent if they are M-subnets of each other. Then they generate the same filter. However, two nets can generate the same filter without being M-subnets of each other.

(b) The usual definition of ultranet ("universal net" in [3]) involves a property of ultrafilters: S is an ultranet if for each subset A of X, S is eventually in A or in $X \setminus A$. Moreover, an ultrafilter has no proper subfilters, an ultranet may have proper M-subnets.

For a rather extensive list of papers related to the present subject we refer the reader to [1].

2. Subnets.

2.1 DEFINITION. Let (S,D) be a net in a set X. The net (T,E) in X is called a *subnet* of (S,D) if for each $n \in D$ there exists $m \in E$ such that $T[E_m] \subset S[D_n]$.

We first take care of the irregularity mentioned in (a) in the introduction.

2.2. LEMMA. *Let $(S,D$ and (T,E) be two nets in a set X. The following two statements are equivalent:*

(i) *T is a subnet of S.*

(ii) *If S is eventually in a subset A of X, then T is also eventually in A.*

PROOF. Obvious.

We shall say that two nets are *equivalent* if they are subnets of each other. Then the above lemma immediately gives the following corollary.

2.3. COROLLARY. *Two nets S and T are equivalent if and only if they generate the same filter, i.e. S and T are eventually in the same sets.*

We now turn to some comments on definition 2.1. Clearly, if T is an M-subnet of S, it is also a subnet in our sense. Thus the standard results on subnets and cluster points (whose proofs are based on the existence of M-subnets) carry over (cf. [3, pp. 70–71]). The main feature in definition 2.1 is, of course, the absence of the function N in the definition of an M-subnet. Condition (i) in the latter definition implies that $T[E] \subset S[D]$, while our definition impose no constraints whatsoever on the initial terms of T. The following trivial example exposes this point quite clearly.

2.4. Example. Let $D=E=\omega$ and put $S_n=T_n=0$ for $n \geqq 1$ and $S_0=-1, T_0=1$. Then evidently S and T are equivalent, nevertheless neither is an M-subnet of the other since $T[E] \setminus S[D]$ and $S[D] \setminus T[E]$ are both non-empty.

From this example one may perhaps get the impression that definition 2.1 is only an adjustment of the definition of an M-subnet necessary to take care of the case of different initial terms in the two nets in question. This is not so. The next example will reveal more clearly the role played by the function N.

2.5. Example. Let (S,ω) be a sequence in a set X. Let F be the set of all functions $f:\omega \to \omega$ and put $E=\omega \times F$. F is given the usual pointwise order and E the product order, that is $(m,g) \geqq (n,f)$ if $m \geqq n$ and $g \geqq f$. We now define $T:E \to X$ by

$$T(n,f) = S(n) .$$

Then evidently the two nets S and T are equivalent, because for each $n \in \omega$ we have (with an arbitrary $f \in F$)

$$T[E_{(n,f)}] = S[\omega_n] .$$

T is also clearly an M-subnet of S, (it is trivial to verify that $N:E \to \omega$ given by $N(n,f)=n$ does the job). But S is not an M-subnet of T. Indeed, suppose that this is the case. Then there is a function $N:\omega \to E$ satisfying (i) and (ii) in the definition of an M-subnet. (Actually, condition (i) will not be needed.) Put $N(n)=(m_n,f_n)$ and let $(p,g) \in E$ be given. By condition (ii) there exists $n_0 \in \omega$ such that $(m_n,f_n) \geqq (p,g)$ if $n \geqq n_0$. In particular we have $f_n \geqq g$ for $n \geqq n_0$. But this immediately yields a contradiction, for if we define $g:\omega \to \omega$ by

$$g(n) = f_n(n)+1 ,$$

the inequality $f_n \geqq g$ is impossible for each $n \in \omega$.

We include some remarks on subsequences. In [3] (T,ω) is called a subsequence of (S,ω) if T is an M-subnet of S. In this case we shall say that T is an M-*subsequence* of S. If the function N is strictly isotone, that is $N_m > N_n$ if $m > n$, we call T an *ordinary subsequence* of S. We shall use the following definition: (T,ω) is called a *subsequence* of (S,ω) (or, more generally, of the net (S,D)) if T is a subnet of S. Clearly, a subsequence in this sense need not be an M-subsequence, cf. example 2.4. This also means that we have to abandon the notation $\{S_{n_k}\}$ for subsequences. At this point the reader may possibly feel a shiver of disgust at our tampering with the definition of the almost sacrosanct concept of an ordinary subsequence. We therefore hasten to point out that our definition only serves to remove some purely *formal* nuisances. For practical puposes it is quite immatrial which one of the three definitions one uses, and thus one can stick to ordinary subsequences. This statement is made precise in our next proposition.

2.6. PROPOSITION. *Let (T,ω) be a subsequence of (S,ω). Then there exists an ordinary subsequence U of S such that T and U are equivalent.*

PROOF. There exists $m \in \omega$ such that $T[\omega_m] \subset S[\omega]$. Now, fix $x \in T[\omega_m]$. If $T^{-1}[x]$ is infinite, so is $S^{-1}[x]$ since T is a subnet of S. In this case we put $N_x = S^{-1}[x]$. If on the other hand $T^{-1}[x]$ is finite, we put $N_x = \{p\}$ where p is the first member of $S^{-1}[x]$. Then

$$N = \bigcup \{N_x : x \in T[\omega_m]\}$$

is an infinite subset of ω Let n_k be the kth element of N (in the order induced from ω) and put $U_k = S_{n_k}$. Then clearly (U,ω) is an ordinary subsequence of S. Furthermore, we observe that $U[\omega] = T[\omega_m]$, and if x is in this set, then the sets $U^{-1}[x]$ and $T^{-1}[x]$ are both finite or both infinite. Now, let $p \in \omega$, $p > m$, be given. For any of the finitely many $x \in T[\omega_m] \setminus T[\omega_p]$ we necessarily have that $T^{-1}[x]$ is finite, but then $U^{-1}[x]$ is also finite. Thus, for sufficiently large $k \in \omega$ we must have $U[\omega_k] \subset T[\omega_p]$, and we have proved that U is a subnet of T. In the same way we show that T is also a subnet of U, and the proof is complete.

3. Ultranets.

We now turn to the investigation of ultranets. If S and T are two nets in a set X such that T is a subnet of S, but S is not a subnet of T, we call T a *proper* subnet of S.

3.1. DEFINITION. A net S is called an *ultranet* if it has no proper subnets.

The simplest examples of ultranets are, of course, those corresponding to trivial ultrafilters: If $x \in X$ we define $S^x : \{1\} \to x$ by $S^x(1) = x$. Then, (giving $\{1\}$ the only possible order), S^x is evidently equivalent with any of its subnets. We call S^x a *trivial ultranet*. It should be noted that each subnet T of S^x is also an M-subnet, but usually S^x is not an M-subnet of T. Thus ultranets may have proper M-subnets.

3.2. PROPOSITION. *Each net S has a subnet T which also is an ultranet.*

PROOF. Let $\mathscr{S}$ denote the family of all subnets of S. $\mathscr{S}$ is not empty since $S \in \mathscr{S}$. If $T_1, T_2 \in \mathscr{S}$ we write $T_1 \succ T_2$ if T_1 is a subnet of T_2. Then $\succ$ is a partial order on $\mathscr{S}$. Let $\{(T_i, E^i)\}_{i \in I}$ be a chain in $\mathscr{S}$. We put

$$E = \{E^i{}_{m_i} : m_i \in E^i,\ i \in I\}$$

and order E by defining

$$E^j{}_{m_j} \geqq E^i{}_{m_i} \quad \text{if} \quad T_j \succ T_i \text{ and } T_j[E^j{}_{m_j}] \subset T_i[E^i{}_{m_i}]\,.$$

Given $E^i{}_{m_i}$ and $E^j{}_{m_j}$ and supposing $T_j \succ T_i$ we can find $m_j' \in E^j$ such that

$$m_j' \geqq m_j \quad \text{and} \quad T_j[E^j{}_{m_j'}] \subset T_i[E^i{}_{m_i}]\,.$$

This means that $E^j{}_{m_j'} \geqq E^i{}_{m_i}$, $E^j{}_{m_j'} \geqq E^j{}_{m_j}$, thus $\geqq$ directs E. We now define $T^* : E \to X$ by

$$T^*(E^i{}_{m_i}) = T_i(m_i)\,.$$

For $E^j{}_{m_j} \geqq E^i{}_{m_i}$ we then have $T^*(E^j{}_{m_j}) \in T_i[E^i{}_{m_i}]$. Thus $T^* \succ T^i$ for all $i \in I$, that is T^* is an upper bound for the chain $\{T_i\}$ since T^* is clearly a subnet of S. Thus Zorn's lemma applies and there exists a maximal element T in $\mathscr{S}$. T is then clearly an ultranet, and the proof is complete.

The above proof illustrates clearly how we may adopt ultrafilter proofs to obtain corresponding statements about ultranets. We give another typical example. *If an ultranet S in a topological space X has a cluster point x, then S converges to x.* For there exists a subnet T of S converging to x. Since S is an ultranet S is also a subnet of T, and the result follows.

We now give a characterization of ultranets showing that they are identical with the universal nets in [3].

3.3 PROPOSITION. *A net S in X is an ultranet if and only if for each subset A of X, S is eventually in either A or $X \setminus A$.*

PROOF. We first assume that there exists a subset A of X such that S is frequently in both A and $X \setminus A$. Then $S \mid S^{-1}[A]$ is a proper subnet of S, and thus S is not an ultranet. Conversely, let (T,E) be a proper subnet of (S,D) and assume that S is eventually in A or $X \setminus A$ for any $A \subset X$. Since T is a proper subnet of S there exists $m \in E$ such that S is not eventually in $T[E_m]$. Then S is eventually in $X \setminus T[E_m]$, that is $S[D_n] \cap T[E_m] = \emptyset$ for some $n \in D$. This is obviously absurd since T is a subnet of S.

4. Convergence classes.

In [3] convergence classes are defined in the following way: Let X be a (non-empty) set. A class $\mathscr{C}$ of ordered pairs (S,x) where S is a net in X and x a point in X, is called a *convergence class* for X if it satisfies the conditions listed below. For convenience we write $S \to x(\mathscr{C})$ or $\lim_{n \in D} S_n \equiv x(\mathscr{C})$ if $(S,x) \in \mathscr{C}$, $S \not\to x(\mathscr{C})$ if $(S,x) \notin \mathscr{C}$.

(a) If (S,D) is a net and $S_n = x$ for each $n \in D$, then $S \to x(\mathscr{C})$.

$(\mathrm{b})_{\mathrm{M}}$ If $S \to x(\mathscr{C})$, then $T \to x(\mathscr{C})$ for each M-subnet T of S.

$(\mathrm{c})_{\mathrm{M}}$ If $S \not\to x(\mathscr{C})$, then there is an M-subnet T of S such that $U \not\to x(\mathscr{C})$ for any M-subnet U of T.

(d) Let D be a directed set, let E^m be a directed set for each $m \in D$, let $F = D \times \times\{E^m : m \in D\}$ and for $(m,f) \in F$ let $R(m,f) = (m, f(m))$. If

$$\lim_{m \in D} \lim_{n \in E^m} S(m,n) \equiv x(\mathscr{C}) ,$$

then $S \circ R \to x(\mathscr{C})$.

If we change $(\mathrm{b})_{\mathrm{M}}$ and $(\mathrm{c})_{\mathrm{M}}$ by replacing M-subnets by subnets the resulting axioms are labelled (b) and (c), respectively.

In the definition of a convergence class of pairs $(\mathscr{F}, x)$ where $\mathscr{F}$ is a filter on X we require:

(A) $\mathscr{F}_x \to x(\mathscr{C})$ (here $\mathscr{F}_x$ is the filter with base $\{x\}$).

(B) If $\mathscr{F} \to x(\mathscr{C})$ and $\mathscr{G} \supset \mathscr{F}$, then $\mathscr{G} \to x(\mathscr{C})$.

(D) Let D be a directed set and suppose that $\mathscr{F}_n \to S_n(\mathscr{C})$ for each $n \in D$. Let $\mathscr{F}_S$ be the filter generated by the net S thus defined and suppose that $\mathscr{F}_S \to x(\mathscr{C})$. Then

$$\liminf_{n \in D} \mathscr{F}_n = \bigcup_{m \in D} \bigcap_{n \geq m} \mathscr{F}_n \to x(\mathscr{C}) ,$$

These axioms were given to us by E. M. Alfsen (oral communication). Another set of axioms for convergence classes using filters can be found in [2].

We obviously have a correspondence between (a) and (A), $(\mathrm{b})_{\mathrm{M}}$ and (B), and finally between (d) and (D). However, $(\mathrm{c})_{\mathrm{M}}$ is not deducible from (a), $(b)_M$ and (d). This is shown by the following rather trivial example: Let X be a set with more than one point and let $\mathscr{C}$ consist of all (S,x) such that $S_n = x$ for all n. Then (a), $(b)_M$ and (d) are trivially satisfied. Now, let $x,y \in X$, $x \neq y$, and define a sequence (S,ω) by $S_0 = y$, $S_n = x$ for $n \geqq 1$. Then $S \not\rightarrow x(\mathscr{C})$, but clearly any M-subnet T of S must have an M-subnet U such that $U \rightarrow x(\mathscr{C})$. Thus $(c)_M$ is not satisfied.

Our next goal is to prove that (c) follows from (a), (b), and (d). On the basis of this fact it seems natural to take (a), (b), and (d) as axioms for a convergence class. To see that $\mathscr{C}$-convergence is then a topological convergence we can repeat the proof of the corresponding theorem in [3, pp. 74–75] almost verbatim.

We start with a lemma which is the net analogue of the statement that to a given family $\{\mathscr{F}_\alpha\}$ of filters on a set X, $\bigcap_\alpha \mathscr{F}_\alpha$ is the finest filter coarser than every $\mathscr{F}_\alpha$.

4.1. Lemma. *Let $\mathscr{S}$ be a family of nets in a sat X. Then there exists a net T in X satisfying:*

1) *S is a subnet of T for every $S \in \mathscr{S}$.*
2) *If every $S \in \mathscr{S}$ is a subnet of some net R, then T is also a subnet of R.*

Proof. Let $\mathscr{F} = \bigcap_{S \in \mathscr{S}} \mathscr{F}_S$ and define

$$E = \{(F,x) : F \in \mathscr{F}, x \in F\} .$$

We then direct E by $(G,y) \geqq (F,x)$ if $G \subset F$. Now define the net (T,E) by $T(F,x) = x$. Clearly $T[E_{(F,x)}] = F$, thus $\mathscr{F}_T = \mathscr{F}$. For any $S \in \mathscr{S}$ we then have $\mathscr{F}_S \supset \mathscr{F}_T$, and we conclude from lemma 2.2 that S is a subnet of T. Similarly, if S is a subnet of R for every $S \in \mathscr{S}$, we have $\mathscr{F}_T = \bigcap \mathscr{F}_S \supset \mathscr{F}_R$, and it follows that T is a subnet of R.

If $\mathscr{S}$ is the family of all nets S in X such that $S \rightarrow x(\mathscr{C})$, we write S_x for the net T constructed in the previous lemma. Our next lemma requires (a) and (d), but not (b)

4.2. Lemma. *Let D be an index set and let $\{(S^m, E^m)\}_{m \in D}$ be a family of nets such that $S^m \rightarrow x(\mathscr{C})$ for each $m \in D$. Then there exists a net T_x such that*

1) $T_x \rightarrow x(\mathscr{C})$,
2) *S^m is a subnet of T_x for each $m \in D$.*

Proof. Direct D by the trivial order, $m \geqq n$ for all $m,n \in D$. Using (a) we then have

$$\lim_{m \in D} \lim_{n \in E^m} S^m(n) \equiv x(\mathscr{C}) .$$

(d) then ensures the existence of a net $T_x = S \circ R$ such that $T_x \to x(\mathscr{C})$ (where $S(m,n) \equiv S^m(n)$). The domain of T_x is the directed set $F = D \times \times \{E^m : m \in D\}$ and $T_x(m,f) = S(m,f(m)) = S^m(f(m))$ for $(m,f) \in F$. But then, given $(m_0,f_0) \in F$, we have

$$S^m[E^m{}_{f_0(m)}] \subset T_x[F_{(m_0,f_0)}] .$$

(Remember that for any $m \in D$ we have $m \geqq m_0$.) This means that S^m is a subnet of T_x, and the lemma is proved.

We now let $\{(S^m,E^m)\}$ above be the family $\mathscr{S}$ of all nets S such that $S \to x(\mathscr{C})$. It then follows from lemmata 4.1 and 4.2 that S_x must be a subnet of T_x. Using (b) we obtain the following important result:

4.3. Corollary. $S_x \to x(\mathscr{C})$.

Translating into filter terminology this corollary states that

$$\bigcap_{\mathscr{F} \to x(\mathscr{C})} \mathscr{F} \to x(\mathscr{C})$$

hence, speaking informally, the natural candidate for the neighbourhood filter at x does really converge to x.

4.4. Proposition. (a), (b), *and* (d) *imply* (c).

Proof. We have used (a), (b), and (d) to derive corollary 4.3. From (b) and the first part of lemma 4.1 we then infer that $S \to x(\mathscr{C})$ if and only if S is a subnet of S_x. To prove (c), assume that $S \not\to x(\mathscr{C})$. Then S is not a subnet of S_x. From lemma 2.2 it follows that there exists a subset A of X such that S_x is eventually in A while S is frequently in $X \setminus A$. Then the subnet $T = S \mid S^{-1}[X \setminus A]$ is eventually in $X \setminus A$ and so is any subnet U of T. Then U cannot be a subnet of S_x, and according to the above remark we infer that $U \not\to x(\mathscr{C})$. This proves (c).

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