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ON THE ANALYTIC VECTOR VARIANT OF THE HILLE-YOSIDA THEOREM

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Abstract.

We show that if the operators $(\lambda - A)^k$ have the same lower bounds as in the classical Hille-Yosida-Feller Theorem and A has a dense set of analytic vectors (i.e., vectors x for which $e^{tA}x$ as a power series are summable at least for small t's) then A is closable and $\bar{A}$ is the infinitesimal generator of a continuous semigroup. We also prove a variant of this result in locally convex spaces. We show the denseness of "super-analytic" vectors for a kind of one-parameter groups including the continuous groups on Banach spaces. Finally, we give an application about representations of Lie groups.

The classical Hille-Yosida-Feller theorem states that an operator A in a Banach space $\mathscr{X}$ is the generator of a so-called C_0-semigroup if and only if for large positive λ the resolvents $R_\lambda = (\lambda - A)^{-1}$ exist and satisfy the following estimate for $k = 1, 2, 3, \ldots$: $\|R_\lambda^k\| \cdot |\lambda - C|^k \leqq M$ with suitable constants M, C. The second condition can be reformulated as follows: there are M, $C \geqq 0$ and $\delta < \dfrac{1}{C}$ such that

(1) $M\|(I - \alpha A)^k x\| \geqq (1 - \alpha C)^k \cdot \|x\|$ for $\alpha \in (0, \delta)$, $k = 1, 2, 3, \ldots$ and $x \in \mathscr{D}(A^k)$.

It is not too hard to see that, having (1) satisfied, it is enough to know the existence of R_λ for one $\lambda > \dfrac{1}{\delta}$ $\left(\text{one should use the expansion } R_\mu = \sum_{k=0}^{\infty} (\lambda - \mu)^k R_\lambda^{k+1}\right)$. If we want a version of the theorem which does not use resolvents at all, we must stipulate some other thing about A. A possibility is to require the denseness of the set of analytic vectors for A (see detailed definitions below). It is stated in [1] that if A is a *closed* operator satisfying (1) and having a dense set of analytic vectors then A is a generator. The proof is included there only for the case $C = 0$, $M = 1$. It seems to be a good idea to get rid of the closedness condition, but (1) by itself presumably does not extend to the closure of A (even if it does, this is not easy to see). But if we know, in addition, the analytic vectors

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form a dense set then A turns out to be a pre-generator (see Theorem 1 below). Essentially the same theorem was proved (unknown to the author when preparing the manuscript) by J. Rusinek in 1983 (see [5]). This proof is very similar to ours. Nevertheless, for the sake of exposition, we shall prove it along with the LCS-version (Theorem 2).

Unfortunately, this analytic-vector-variant of the Hille-Yosida Theorem is not "if and only if" because a C_0-semigroup may not have any nonzero analytic vectors, as the example of the translation semigroup on $L^2(0, +\infty)$ shows (cf. [3], p. 600). But in the case of groups (i.e., strongly continuous representations of R) we have a dense set of analytic vectors, as it was shown by I. Gelfand in 1939. Moreover, using his method, we are able to prove the denseness of a kind of "super-analytic" vectors in a more general setting (see Theorem 3 below).

The author has not been able to prove the analogue of Theorem 1 if $\mathscr{X}$ is a general locally convex space rather than a Banach space but only a slightly weaker variant (see Theorem 2) which, in the light of Theorem 3, is enough to formulate an "if and only if" statement at least for *groups* "with uniform exponential growth."

The above-mentioned analogue seems to be true; we shall include some comment about it after the proofs.

For the convenience of the reader, this paper is relatively self-contained: we shall sketch the known proofs of some lemmas we need.

DEFINITIONS. A mapping $V: [0, +\infty) \mapsto B(\mathscr{X})$, where $\mathscr{X}$ is a Banach space and $B(\mathscr{X})$ is the set of the continuous linear operators, is called a C_0-semigroup if $V(t+s) = V(t)V(s)$, $V(0) = I$ and the functions $t \to V(t)x$ are continuous for all $x \in \mathscr{X}$.

If we assume $\mathscr{X}$ to be only a LCS (meaning locally convex Hausdorff space in this paper) then we require in addition that V be locally equicontinuous (i.e., the set of operators $V([0,t])$ be equicontinuous for all t) and call this a "cle" (continuous locally equicontinuous) semigroup.

We can see from the Banach-Steunhaus Theorem that a C_0-semigroup on a Banach space is a cle semigroup as well.

The *generator* of a cle semigroup is simply the strong derivative at 0:

$$Ax = \lim_{t \to 0} \frac{V(t)x - x}{t}.$$

For any linear operator A in a LCS $\mathscr{X}$ we define the s-analytic vectors of A as follows:

$$\mathscr{A}_s(A) := \left\{ x \in \mathscr{X};\ \text{the sequence} \left\{ \frac{t^n}{n!} A^n x \right\} \text{is bounded } \forall \text{ positive } t < s \right\}.$$

It is easy to see that $\mathscr{A}_s(A)$ is an A-invariant subspace. We shall call the union

$\mathscr{A}(A) = \bigcup_{s>0} \mathscr{A}_s(A)$ the set of *analytic* vectors, and the intersection $\mathscr{E}(A) = \bigcap_{s>0} \mathscr{A}_s(A)$ the set of *entire* vectors.

We shall say that V is a cle group if it is defined on R rather than R_+ and satisfies the corresponding conditions.

A cle group is said to be of *exponential growth* if there is a dense subset $\mathscr{X}_0$ of "exponential vectors" such that for any continuous seminorm p on $\mathscr{X}$ and for any $x \in \mathscr{X}_0$ we can find a constant K (depending on p and x) such that

(2) $$p(V(t)x) \leqq e^{K(|t|+1)} \quad \text{for all } t.$$

It is well known that a C_0-group on a Banach space is of exponential growth, namely with $\mathscr{X}_0 = \mathscr{X}$.

If V is a cle semigroup over the LCS $\mathscr{X}$ then we can define, in an obvious manner, a semigroup $\tilde{V}$ over the completion $\bar{\mathscr{X}}$ of $\mathscr{X}$. Denoting by $\tilde{A}$ the generator of $\tilde{V}$ we define the C^∞-space of V to be $\bigcap_{n=1}^{\infty} \mathscr{D}(\tilde{A}^n)$ in $\bar{\mathscr{X}}$ endowed with the topology defined by the following seminorms:

$$\{x \to p(\tilde{A}^n x);\ p \text{ is continuous seminorm in } \bar{\mathscr{X}},\ n = 0, 1, 2, 3, \ldots\}.$$

This is called the C^∞-topology.

THEOREM 1. *Assume A is an operator in a Banach space $\mathscr{X}$ such that, with a suitable positive constant M, real number C and $\delta > 0$ we have*

(1) $$M\|(I - \alpha A)^k x\| \geqq (1 - \alpha C)^k \|x\| \quad \text{for } \alpha \in (0, \delta),\ k = 1, 2, 3, \ldots \text{ and } x \in \mathscr{D}(A^k).$$

Assume further that $\mathscr{A}(A)$ is dense in $\mathscr{X}$. Then A is closable and $\bar{A}$ is the generator of a C_0-semigroup $V(t)$ such that $e^{-tC} \cdot V(t)$ is a bounded semigroup. Further, $\mathscr{A}(A)$ is dense in $C^\infty(V)$ with respect to the C^∞-topology.

THEOREM 2. *Let A be an operator in a* LCS *$\mathscr{X}$ such that, with suitable constants C and δ, we have*

(1′) *$\forall$ neighborhood of zero W $\exists$ a neighborhood of zero U such that $(1 - \alpha C)^{-k}(I - \alpha A)^k x \in U$ imply $x \in W$ for any $k \in 1, 2, 3, \ldots$, $x \in \mathscr{D}(A^k)$ and $\alpha \in (0, \delta)$.*

Assume further that $\mathscr{E}(A)$ is dense in $\mathscr{X}$. Then A is closable in $\bar{\mathscr{X}}$ and $\bar{A}$ is the generator of a cle semigroup V in $\mathscr{X}$ such that $e^{-tC}V(t)$ is an equicontinuous semigroup. Moreover, $\mathscr{E}(A)$ is dense in $C^\infty(V)$.

THEOREM 3. *Let V be a cle group of exponential growth on a sequentially complete* LCS *$\mathscr{X}$, and A be the generator of V. Then the set $\zeta := \{x \in \mathscr{X};$ the* *sequence* $\left\{\left(\frac{s}{\sqrt{k}}\right)^k \cdot A^k x\right\}$ *is bounded for some* $s > 0\}$ *is dense in* $\mathscr{X}$.

COROLLARY. *Then* $\mathscr{E}(A)$ *is dense, for* $\mathscr{E}(A) \supset \zeta$.

PROOF OF THEOREMS 1 AND 2. First we note if $\mathscr{F}$ is an index set, $\{x_j; j \in \mathscr{F}\}$ is a bounded set in $\bar{\mathscr{X}}$ and $(\lambda_j)_j$ is an absolutely summable family of complex numbers, then the finite sub-sums of $\sum \lambda_j x_j$ form a Cauchy-net, and therefore $\sum \lambda_j x_j \in \bar{\mathscr{X}}$ exists; moreover, taking any partition of the index set, the corresponding double summation $\sum_{i \in I}\left(\sum_{j \in \mathscr{F}_i} \lambda_j x_j\right)$ yields the same result. Furthermore, the mapping ρ: $l^1(\mathscr{F}) \mapsto \bar{\mathscr{X}}$, $\rho((\lambda_j)_j) := \sum_j \lambda_j x_j$ is continuous. These observations are the base of the proof of Lemmas 1 and 2 below.

$$\text{Let } B = A - C \cdot I. \tag{3}$$

Then it is easy to see that $\mathscr{A}_s(B) = \mathscr{A}_s(A)$ for all s. For $x \in \mathscr{A}_s(A)$ and $t \in [0, s)$ let $e^{tB}x := \sum_{k=0}^{\infty} \frac{t^k}{k!} B^k x$ $\left(\text{here the bounded set is } \left\{\frac{q^k}{k!} B^k x\right\} \text{ for some } t < q < s \text{ and } \left(\left(\frac{t}{q}\right)^k\right)_k \text{ is the summable sequence}\right)$.

LEMMA 1. *Let* $x \in \mathscr{A}_s(A)$. *Then for* $t \in \left[0, \frac{s}{2}\right)$ *we can find a sequence* $x_n(t) \in \mathscr{A}_s(A)$ *such that* $x_n(t) \to e^{tB}x$ *and* $\left(I - \frac{t}{n} B\right)^n x_n(t) \to x$.

PROOF. Fix a q such that $2t < q < s$ and let $g(j, k) := \binom{j+k}{j}\left(\frac{t}{q}\right)^{j+k}(-1)^k$, $f_n(j, k) = g(j, k) \cdot \prod_{i=0}^{k-1} \frac{n-i}{n}$ and finally $f_{n,m}(j, k) = \begin{cases} f_n(j, k) & \text{if } j \leqq m \\ 0 & \text{if } j > m \end{cases}$ where j, k are non-negative integers. Then clearly $g \in l^1(\mathsf{N}^2)$ and $|f_n| \leqq |g|$, and $\lim_{n \to \infty} f_{n,n}(j, k) = g(j, k)$, therefore $f_{n,n} \to g$ in $l^1(\mathsf{N}^2)$. Let $x_{j,k} = \frac{q^{j+k}}{(j+k)!} B^{j+k} x$. Then the function $\rho := l^1(\mathsf{N}^2) \mapsto \bar{\mathscr{X}}$, $\rho(f) = \sum f(j, k) x_{j,k}$ is continuous. Let $x_n(t) = \sum_{j=0}^{n} \frac{t^j}{j!} B^j x$ (this is in $\mathscr{A}_s(A)$ because $\mathscr{A}_s(A) = \mathscr{A}_s(B)$ is a B-invariant subspace); then $\left(I - \frac{t}{n} B\right)^n x_n(t) = \rho(f_{n,n})$ and therefore tends to $\rho(g) = x$.

COROLLARY. *The operators* $U_s(t) := e^{tB}|_{\mathscr{A}_s(A)}$, $2t < s$, *are equicontinuous.*

PROOF. Clearly (1) is the special case of (1′) from which we get $(I - \beta B)^k y \in U$ implies $y \in W$ if β is small enough. Writing $y = x_n(t)$ we have $e^{tB}x \in \bar{W}$ (in $\bar{\mathscr{X}}$) whenever $x \in \operatorname{int} U \cap \mathscr{A}_s(A)$ and $t \in \left[0, \frac{s}{2}\right)$.

LEMMA 2. *For* $t_1, t_2 \in \left[0, \frac{s}{2}\right)$, $x \in \mathscr{A}_s(A)$ *we have* $\overline{U_s(t_1)}e^{t_2 B}x = e^{(t_1+t_2)B}x$, *where* $\overline{U_s(t)}$ *is the closure of the continuous operator* $U_s(t)$ (*in* $\bar{\mathscr{X}} \times \bar{\mathscr{X}}$).

PROOF. Straightforward.

Because of Lemma 2 and the B-invariance of $\mathscr{A}_s(A)$ we have a cle semigroup $V_s(t)$ on $\mathscr{X}_s = \overline{\mathscr{A}_s(A)}$ (closure in $\bar{\mathscr{X}}$) by setting

$$V_s(t) = \left(U_s\left(\frac{t}{n}\right)\right)^n \text{ for any } n \text{ such that } \frac{t}{n} < \frac{s}{2}. \tag{4}$$

It is easy to check that $V_s(t) \subset V_r(t)$ if $s > r$.

Now if $\mathscr{E}(A)$ is dense (or even if all of the $\mathscr{A}_s$ are dense), then $\mathscr{X}_s = \bar{\mathscr{X}}$ for all s and $V_s(t)$ is an equicontinuous semigroup.

If we just know $\mathscr{A}(A)$ is dense, then we must work hard for a similar conclusion.

LEMMA 3. *Let* $\mathscr{Y}$ *be a sequentially complete* LCS, u: $[0, a] \mapsto \mathscr{Y}$ *a* C^1-*function,* T *an operator in* $\mathscr{Y}$ *such that*

$$(u(s), u'(s)) \in \overline{\operatorname{graph} T} \quad \text{for} \quad s \in [0, a] \tag{5}$$

and assume there is a $d > 0$ *such that for* $h \in (0, d)$ $I - hT$ *is injective and* $(I - hT)^{-1}$ *is extendible to an everywhere defined* K_h *such that* $\{K_h^k; h \in (0, d), k = 1, 2, 3, \ldots\}$ *is an equicontinuous set of operators.*

Then $\lim_{n \to \infty} (K_{r/n})^n u(0) = u(r)$ *uniformly for* $r \in [0, a]$.

REMARK. If T is the generator of a C_0-semigroup in a Banach space then this lemma is the classical result of E. Hille, i.e., for the sake of Theorem 1 only it is superfluous to prove it; nevertheless, we include the proof, thinking about a possible improvement of Theorem 2 in which the author has so far been unsuccessful.

PROOF. If $0 \leqq t < t + h \leqq a$ then

$$u(t + h) - hu'(t + h) = u(t) + \int_t^{t+h} (u'(s) - u'(t + h))ds.$$

Applying K_h to this equation and using (5) we get

$$u(t+h) - K_h u(t) = K_h \int_t^{t+h} (u'(s) - u'(t+h))ds$$

and hence we can infer (using u is C^1 and $\mathscr{Y}$ is a LCS) $\forall\, U$ neighborhood of zero in $\mathscr{Y}$ $\exists\, \varepsilon > 0$ such that $u(t+h) - K_h u(t) \in h \cdot K_h(U)$ whenever $h \leqq \varepsilon$. It is easy to deduce from this,

$$u(t+nh) - K_h^n u(t) \in h \sum_{j=1}^{n} K_h^j(U).$$

Substitute $t = 0$, $y = \frac{r}{n}$ and use the assumption on K_h^k and the fact $\mathscr{Y}$ is a LCS. The lemma is thus proved.

LEMMA 4. *$\mathscr{A}_s(A)$ is dense in $C^\infty(V_s)$ (with respect to the C^∞-topology).*

PROOF. Let $\mathscr{H}$ denote the closure of $\mathscr{A}_s(A)$ in $C^\infty(V_s)$. Let $x \in \mathscr{A}_s(A)$ and $t \in \left[0, \frac{s}{2}\right)$. Denote the generator of V_s by B_1. Clearly $B_1 \supset B|_{\mathscr{A}_s(A)}$. We want to show first $V_s(t)\,x \in \mathscr{H}$. To this end, consider $y_n = \sum_{k=0}^{n} \frac{t^k}{k!} B^k x$. Then $B_1^j\, y_n = \sum_{k=0}^{n} \frac{t^k}{k!}$ $B^{j+k}x$ and $B^j x \in \mathscr{A}_s(A)$, hence $B_1^j\, y_n \xrightarrow{\text{in } \mathscr{X}_s} e^{tB} B^j x = V_s(t)\, B_1^j x = B_1^j\, V_s(t)\, x$ since B_1 is the generator of V_s. This amounts to $y_n \xrightarrow{\text{in } C^\infty(V_s)} V_s(t)x$, but $y_n \in \mathscr{A}_s(A)$, hence $V_s(t)x \in \mathscr{H}$. Since $V_s(t)$ is clearly continuous on $C^\infty(V)$, we get $\mathscr{H}$ is $V_s(t)$-invariant for $t \in \left[0, \frac{s}{2}\right)$. But this is enough, because if we have a dense subspace which is invariant under the semigroup and is contained in the C^∞-space then that subspace must be C^∞-dense (this result was stated in [4] for Banach spaces and groups; but the easy proof works in general: one should consider a sequence $\varphi_n \in C_c^\infty((0, \infty))$ such that $\varphi_n \geqq 0$, $\int \varphi_n = 1$ and the supports then $y_{k,n} = \int_0^\infty \varphi_k(t)\, V_s(t)\, x_n\, dt \in \mathscr{H}$ and

$$B_1^j\, y_{k,n} \xrightarrow{\mathscr{X}_s} \int_0^\infty \varphi_k^{(j)}(t)(-1)^j\, V_s(t)x\, dt = B_1^j\, y_k$$

if $n \to \infty$, where $y_k = \int_0^\infty \varphi_k(t)\, V_s(t)x\, dt$, thus $y_k \in \mathscr{H}$ and $y_k \xrightarrow{C^\infty} x$).

COROLLARY. *With a suitable $d > 0$, the set $\{(I - \beta B_s)^{-k};\ \beta \in (0, d)$, $k = 1, 2, 3, \ldots$, $s > 0$, B_s is the restriction of the generator of V_s to $C^\infty(V_s)\}$ is equicontinuous (with respect to the original topology of $\bar{\mathscr{X}}$).*

PROOF. If $x_n \in \mathscr{A}_s(A)$, and $x_n \xrightarrow{C^\infty} x$ then $B_s^j x_n \xrightarrow{\mathscr{X}_s} B_s^j x$ for all j, hence $(1 - \beta B)^k x_n \to (I - \beta B_s)^k x$ for all k. Therefore $(I - \beta B_s)^k x \in \operatorname{int} U$ implies $x \in \bar{W}$ if U, W are taken from (1′), and β is small enough.

LEMMA 5. *If $\mathscr{X}$ is a Banach space then the $V_s(t)$ are equicontinuous,* $\| V_s(t) \| \leqq M$.

PROOF. Denote the generator of V_s by D. Now $\mathscr{X}_s$ is a Banach space, and hence by the classical theory $I - \beta D$ is surjective if β is small $\left((\lambda - D)^{-1} = \int_0^\infty e^{-\lambda t} V_s(t)\, dt \right)$.

We also have $(I - \beta D)^{-1} D \subset D(I - \beta D)^{-1}$ and hence $(I - \beta D)^{-1} D^k \subset D^k (I - \beta D)^{-1}$ for all k, which implies $(I - \beta D)^{-1}$ leaves $C^\infty(V_s)$ invariant. The same is true for $I - \beta D$, thus $I - \beta D$ is a bijection of $C^\infty(V_s)$ onto itself. By the former corollary, $\|(I - \beta B_s)^{-k}\| \leqq M$ but we now know $(1 - \beta B_s)^{-k} = (I - \beta D)^{-k}|_{C^\infty(V_s)}$ and $C^\infty(V_s)$ is dense in $\mathscr{X}_s$, so we have $\|(I - \beta D)^{-k}\| \leqq M$ which implies $\| V_s(t) \| \leqq M$ by Lemma 3.

Define the operators $T(t)$ on $\bigcup_{s>0} \mathscr{X}_s$ by setting $T(t)x = V_s(t)x$ if $x \in \mathscr{X}_s$. Now we can see this family of operators is equicontinuous. Let $V(t) = e^{tC} \cdot \overline{T(t)}$. Denote the generator of V by G. It remains to prove that $G = \bar{A}$ and $\mathscr{A}(A)$ (or $\mathscr{E}(A)$ in the second case) is dense in $C^\infty(V)$. Denote $A|_{\mathscr{A}(A)}$ by A_1, then $G \supset A_1$. We know that for $x \in \mathscr{A}_s(A)$ $V_s(t)x$ belongs to the closure of $\mathscr{A}_s(A)$ in $C^\infty(V_s)$. Hence $V(t)(\mathscr{A}(A)) \subset \overline{\mathscr{A}(A)}$ in $C^\infty(V)$. A repetition of the argument of Lemma 4 shows that $V_s(t)$ leaves the C^∞-closure of $\mathscr{E}(A)$ invariant, hence $V(t)$ does the same. Thus we can see $\mathscr{A}(A)$ (or $\mathscr{E}(A)$ if the conditions of Theorem 2 hold) is dense in $C^\infty(V)$ (cf. the proof of Lemma 4).

Since G is a continuous operator on $C^\infty(V)$, thus A_1 is C^∞-dense in $G|_{C^\infty(V)}$ which, in turn, is dense in G; therefore A_1 is dense in G which is closed, being the generator of a cle semigroup in a complete LCS. We can see now $G = \bar{A}_1$. Hence $(I - \alpha A_1)^{-1}$ is dense in $(I - \alpha G)^{-1}$ which is an everywhere defined continuous operator for small α (in the second case, too, for $e^{-Ct} V(t)$ is equicontinuous). On the other hand, $(I - \alpha A_1)^{-1} \subset (I - \alpha A)^{-1}$, which is a continuous operator for small α (by (1′)). Thus $(I - \alpha A)^{-1} \subset (I - \alpha G)^{-1}$, $A_1 \subset A \subset G$, $G = \bar{A}$.

PROOF OF THEOREM 3. Let $x \in \mathscr{X}_0$, the set of exponential vectors of V (see our Definitions). We assert that

$$x_c := \int_{-\infty}^{\infty} e^{-ct^2} V(t)x\, dt \in \zeta \text{ for any } c > 0. \tag{6}$$

On the other hand,

$$\sqrt{\frac{c}{\pi}}\, x_c \to x \text{ if } c \to \infty. \tag{7}$$

Clearly (6) and (7) yield Theorem 3. Note first that for any $s > 0$, $e^{-st^2} V(t)x$ is a bounded function in $\mathscr{X}$ because $x \in \mathscr{X}_0$ and e^{-st^2} decays more rapidly than $e^{-K(1+|t|)}$. Hence x_c exists as the Riemann-type integral of the bounded continuous function $e^{-st^2} V(t)x$ with respect to the finite measure $e^{(s-c)t^2} \cdot dt$ with some $s < c$. Then $\sqrt{\frac{c}{\pi}} x_c - x = \sqrt{\frac{c}{\pi}} \int_{-\infty}^{\infty} e^{-ct^2} (V(t)x - x) dt$, hence $p\left(\sqrt{\frac{c}{\pi}} x_c - x\right) \leqq \sqrt{\frac{c}{\pi}} \int_{-\infty}^{\infty} e^{-ct^2} p(V(t)x - x) dt$ for any continuous seminorm p, and $h(t) = p(V(t)x - x)$ is a continuous function of at most exponential growth and $h(0) = 0$. Hence $e^{-t^2} h(t)$ is bounded, continuous and vanishes at 0, while $\left\| \sqrt{\frac{c}{\pi}} e^{(1-c)t^2} \right\|_1 = \sqrt{\frac{c}{c-1}}$ and the vast majority of this is concentrated in a small neighborhood of zero if c is large. Thus (7) is proved.

If φ is any C^1-function such that the improper integrals $u = \int_{-\infty}^{\infty} \varphi(t) V(t)x \, dt$ and $v = \int_{-\infty}^{\infty} \varphi'(t) V(t)x \, dt$ exist and if $\varphi(t) V(t)x \to 0$ for $|t| \to \infty$ then it is not hard to see that $Au = -v$. Using this and the fact the derivatives of e^{-ct^2} are polynomial multiples of it, we infer $(-A)^n x_c = \int_{-\infty}^{\infty} (e^{-ct^2})^{(n)} V(t)x \, dt$, and $p(A^n x_c) \leqq \int_{-\infty}^{\infty} |(e^{-ct^2})^{(n)}| \cdot e^{K(|t|+1)} dt$. Since $e^{-st^2} e^{K(|t|+1)}$ is bounded for $s > 0$, we shall achieve (6) if we prove the following lemma.

LEMMA 6. *The sequence* $\dfrac{\|(e^{-ct^2})^{(n)} \cdot e^{st^2}\|_1^{1/n}}{\sqrt{n}}$ *is bounded for any* $0 \leqq s < c$.

REMARK. It would be enough to know this result for one s, and in the case $s < \frac{c}{2}$, $\|(e^{-ct^2})^{(n)} e^{st^2}\|_1 \leqq \|e^{(s-\frac{c}{2})t^2}\|_2 \cdot \|(e^{-ct^2})^{(n)} e^{\frac{ct^2}{2}}\|_2$ and the latter factor is exactly known from the theory of Hermite-functions. But it is possible to give an elementary proof as follows.

PROOF. Denote the polynomial $(e^{-\frac{t^2}{2}})^{(n)} e^{\frac{t^2}{2}}$ by $p_n(t)$. Then clearly

$$(e^{-ct^2})^{(n)} e^{st^2} = (2c)^{\frac{n}{2}} p_n(\sqrt{2c} \cdot t) e^{(s-c)t^2}. \tag{8}$$

On the other hand,

$$p_0 \equiv 1, \quad p_{n+1}(t) = p_n'(t) - t p_n(t). \tag{9}$$

Therefore $p_n(t)$ is the sum of 2^n terms, each of which is a result of k derivations and $n-k$ multiplications by $(-t)$ applied on p_0, $k = 0, 1, \ldots, n$. The terms for which $2k > n$ are zero, the other terms can be estimated by $n^k |t|^{n-2k}$. Now if $r > 0$ is arbitrary and $p \geqq 0$ then it is easy to check that $\max\limits_{t \in \mathbf{R}} |t|^p e^{-rt^2} = \left(\frac{p}{2re}\right)^{p/2}$ $\left(\text{this maximum is achieved at } |t| = \sqrt{\frac{p}{2r}}\right)$. Therefore, with $u > 0$, we have

$$\max |p_n(ut)\, e^{-rt^2}| \leqq 2^n \max \left\{ n^k \cdot \left(\frac{u^2(n-2k)}{2re}\right)^{\frac{n}{2}-k}; \qquad k = 0, 1, \ldots, \left[\frac{n}{2}\right] \right\}$$

and hence $|p_n(ut)| \leqq C(u,r)^n n^{\frac{n}{2}} \cdot e^{rt^2}$ where $C(u,r)$ does not depend on n. Writing $u = \sqrt{2c}$ and choosing $r \in (0, c-s)$ we get the result.

The proof of Theorem 3 is thus complete.

REMARK ON LEMMA 6. This result is sharp in the sense that this sequence has a positive lower bound. Clearly $\|(e^{-ct^2})^{(n)} e^{st^2}\|_1 \geqq \|(e^{-ct^2})^{(n)}\|_1 \geqq$

$$\geqq \max_{x \in \mathbf{R}} \left| \int (e^{-ct^2})^{(n)} e^{-itx}\, dt \right| = \max_{x \in \mathbf{R}} |x|^n \cdot \sqrt{\frac{\pi}{c}} \cdot e^{-\frac{x^2}{4c}} = \sqrt{\frac{\pi}{c}} \cdot \left(\frac{2cn}{e}\right)^{n/2}.$$

COMMENTS ON THE IMPROVING OF THEOREM 2. The conjecture is the following: replacing $\mathscr{E}(A)$ by $\mathscr{A}(A)$, Theorem 2 remains valid. This is true in the case when all cle semigroups on closed subspaces of $\bar{\mathscr{X}}$ have resolvents. In any case, by Lemma 4, we have (1′) for the generator of V_s instead of A (with $C = 0$). Does this imply the equicontinuity of V_s?

In general, Hille's Formula $\left(\left(I - \frac{t}{n}A\right)^{-n} \to V(t) \text{ if } A \text{ is the generator of } V\right)$ does not hold, even $I - \alpha A$ need not be injective. On the other hand, we can have continuous $(I - \alpha A)^{-1}$ such that $I - \alpha A$ is not surjective (e.g., if $\mathscr{X}$ is the space of entire functions endowed with the compact-open topology and $V(t)f(z) = e^{tz}f(z)$). The author's attempts to construct a counterexample for which (1′) holds for the generator but the semigroup is not of exponential growth, have failed.

SOME CONSEQUENCES OF OUR RESULTS. If we assume (1′) for $-A$ and A then we get another semigroup V_1 commuting with V and having generator $\overline{-A}$. Therefore the generator of $V \cdot V_1$ is 0, i.e., $V \cdot V_1 \equiv I$. Thus we can see if $M\|(I - \alpha A)^k x\| \geqq (1 - |\alpha| C)^k \|x\|$ for $|\alpha| < \delta$ and A has a dense set of analytic vectors then $\bar{A}$ is the generator of a *group*. Similarly, from Theorems 2 and 3 we get this result: the generators of groups V on sequentially complete LCS-es satisfying $e^{-C|t|} V(t)$ are equicontinuous are exactly those closed operators A for which $(1 - |\alpha| C)^k (I - \alpha A)^{-k}$ are equicontinuous for $|\alpha| < \delta$ and $\mathscr{E}(A)$ is dense.

We have now an improvement of Rusinek's theorem (cf. [2], [6]): Let $\mathscr{D}$ be a dense subspace of a Banach space, $\mathrm{End}(\mathscr{D})$ be the endomorphisms of $\mathscr{D}$ as a linear space and $\mathscr{L}$ be a finite dimensional Lie-subalgebra of $\mathrm{End}(\mathscr{D})$. If $A_1, \ldots A_k \in \mathscr{L}$ is a Lie-generating subset such that

a) $M \cdot \|\lambda - A_j)^n x\| \geqq (|\lambda| - C)^n \|x\|$ for large $|\lambda|$,

b) $\mathscr{D} = \mathscr{A}(A_j)$ for all j,

then there is a (unique) representation V of the corresponding simply connected Lie group such that $\partial V(T) = \bar{T}$ for all $T \in \mathscr{L}$.

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