NEAREST POINTS ON TORIC VARIETIES

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(Dedicated to Alicia Dickenstein on the occasion of her 60th birthday)

Abstract

We determine the Euclidean distance degree of a projective toric variety. This extends the formula of Matsui and Takeuchi for the degree of the A-discriminant in terms of Euler obstructions. Our primary goal is the development of reliable algorithmic tools for computing the points on a real toric variety that are closest to a given data point.

1. Introduction

We are interested in the best approximation of data points in $\mathbb{R}^n$ by a model that is given by a monomial parametrization. Such a model corresponds to a projective toric variety. Our result is a formula for the generic Euclidean distance degree (gED degree [10]) of that variety.

Consider the problem of identifying d unknown real numbers $t_1, t_2, \ldots, t_d$ by sampling noisy products of any k of these numbers. The input data consists of $\binom{d}{k}$ measurements $u_{i_1 i_2 \cdots i_k}$ that are supposed to be approximations of $t_{i_1} t_{i_2} \cdots t_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq d$. The least squares paradigm suggests the unconstrained polynomial optimization problem

$$\text{Minimize the function } L(t_1, \ldots, t_d) = \sum_{1 \leq i_1 < \cdots < i_k \leq d} (t_{i_1} t_{i_2} \cdots t_{i_k} - u_{i_1 i_2 \cdots i_k})^2. \quad (1)$$

The critical points of this problem are solutions of the system of polynomial equations

$$\frac{\partial L}{\partial t_1} = \frac{\partial L}{\partial t_2} = \cdots = \frac{\partial L}{\partial t_d} = 0. \quad (2)$$

The non-zero complex solutions to (2) come in clusters of k solutions that differ by multiplication with a k-th root of unity. The number of such clusters for generic data $u_{i_1 i_2 \cdots i_k}$ is the algebraic degree of the optimization problem (1). For instance, if $d = 4$, $k = 2$ then (2) is a system of 4 cubics in 4 unknowns. Using `Macaulay2` [15], we find that it has 28 pairs of solutions $\{t, -t\}$.

Thus, for $d = 4$, $k = 2$, the algebraic degree of the problem (1) equals 28. Proposition 4.7 generalizes that number to a combinatorial formula in terms of d and k.

The models in this paper are as follows. We fix an integer $d \times n$-matrix $A = (a_1, a_2, \ldots, a_n)$ of rank d such that $(1, 1, \ldots, 1)$ lies in the row space of A over $\mathbb{Q}$. We allow for A to have negative entries. Each column vector a_i corresponds to a (Laurent) monomial $t^{a_i} = t_1^{a_{1i}} t_2^{a_{2i}} \cdots t_d^{a_{di}}$. The *affine toric variety* $\tilde{X}_A$ is the closure in $\mathbb{C}^n$ of the set $\{(t^{a_1}, \ldots, t^{a_n}) : t \in (\mathbb{C}^*)^d\}$, where $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. This is the affine cone over the *projective toric variety* $X_A \subset \mathbb{P}^{n-1}$ with the same parametrization. Note that $\dim(X_A) = d-1$ and $\dim(\tilde{X}_A) = d$. For basics on toric geometry and toric algebra we refer to the books [7] and [30].

Fix a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive reals and consider the λ-weighted Euclidean norm on $\mathbb{R}^n$, defined by $\|x\|_\lambda = (\sum_{i=1}^n \lambda_i x_i^2)^{1/2}$. Given $u \in \mathbb{R}^n$, we seek to find a real point $v \in \tilde{X}_A$ that is closest to u. Thus, our aim is to solve the constrained optimization problem

$$\text{Minimize } \|u - v\|_\lambda \quad \text{subject to } v \in \tilde{X}_A \cap \mathbb{R}^n. \tag{3}$$

This is equivalent to the unconstrained optimization problem

$$\text{Minimize } \sum_{i=1}^n \lambda_i (u_i - t^{a_i})^2 \quad \text{over all } t = (t_1, \ldots, t_d) \in \mathbb{R}^d. \tag{4}$$

The number of complex critical points of (3) is denoted by $\text{EDdegree}_\lambda(X_A)$. This is the ED degree (cf. [10], [24]) of the toric variety X_A. It depends on λ but is independent of u, since u is generic. It governs the intrinsic algebraic complexity of finding and representing the exact solutions to (3) and (4). In particular, it is an upper bound for the number of local minima. The number of complex critical points of (4) is the product $\text{EDdegree}_\lambda(X_A) \cdot [\mathbb{Z}^d : \mathbb{Z}A]$. The index arises as a factor because it is the degree of the monomial parametrization of X_A.

If the weight vector λ is chosen generically then $\text{EDdegree}_\lambda(X_A)$ is independent of λ. We call this the *generic ED degree* of the toric variety X_A and we denote it by $\text{gEDdegree}(X_A)$. For instance, in (1) we saw that $\text{gEDdegree}(X_A) = 28$ for the threefold $X_A \subset \mathbb{P}^5$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} = \text{the octahedron.}$$

The following formula, inspired by Aluffi [2], will be derived and used in this paper.

THEOREM 1.1. *The generic Euclidean distance degree of the projective toric variety X_A is*

$$\text{gEDdegree}(X_A) = \sum_{j=0}^{d-1} (-1)^{d-j-1} \cdot (2^{j+1} - 1) \cdot V_j, \tag{5}$$

where V_j is the sum of the Chern-Mather volumes of all j-dimensional faces of $P = \text{conv}(A)$.

The lattice polytope $P = \text{conv}(A)$ has dimension $d-1$ since $\text{rank}(A) = d$. If the toric variety X_A is smooth then P is simple and V_j is the sum of the normalized lattice volumes of the j-faces of P. In the smooth case, Theorem 1.1 is precisely the formula given in [10, Corollary 5.11]. What is new here is the extension to the singular case. Indeed, X_A is an arbitrary singular projective toric variety in $\mathbb{P}^{n-1}$. In particular, X_A is generally not normal.

Theorem 1.1 rests on work by Aluffi [2], Esterov [12], and Matsui-Takeuchi [22]. The key notion is the *Chern-Mather volume* (or CM volume for short). We will define this in Section 2. One ingredient is the *local Euler obstruction* [6, Chapter 8] of singular strata on X_A. We now present a formula for the dimension and degree of the *A-discriminant* [14], that is, the variety $X_A^\vee$ projectively dual to X_A. The following is a variant of [22, Theorem 1.4]:

THEOREM 1.2. *Using notation as above, the polar degrees of the projective toric variety are*

$$\delta_i(X_A) = \sum_{j=i+1}^{d} (-1)^{d-j} \binom{j}{i+1} V_{j-1}. \tag{6}$$

The codimension of the A-discriminant is $\min\{c : \delta_{c-1} \neq 0\}$. For that c, $\text{degree}(X_A^\vee) = \delta_{c-1}$.

We note that the polar degrees of projective varieties are of independent interest in the study of algorithms for real algebraic geometry. They govern the complexity of methods for reliably sampling points in each connected component of a semi-algebraic set (cf. [3], [28]). The polar degrees δ_i can also be seen as the degrees of polar varieties. Foundational results on this topic can be found in the work of Kleiman [19], Piene [25], [26] and Bank et al. [4].

Our focus in this paper is on tools for concrete computations, starting from an integer matrix A. We implemented the formulas for the polar degrees and the gED degree in `Macaulay2` [15]. Given an arbitrary integer matrix A as above,

our software computes the quantities in (5)–(6). The code and accompanying discussion can be found at the supplementary website [1].

For a concrete illustration consider the case $d = 2$, when X_A is a toric curve in $\mathbb{P}^{n-1}$. After row operations and column permutations, we may assume that our input has the form

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

where $0 \leq \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_{n-1} < \alpha_n$ and the differences $\alpha_i - \alpha_j$ are relatively prime. One finds that the generic ED degree of the toric curve X_A equals $2\alpha_n + \alpha_{n-1} - \alpha_2 - 2\alpha_1$. This quantity is the expected number of complex solutions to the polynomial system

$$\frac{1}{t} \frac{\partial L}{\partial s} = \frac{\partial L}{\partial t} = 0,$$

where

$$L(s, t) = \lambda_1 (s^{\alpha_1} t - u_1)^2 + \lambda_2 (s^{\alpha_2} t - u_2)^2 + \cdots + \lambda_n (s^{\alpha_n} t - u_n)^2. \tag{7}$$

A priori knowledge of the ED degree is useful for optimization because it furnishes an upper bound on the number of local minima of L. The following numerical example illustrates this.

EXAMPLE 1.3. Let $n = 7$ and $\alpha = (0, 1, 2, 3, 4, 5, 6)$, so X_A is the rational normal curve in $\mathbb{P}^6$. The ED degree is 16. The weight vector $\lambda = (1, 1, 1, 1, 1, 1, 1)$ exhibits the generic behavior, by Proposition 4.1. So, we fix unit weights and use standard Euclidean distance.

Consider the data vector $u = (11, 1, 3, 1, 3, 1, 11)$ in $\mathbb{R}^7$. We seek to find the real point on the surface $\tilde{X}_A \cap \mathbb{R}^7$ that is located closest to u. Note that we may regard u as the vector of coefficients of a binary sextic, and hence as a symmetric tensor of format $2 \times 2 \times 2 \times 2 \times 2 \times 2$. See [10, §8] or [24, §4]. In that interpretation, our goal would be to find the *best rank* 1 *approximation of the tensor u*. We do this by minimizing the squared-distance function

$$\begin{aligned} L(s, t) = {}&(t - 11)^2 + (st - 1)^2 + (s^2 t - 3)^2 \\ &+ (s^3 t - 1)^2 + (s^4 t - 3)^2 + (s^5 t - 1)^2 + (s^6 t - 11)^2. \end{aligned}$$

As expected, the system (7) has 16 complex solutions. Precisely eight of these 16 are real. By the Second Derivative Test, four of these eight are found to be

local minima. They are

s	t	$L(s,t)$
1	4.4285714285714285714	125.71428571428571428
4.5086875578349189693	0.0012891163419679352	139.66300592712833700
.22179403366779357295	10.829114809514133306	139.66300592712833700
-1	3.5714285714285714283	173.71428571428571429

The global minimum is attained at $(s,t) = (1, 31/7)$, with value $L(s,t) = 880/7$.

Section 2 develops the relevant results from algebraic geometry. After defining polar degrees, Euler obstructions, and CM volumes, we prove Theorems 1.1 and 1.2. Section 3 starts by illustrating these results for toric surfaces ($d = 3$). We then focus on toric hypersurfaces in $\mathbb{P}^{n-1}$. These are defined by a single binomial, and their conormal varieties are toric too. We write these in terms of a Cayley polytope, and we express (5)–(6) in terms of the binomial's exponents. In Section 4 we derive the discriminants in λ and u whose non-vanishing ensures that gEDdegree(X_A) correctly counts the complex critical points of (3). We also discuss the tropicalization of the conormal variety of X_A, along the lines of [8], [9]. We end the paper by returning to its beginning: a formula for the generic ED degree of the hypersimplex reveals the intrinsic algebraic complexity of learning d numbers from noisy k-fold products.

2. Euler obstructions and Chern-Mather volumes

The (generic) ED degree of a projective variety $X \subset \mathbb{P}^{n-1}$ is the sum of the polar degrees of X. The following formula was derived in [10, Theorem 5.4] and used in [24, Corollary 3.2]:

$$\text{gEDdegree}(X) = \delta_0(X) + \delta_1(X) + \cdots + \delta_{n-1}(X). \tag{8}$$

Many authors, including Fulton [13], Holme [16] and Piene [25], define $\delta_j(X)$ as the degree of the j-th *polar variety* of X with respect to a general linear subspace $\ell_j = \mathbb{P}^{j+\text{codim}(X)} \subset \mathbb{P}^{n-1}$. The polar variety equals

$$P_j = \overline{\{x \in X_{\text{smooth}} \mid \dim(T_x X \cap \ell_j) \geq j+1\}} \subset \mathbb{P}^{n-1}.$$

Following Kleiman [19], we can also define $\delta_j(X)$ using the multidegree of the conormal variety Con(X). This approach is used in [2]. It is explained in [10, §5] after equation (5.3). In practice, we can use the command `multidegree` in `Macaulay2`, as shown in Example 3.3.

If $X \subset \mathbb{P}^{n-1}$ is smooth then its polar degrees can be expressed in terms of the Chern classes of the tangent bundle. Holme [16, p. 150] and Piene [26, Thm. 3] give the formula

$$\delta_i(X) = \sum_{j=i+1}^{d} (-1)^{d-j} \cdot \binom{j}{i+1} \cdot \deg(c_{d-j}(X)). \qquad (9)$$

This formula also covers the singular case (as shown by Piene [26]) if we replace the Chern class with the Chern-Mather class. This is the approach to be pursued in this section. We shall develop the combinatorial meaning of the formula (9) in the case where X_A is an arbitrary singular projective toric variety. As a consequence, we obtain a practical algorithm, made available in [1], for computing the polar degrees and the generic ED degree of X_A. We begin by explaining the relevant results of Esterov [12] and Matsui and Takeuchi [22]. These will enable us to derive Theorems 1.1 and 1.2.

As above, $A = (a_1, a_2, \ldots, a_n)$ is an integer $d \times n$-matrix of rank d with $(1, 1, \ldots, 1)$ in its row space. The columns a_i span the semigroup $\mathbb{N}A$ and the lattice $\mathbb{Z}A$, both in $\mathbb{Z}^d$. The polytope $P = \text{conv}(A)$ has dimension $d - 1$ and it lives in $\mathbb{R}^d$. Let α be an $(s - 1)$-dimensional face of P. Its span $\mathbb{R}\alpha$ is a linear subspace of dimension s in $\mathbb{R}^d$. The intersection $M_\alpha := \mathbb{R}\alpha \cap \mathbb{Z}^d$ is a lattice of rank s. The quotient group is also free abelian: $\mathbb{Z}^d / M_\alpha \simeq \mathbb{Z}^{d-s}$.

Let A_α denote the set of all columns a_i of A that lie in α. The lattice $\mathbb{Z}A_\alpha$ spanned by that set is a subgroup of finite index in M_α. We also consider the image of the set of columns of A in $\mathbb{Z}^d / M_\alpha$. This is a $(d-s)$-dimensional vector configuration, to be denoted by A/α. We wish to stress that the toric varieties in this paper are generally not normal, and all our volumes are understood in the normalized integer sense that is customary in toric geometry.

DEFINITION 2.1. Fix two faces α, β of P such that $\beta \subset \alpha$. After a change of coordinates, we may assume that the origin in $\mathbb{Z}^d$ is contained in the face β. We write A_α/β for the image of the finite set A_α in the free abelian group M_α/M_β. Its convex hull $\text{conv}(A_\alpha/\beta)$ is a polytope of dimension $r = \dim(\alpha) - \dim(\beta)$ in the real vector space $(M_\alpha/M_\beta) \otimes_\mathbb{Z} \mathbb{R} = \mathbb{R}\alpha/\mathbb{R}\beta \simeq \mathbb{R}^r$.

We define the *subdiagram volume* of β in α to be the positive integer

$$\mu(\alpha/\beta) = \text{Vol}\big(\text{conv}(A_\alpha/\beta) \setminus \text{conv}((A_\alpha/\beta)\setminus\{0\})\big) \qquad (10)$$

where Vol is the r-dimensional volume that is normalized with respect to the lattice M_α/M_β.

The notion of subdiagram volume is also defined in [14, Definition 3.8] and in [22, Definition 4.5], but their notation and normalization conventions are slightly different.

REMARK 2.2. To compute the subdiagram volume in (10), we use coordinates on $\mathbb{Z}^d$ that are adapted to the inclusions $M_\beta \subset M_\alpha \subset \mathbb{Z}^d$. Changing coordinates on $\mathbb{Z}^d$ corresponds to integer row operations on A. We shall use the following procedure to carry this out:

- First reorder the columns of A so that those in β come first, followed by those in $\alpha \backslash \beta$, and the remaining columns last. In other words, we write A in block form as

$$A = \left(A_\beta, \ A_{\alpha \backslash \beta}, \ A_{P \backslash \alpha} \right).$$

- Next compute the *Hermite normal form* of A. It has the triangular block structure

$$A' = \begin{matrix} & \beta & \alpha \backslash \beta & P \backslash \alpha \\ & \begin{pmatrix} * & * & * \\ 0 & C & * \\ 0 & 0 & * \end{pmatrix} \end{matrix}.$$

Note that $X_A = X_{A'}$. The integer matrix C has r rows where $r = \dim(\alpha) - \dim(\beta)$. Restricting to these r rows corresponds to the appropriate projection $\mathbb{Z}^n \to \mathbb{Z}^r \simeq M_\alpha / M_\beta$. To find the subdiagram volume in (10), we may use the normalized r-dimensional volumes of the polytopes $\mathrm{conv}(C \cup \{0\})$ and $\mathrm{conv}(C)$. These considerations imply the following formula:

$$\mu(\alpha / \beta) = \mathrm{Vol}(\mathrm{conv}(C \cup \{0\})) - \mathrm{Vol}(\mathrm{conv}(C)). \tag{11}$$

MacPherson [21] introduced the local Euler obstructions in singularity theory. See the book [6] for subsequent developments. Ernström [11] related this to polar degrees and dual varieties. For the case of toric varieties, the local Euler obstructions admit a combinatorial description in terms of subdiagram volumes. This was developed by Esterov [12, §2.5] and refined by Matsui and Takeuchi [22, §4.2]. We shall present a review of these results, modified to use the notation above. The matrix A and the polytope $P = \mathrm{conv}(A)$ are as before.

DEFINITION 2.3. Let β be a face of P. The *Euler obstruction* of β is an integer $\mathrm{Eu}(\beta)$ that depends on the point configuration A. It is defined recursively by the following relations:

(1) $\mathrm{Eu}(P) = 1$,

(2) $\mathrm{Eu}(\beta) = \displaystyle\sum_{\substack{\alpha \text{ s.t. } \beta \text{ is a} \\ \text{proper face of } \alpha}} (-1)^{\dim(\alpha) - \dim(\beta) - 1} \cdot \mu(\alpha / \beta) \cdot \mathrm{Eu}(\alpha).$

If X_A is smooth along the orbit given by the face β then $\mathrm{Eu}(\beta) = 1$. We note that, as discussed above, the lattice indices in [22, Theorem 4.7] are subsumed in Definition 2.1. See also [23, Corollary 1.11.3].

Let β be a face of $P = \mathrm{conv}(A)$ and T_β the corresponding orbit. Let $\mathrm{Eu}_{X_A} \colon X_A \to \mathbb{Z}$ be the local Euler obstruction of X_A as defined by [21] and [6, Chapter 8]. Note that Eu_{X_A} is constant on the orbits given by the faces of P. Let $\mathrm{Eu}_{X_A}(T_\beta)$ denote the value of Eu_{X_A} for any point in T_β. By Theorem 4.7 of Matsui and Takeuchi [22] we have that

$$\mathrm{Eu}_{X_A}(T_\beta) = \mathrm{Eu}(\beta) \cdot [M_\beta : \mathbb{Z}A_\beta]. \tag{12}$$

Using the Euler obstruction of Definition 2.3, we now define the Chern-Mather (CM) volume.

DEFINITION 2.4. The *Chern-Mather volume* of a face β of P is an integer that depends on A. It is the product $\mathrm{Vol}(\beta)\,\mathrm{Eu}(\beta)$ of the normalized volume and the Euler obstruction of β. As in Theorem 1.1, we write V_j for the sum of the CM volumes of the j-dimensional faces of P:

$$V_j = \sum_{\substack{\beta \text{ face of } P \\ \dim(\beta) = j}} \mathrm{Vol}(\beta)\,\mathrm{Eu}(\beta). \tag{13}$$

We chose to use the term "volume" even though the integers $\mathrm{Eu}(\beta)$ and V_j can be negative.

REMARK 2.5. The primary aim of Matsui and Takeuchi in [22] was to compute the dimension and degree of the A-discriminant $X_A^\vee$. These are given by the first non-zero polar degree: if $\delta_0 = \cdots = \delta_{c-2} = 0$ and $\delta_{c-1} > 0$ then $\mathrm{codim}(X_A^\vee) = c$ and $\mathrm{degree}(X_A^\vee) = \delta_{c-1}$. This is essentially the content of [22, Theorem 1.4]. However, it is important to note that the quantities $\delta_\bullet$ in [22, (1.6)] are *not* the polar degrees of X_A. Instead, they are the alternating sums

$$\delta_0, \ \delta_1 - 2\delta_0, \ \delta_2 - 2\delta_1 + 3\delta_0, \ \delta_3 - 2\delta_2 + 3\delta_1 - 4\delta_0, \ \delta_4 - 2\delta_3 + 3\delta_2 - 4\delta_1 + 5\delta_0, \ \ldots.$$

Note that the first non-zero number in this list also gives the codimension and degree of $X_A^\vee$.

We prefer the direct formulation, just using the polar degrees, given in the second and third sentence of Theorem 1.2. Formula (6) writes the polar degrees in terms of CM volumes.

PROOF OF THEOREM 1.2. For any subvariety X of $\mathbb{P}^{n-1}$, the ith polar degree can be expressed in terms of the Euler obstructions of linear sections of X.

Ernström [11, Theorem 2.2] proves

$$\delta_i(X) = (-1)^{\dim(X)-i}\big(\chi(\mathrm{Eu}_{X^{(i)}}) - 2\chi(\mathrm{Eu}_{X^{(i+1)}}) + \chi(\mathrm{Eu}_{X^{(i+2)}})\big), \qquad (14)$$

where $X^{(j)} = X \cap H_1 \cap \cdots \cap H_j$ for general hyperplanes H_ℓ in $\mathbb{P}^{n-1}$. In their proof of [22, Theorem 1.4], Matsui and Takeuchi give an explicit expression for the terms in (14) when $X = X_A$ and $\dim(X) = d-1$. Specifically, equations (3.16) and (3.10) in [22] show that

$$\chi(\mathrm{Eu}_{X_A^{(0)}}) = \chi(\mathrm{Eu}_{X_A}) = V_0 \qquad (15)$$

and

$$\chi(\mathrm{Eu}_{X_A^{(i)}}) = \sum_{j=i}^{d-1}(-1)^{j-i}\binom{j-1}{i-1}V_j \qquad \text{for } i = 1, \ldots, d-1. \qquad (16)$$

Substituting (15) and (16) into (14) gives the formula

$$\delta_0(X_A) = (-1)^{d-1}\bigg(V_0 - 2\sum_{j=1}^{d-1}(-1)^{j-1}V_j + \sum_{j=2}^{d-1}(-1)^j(j-1)V_j\bigg).$$

Similarly, for $i = 1, \ldots, d-1$ we obtain

$$\delta_i(X_A) = (-1)^{d-1}\bigg(\sum_{j=i}^{d-1}(-1)^j\binom{j-1}{i-1}V_j$$

$$- 2\sum_{j=i+1}^{d-1}(-1)^{j-1}\binom{j-1}{i}V_j + \sum_{j=i+2}^{d-1}(-1)^j\binom{j-1}{i+1}V_j\bigg).$$

By reindexing the two summations above, and by collecting terms, we obtain the more compact expression for the polar degrees given in (6). This completes the proof.

PROOF OF THEOREM 1.1. This follows from Theorem 1.2 using formula (8).

We next justify why we chose the term "Chern-Mather volume" for the quantities V_j in Definition 2.4. The *Chern-Mather class* is a generalization of the total Chern class (of the tangent bundle) to singular varieties. See [6, Section 10.6] or [13, Example 4.29] for the definition. Piene [26] expressed the Chern-Mather class of a projective variety as an alternating sum of polar degrees. Her formula leads to the following identification of the Chern-Mather class of a toric variety X_A with the Chern-Mather volumes V_j of its matrix A. We regard the Chern-Mather class of X_A as an element in the Chow ring

$A^*(\mathbb{P}^{n-1}) \cong \mathbb{Z}[h]/\langle h^n \rangle$ of the ambient projective space $\mathbb{P}^{n-1}$. Here h denotes the hyperplane class.

PROPOSITION 2.6. *The Chern-Mather class of the projective toric variety* $X_A \subset \mathbb{P}^{n-1}$ *equals*

$$c_M(X_A) = \sum_{j=0}^{d-1} V_j \cdot h^{n-j-1} \ \in A^*(\mathbb{P}^{n-1}) \cong \mathbb{Z}[h]/\langle h^n \rangle. \qquad (17)$$

In particular, the CM volume V_j is the degree of the dimension j Chern-Mather class of X_A.

PROOF. In light of Theorem 1.2, this follows immediately from Piene's formula [26, Theorem 3] for the Chern-Mather class of a projective variety in terms of polar degrees. The simplification of the summations required to arrive at the formula (17) is aided considerably by employing the Chern-Mather involution formulas of Aluffi [2].

The result of Proposition 2.6 may also be expressed in the Chow ring of X_A as

$$c_M(X_A) = \sum_{\alpha \text{ a face of } P} \text{Eu}(\alpha) \cdot [M_\alpha : \mathbb{Z}A_\alpha][\overline{T_\alpha}] \ \in A^*(X_A),$$

where $[\overline{T_\alpha}]$ is the class in $A^*(X_A)$ of the orbit closure associated to a face α of P. This reformulation follows from Proposition 2.6 and (12). A direct proof is given in [27, Theorem 2].

Theorem 1.1 is now a special case of [2, Proposition 2.9]. Aluffi's result expresses the ED degree of an arbitrary projective variety in terms of the Chern-Mather class. While this does encompass our situation, it does not provide new tools for actually computing polar degrees, Chern-Mather classes, or ED degrees. Our contribution fills this gap in the toric case. We furnish an algorithm for computing these quantities for an arbitrary projective toric variety X_A, not necessarily normal. Our method is implemented in the `Macaulay2` package at [1]. Its input is the $d \times n$-integer matrix A, and its output is the numbers in (5) and (6).

Our implementation allows for relatively efficient and extremely scalable computation. The running time is almost entirely determined by the facial structure of $P = \text{conv}(A)$. While this may make the computation difficult for high-dimensional polytopes with many faces, it has several important advantages over algebraic methods. First, the running time of our code has very little direct dependence on the degree of X_A. For algebraic methods (both numerical and symbolic), this will be a bottleneck: computations become infeasible as degree(X_A) grows. Second, for fixed d and large n, the toric ideal of A can

become unmanageable quite rapidly, while an iteration over the faces of P is still feasible. Third, our combinatorial method is exact, and many portions of the computation could be parallelized.

We close this section by summarizing the steps of our algorithm. The input is the matrix A. It computes the CM volume for each face of $P = \text{conv}(A)$. The output is the list of CM volumes $V_0, \ldots, V_{d-1}$, the polar degrees $\delta_0(X_A), \ldots,$ $\delta_{d-1}(X_A)$, and the ED degree of X_A.

- Compute the face poset $\mathscr{P}$ of the lattice polytope $P = \text{conv}(A)$.
- Build a second poset $\overline{\mathscr{P}}$, isomorphic to $\mathscr{P}$, whose elements are the pairs (α, A_α) for $\alpha \in \mathscr{P}$
- For each chain $(P, A) \supset (\alpha_1, A_{\alpha_1}) \supset \cdots \supset (\alpha_\ell, A_{\alpha_\ell})$ in the poset $\overline{\mathscr{P}}$, do the following:
  - Reorder the columns of the matrix A according to this chain. The new matrix is

  $$\tilde{A} = \left(A_{\alpha_\ell}, A_{\alpha_{\ell-1}\backslash\alpha_\ell}, A_{\alpha_{\ell-2}\backslash\alpha_{\ell-1}}, \ldots, A_{\alpha_1\backslash\alpha_2}, A_{P\backslash\alpha_1} \right).$$

  - Find the Hermite normal form A' of $\tilde{A}$, as in Remark 2.2.
  - For all pairs $1 \le i < j \le \ell$, compute the relative subdiagram volumes $\mu(\alpha_i \backslash \alpha_j)$, using (11) by selecting the appropriate submatrix C of A'.
- Compute the normalized volumes of all elements in the face poset $\mathscr{P}$.
- Combining all subdiagram volumes and face volumes found above, we now compute the Euler obstruction for each face of P using the formula in Definition 2.3.
- Compute V_j using formula (13). Compute $\delta_i(X_A)$ using (6). Output gEDdegree(X_A).

3. Dimension two and codimension one

In this section we compute the gED degree for instances of low dimension and low codimension. We start with toric surfaces. Here $d = 3$ and we assume that the matrix has the form

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{n-1} & \beta_n \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

The lattice polygon $P = \text{conv}(A)$ has normalized area $V_2 = \text{Vol}(P)$. Its polar degrees are

$$\delta_0 = 3V_2 - 2V_1 + V_0, \quad \delta_1 = 3V_2 - V_1 \quad \text{and} \quad \delta_2 = V_2.$$

The generic ED degree is equal to the sum of the polar degrees:

$$\text{gEDdegree}(X_A) = \delta_0 + \delta_1 + \delta_2 = 7V_2 - 3V_1 + V_0.$$

If X_A is smooth then V_0 and V_1 are positive integers. Namely, V_0 is the number of vertices of P, and V_1 is number of all lattice points in the boundary of P. Here is a simple example.

EXAMPLE 3.1. Let $n = 9$ and $X_A = \mathbb{P}^1 \times \mathbb{P}^1$, embedded in $\mathbb{P}^8$ with the line bundle $\mathcal{O}(2, 2)$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This corresponds to approximating a data vector $u \in \mathbb{R}^9$ by biquadratic monomials. Then $P = \text{conv}(A)$ is a square of side length 2. The face volumes are $V_2 = 8$, $V_1 = 8$ and $V_0 = 4$, and hence $\text{gEDdegree}(X_A) = 36$. For instance, if the weights are $\lambda = (4, 1, 9, 2, 3, 1, 7, 6, 5)$ and data point is $u = (29, 14, 46, 13, -5, 42, 42, 5, 23)$ then precisely 14 of the 36 complex critical points are real. This choice of λ exhibits the generic behavior. The ED degree drops from 36 to 20 if we take $\lambda = (1, 1, 1, 1, 1, 1, 1, 1, 1)$; here the unit weights are not generic. This degree drop is explained by the criterion we shall derive in Proposition 4.1.

For singular toric surfaces X_A, we must consider the CM volumes of the edges and vertices of the planar configuration A. If X_A is normal then the following formula can be used:

COROLLARY 3.2. *Suppose that X_A is a toric surface with isolated singularities in $\mathbb{P}^n$. Then V_1 is the number of lattice points in the boundary of $P = \text{conv}(A)$, and the CM volume of a vertex a_i of A equals $\text{Vol}(\text{conv}(A \backslash \{a_i\})) + 2 - \text{Vol}(P)$, where Vol denotes normalized area. Hence V_0 is the sum of these (possibly negative) integers, as a_i ranges over all vertices of P.*

PROOF. This follows from the general results in Section 2. See also [23, Proposition 1.11.7].

The following example illustrates Corollary 3.2. For a non-normal case see Example 3.6. For any such small instance A, we can always verify our combinatorial computation of toric ED degrees using the general algebraic method in [10, (5.3)]. This is done by first computing the bigraded prime ideal of the *conormal variety* $\text{Con}(X_A)$. Recall that $\text{Con}(X_A)$ is an irreducible closed subvariety of dimension $n - 2$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. It is the closure of the set of

pairs (x, y) in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ such that x is a smooth point in X_A and y is a hyperplane tangent to X_A at x. The projection of $\mathrm{Con}(X_A)$ onto the second factor is the *A-discriminant* $X_A^\vee$.

EXAMPLE 3.3. Let $n = 6$ and let X_A be the normal toric surface in $\mathbb{P}^5$ given by

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This is the closure of the image of $(\mathbb{C}^*)^3 \to \mathbb{P}^5$, $(s, t, u) \mapsto (su : tu : stu : s^2tu : s^3tu : st^2u)$.
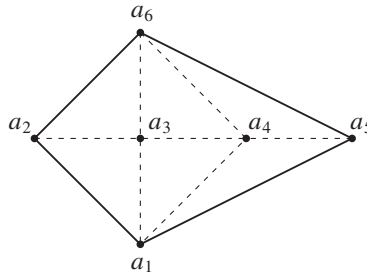

FIGURE 1. The polygon $P = \mathrm{conv}(A)$ has normalized area six. The only lattice points in its boundary are the four vertices. Their CM volumes can be read off from this triangulation.

Figure 1 shows that $V_2 = 6$ and $V_1 = 4$. The four vertices of the polygon P are a_1, a_2, a_5, a_6, and the corresponding complementary areas $\mathrm{Vol}(\mathrm{conv}(A \backslash \{a_i\}))$ are $3, 4, 4, 3$. Hence the CM volumes of the vertices are $-1, 0, 0, -1$, for a total of $V_0 = -2$. We conclude

$$\mathrm{gEDdegree}(X_A) = 7V_2 - 3V_1 + V_0 = 7 \cdot 6 - 3 \cdot 4 + (-2) = 28.$$

We verify this by computing the conormal variety $\mathrm{Con}(X_A) \subset \mathbb{P}^5 \times \mathbb{P}^5$. Each point $y \in X_A^\vee$ represents a singular curve $\{y_1 su + y_2 tu + y_3 stu + y_4 s^2tu + y_5 s^3tu + y_6 st^2u = 0\}$ on the toric surface $X_A \subset \mathbb{P}^5$, and $x = (su : tu : \cdots : st^2u)$ is the singular point. The conormal variety has dimension 4. Its prime ideal $\mathscr{C}$ is minimally generated by 17 polynomials in the $6 + 6$ homogeneous coordinates of $\mathbb{P}^5 \times \mathbb{P}^5$. Among these are four binomial quadrics that generate the toric ideal of X_A. The polar degrees are the coefficients of the *multidegree* of the ideal $\mathscr{C}$, and they are $\delta_0 = 8$, $\delta_1 = 14$, and $\delta_2 = 6$. This is consistent with Theorem 1.2, which says that $\delta_0 = 3V_2 - 2V_1 + V_0$, $\delta_1 = 3V_2 - V_1$ and $\delta_2 = V_2$. The A-discriminant $X_A^\vee$ is a hypersurface of degree 8. Its defining polynomial is found among our 17 ideal generators.

The following code in `Macaulay2` [15] realizes what is described in the previous paragraph.

```
R = QQ[s,t,u,x1,x2,x3,x4,x5,x6,y1,y2,y3,y4,y5,y6,
Degrees=>{{1,1},{1,1},{1,1},{1,0},{1,0},{1,0},{1,0},{1,0},{1,0},
{0,1},{0,1},{0,1},{0,1},{0,1},{0,1}}];
f = y1*s*u+y2*t*u+y3*s*t*u+y4*s^2*t*u+y5*s^3*t*u+y6*s*t^2*u;
I = ideal(diff(s,f),diff(t,f),diff(u,f),
    x1-s*u, x2-t*u, x3-s*t*u, x4-s^2*t*u, x5-s^3*t*u,x6-s*t^2*u);
C = eliminate({s,t,u},I);
C = saturate(C,ideal(x1*x2*x3*x4*x5*x6));
C = saturate(C,ideal(y1*y2*y3*y4*y5*y6));
apply(first entries mingens(C),t->degree(t))
multidegree C
```

The output of the last line is the binary form whose coefficients are the polar degrees.

We next examine toric hypersurfaces. Let $X_A \subset \mathbb{P}^{n-1}$ be defined by one binomial equation

$$x_1^{c_1} \cdots x_r^{c_r} = x_{r+1}^{c_{r+1}} \cdots x_n^{c_n}. \tag{18}$$

Here $c_1, \ldots, c_n$ are positive integers that are relatively prime, and they satisfy

$$c_1 + \cdots + c_r = c_{r+1} + \cdots + c_n = \deg(X_A). \tag{19}$$

Our goal is to express the gED degree and the polar degrees of X_A in terms of $c_1, c_2, \ldots, c_n$.

The integer matrix A has format $(n-1) \times n$, and its kernel is spanned by the column vector $(c_1, \ldots, c_r, -c_{r+1}, \ldots, -c_n)^T$. The associated lattice polytope $P = \text{conv}(A)$ has dimension $n-2$, and it has n vertices provided $2 \le r \le n-2$. We consider the *Cayley polytope* of P and its mirror image $-P$. This is the $(n-1)$-dimensional polytope obtained by placing P and $-P$ into parallel hyperplanes and taking the convex hull. See e.g. [20, Definition 4.6.1]. The integer matrix representing the Cayley polytope has format $n \times 2n$. It equals

$$\text{Cay}(A, -A) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A & -A \end{pmatrix},$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{0} = (0, 0, \ldots, 0)$ in $\mathbb{R}^n$. We shall first derive the following result.

THEOREM 3.4. *The conormal variety* $\text{Con}(X_A)$ *is a toric variety of dimension* $n-2$ *in* $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. *It corresponds to the toric variety of* $\text{Cay}(A, -A)$. *The*

generic ED degree of X_A is the normalized volume of the Cayley polytope. The polar degrees $\delta_i = \delta_i(X_A)$ are given by

$$\text{Vol}(\lambda P + \mu(-P)) = \sum_{i=0}^{n-2} \delta_i \binom{n-2}{i} \lambda^i \mu^{n-2-i}, \quad \text{where } \lambda, \mu \in \mathbb{R}_{>0}. \quad (20)$$

The volume in (20) is the normalized lattice volume. Hence $\delta_0 = \delta_{n-2} = \text{Vol}(P)$ is the integer in (19). The formula (20) confirms the known fact that the polar degrees of a toric hypersurface are symmetric, i.e. $\delta_{i-1} = \delta_{n-1-i}$ for all i. This symmetry of the polar degrees holds for any self-dual projective variety. This is known by results of Kleiman [19]; see also [2]. Before we give the proof of Theorem 3.4, let us present one corollary and one example.

COROLLARY 3.5. *The polar degrees of X_A are piecewise linear functions of $c_1, \ldots, c_n$. Their regions of linearity are the cones in the arrangement of hyperplanes given by equating a subsum of $\{c_1, \ldots, c_r\}$ with a subsum of $\{c_{r+1}, \ldots, c_n\}$, inside the $(n-1)$-space given by (19).*

PROOF. The kernel of the matrix $\text{Cay}(A, -A)$ is the row span of the $n \times 2n$-matrix

$$\begin{pmatrix} \mathbf{c} & \mathbf{0} \\ B & B \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} 1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -1 & & 0 & 0 \\ \vdots & \vdots & & & \ddots & \ddots & \\ \vdots & \vdots & & & & \ddots & \ddots \\ 0 & 0 & 0 & 0 & & 1 & -1 \end{pmatrix} \quad (21)$$

and $\mathbf{c} = (c_1 \quad c_2 \quad \ldots \quad c_r \quad -c_{r+1} \quad \ldots \quad -c_n)$. Each of the $\binom{2n}{n}$ maximal minors of this *Gale dual* matrix is the difference of a subsum of $\{c_1, \ldots, c_r\}$ and a subsum of $\{c_{r+1}, \ldots, c_n\}$. All $2^n - 1$ non-zero such linear forms arise. They define hyperplanes inside the $(n-1)$-space defined by (19). We restrict this hyperplane arrangement to $\mathbb{R}_{>0}^n$. Up to sign, the maximal minors of the matrix (21) are also the maximal minors of $\text{Cay}(A, -A)$. Hence the oriented matroid of $\text{Cay}(A, -A)$ is fixed when $(c_1, \ldots, c_n)$ ranges over any cone of our arrangement in $\mathbb{R}_{>0}^n$. The volume of the Cayley polytope is a sum of certain maximal minors, selected by the oriented matroid. This implies our claim.

EXAMPLE 3.6. Let $n = 4$ and consider the toric surface $X_A = \{x_1^{c_1} x_2^{c_2} = x_3^{c_3} x_4^{c_4}\}$ in $\mathbb{P}^3$. Writing y_1, y_2, y_3, y_4 for the coordinates of the dual $\mathbb{P}^3$, the

conormal variety $\text{Con}(X_A)$ is the irreducible surface in $\mathbb{P}^3 \times \mathbb{P}^3$ that is defined
by $x_1^{c_1} x_2^{c_2} = x_3^{c_3} x_4^{c_4}$ together with the constraint

$$\text{rank} \begin{pmatrix} c_1 x_1^{c_1-1} x_2^{c_2} & c_2 x_1^{c_1} x_2^{c_2-1} & c_3 x_3^{c_3-1} x_4^{c_4} & c_4 x_3^{c_3} x_4^{c_4-1} \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1. \quad (22)$$

This binomial ideal is not prime, but we must saturate with respect to $x_1 x_2 x_3 x_4$
in order to compute the prime ideal of $\text{Con}(X_A)$. Performing this saturation
one obtains the 2×2-minors of the following matrix which has the same row
space as the matrix above:

$$\text{rank} \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 \end{pmatrix} \leq 1. \quad (23)$$

After replacing each variable y_i by $c_i y_i$, we obtain the binomials corresponding
to the rows of the 4×8-matrix in (21). For instance, the second row of this mat-
rix corresponds to the binomial $c_1 x_2 y_2 - c_2 x_1 y_1$. The Gale dual $\text{Cay}(A, -A)$
of (21) represents the 3-dimensional polytope obtained by taking the quad-
rangle $P = \text{conv}(A)$ and placing its mirror image $-P$ on a parallel plane in
3-space. The volume of that 3-dimensional Cayley polytope equals

$$\text{gEDdegree}(X_A) = \delta_0 + \delta_1 + \delta_2 = 3(c_1 + c_2) + \max(|c_1 - c_2|, |c_3 - c_4|).$$

Here, $\delta_0 = \delta_2 = c_1 + c_2 = c_3 + c_4$, and $\delta_1 = \delta_0 + \max(|c_1 - c_2|, |c_3 - c_4|)$.
By (20), we find these formulas by measuring the area of the planar polygon
$\lambda P + \mu(-P)$.

PROOF OF THEOREM 3.4.   The map that attaches tangent hyperplanes to
smooth points of X_A is a birational map from $X_A \subset \mathbb{P}^{n-1}$ to the conormal
variety $\text{Con}(X_A) \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. It is equivariant with respect to the action
of the dense torus of X_A. Hence $\text{Con}(X_A)$ is toric. We find its toric ideal using
a procedure analogous to the transformation from (22) to (23). Let $\mathscr{J}$ be the
ideal given by the 2×2-minors of $(J(X_A) \quad y)^T$ where $y = (y_1, \ldots, y_n)$ and
$J(X_A)$ is the gradient vector of (18). This matrix is analogous to (22). Let I_A
be the ideal of (18).

   The ideal defining $\text{Con}(X_A)$ is $(I_A + \mathscr{J}) : \langle J(X_A)\rangle^\infty$. This is a toric ideal.
It can also be obtained by saturating the binomial ideal $I_A + \mathscr{J}$ with respect
to $x_1 \cdots x_n$ since the singular locus of X_A lies in $\{x_1 \cdots x_n = 0\}$. Among the
generators of that toric ideal are the binomials $c_i x_j y_j - c_j x_i y_i$ as in (23). We
take these for $j = i + 1$ together with (18) and we write their exponents as the
rows of the $n \times 2n$-matrix (21). This matrix is the Gale dual of $\text{Cay}(A, -A)$.
This proves the first two statements in Theorem 3.4. The next conclusions about
the ED degree and the polar degrees of X_A now follow from known results

(cf. [20, Proposition 4.6]) about the relationship between mixed volumes and triangulations of Cayley polytopes.

Theorem 3.4 identified the conormal variety of a toric hypersurface as the toric variety given by the Cayley polytope. The ED degree is the volume of the Cayley polytope. We now use the general result in Theorem 1.1 and 1.2 to derive a formula for that volume.

THEOREM 3.7. *The ith polar degree of the toric hypersurface X_A equals*

$$\delta_i = \binom{n-1}{i+1} \cdot \deg(X_A) - \sum_{\tau : |\tau| = n-i-1} \min\left(\sum_{j \in \tau \cap \{1,\ldots,r\}} c_j, \sum_{j \in \tau \cap \{r+1,\ldots,n\}} c_j \right). \quad (24)$$

PROOF. The $(n-2)$-dimensional polytope $P = \operatorname{conv}(A)$ is simplicial and has n vertices, provided $1 < r < n$. Following [31, Section 6.5], the minimal non-faces of P are $\{1, \ldots, r\}$ and $\{r+1, \ldots, n\}$. For $i \le n-3$, we encode each i-simplex in ∂P by the index set $\tau \subset \{1, 2, \ldots, n\}$ of those columns a_i that are not in that simplex. These τ satisfy $|\tau| = n - 1 - i$, and both $\tau^+ = \tau \cap \{1, \ldots, r\}$ and $\tau^- = \tau \cap \{r+1, \ldots, n\}$ are non-empty.

By Corollary 3.5, the polar degrees of X_A are linear functions on certain full-dimensional polyhedral cones in $\mathbb{R}^n_{>0}$. The lattice points $(c_1, \ldots, c_n)$ with relatively prime coordinates in such a cone are Zariski dense. Every linear function on $\mathbb{R}^n$ is determined by its values on a Zariski dense subset. Hence, in what follows, we may assume that $\gcd(c_i, c_j) = 1$ for all i, j.

Given this assumption, we claim that $\operatorname{Vol}(\tau) = 1$ for every proper face τ of P. Suppose this does not hold. Then $\operatorname{Vol}(\tau) > 1$ for some facet τ, say $\tau = \{r, n\}$ after relabeling. This facet is the simplex with vertex set $\gamma = \{a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_{n-1}\}$. There exists $p \in \mathbb{Z}\gamma$ such that, for some i, the lattice spanned by $(\gamma \backslash \{a_i\}) \cup \{p\}$ has index $i_p \ge 2$ in $\mathbb{Z}\gamma$. We have

$$c_r = \operatorname{Vol}(\gamma \cup \{a_n\}) = i_p \cdot \operatorname{Vol}((\gamma \backslash \{a_i\}) \cup \{p, a_n\})$$

$$\text{and} \quad c_n = \operatorname{Vol}(\gamma \cup \{a_r\}) = i_p \cdot \operatorname{Vol}((\gamma \backslash \{a_i\}) \cup \{p, a_r\}).$$

So, i_p divides $\gcd(c_r, c_n)$, a contradiction. Hence $\operatorname{Vol}(\tau) = 1$ for every proper face τ of P.

For every face σ of P that contains τ, the subdiagram volume in Definition 2.1 equals

$$\mu(\sigma/\tau) = \begin{cases} \min\left(\sum_{i \in \tau^+} c_i, \sum_{j \in \tau^-} c_j \right), & \text{if } \sigma = P, \\ 1, & \text{otherwise.} \end{cases} \quad (25)$$

With this, we can solve the recursion in Definition 2.3. For a face α of P let

$$\min_A^{(r)}(\alpha) = \min\left(\sum_{j\in\alpha\cap\{1,\ldots,r\}} c_j, \sum_{j\in\alpha\cap\{r+1,\ldots,n\}} c_j\right).$$

From (25) and Definition 2.3 we have

$$\mathrm{Eu}(\tau) = \sum_{\substack{\beta\neq P \text{ s.t } \tau \\ \text{is a face of } \beta \text{ and} \\ \dim(\beta)=\dim(\tau)+1}} (-1)^{n-\dim(\beta)-1} \min_A^{(r)}(\beta) + (-1)^{n-\dim(\tau)-1} \min_A^{(r)}(\tau).$$

This results in a formula for the CM volume of τ, as an alternating sum of expressions $\min\left(\sum_{j\in\sigma^+} c_j, \sum_{j\in\sigma^-} c_j\right)$. When we write the sum in (13), and thereafter the sum in (6), a lot of regrouping and cancellation occurs. The final result is the expression for δ_i in (24).

COROLLARY 3.8. *The generic Euclidean distance degree of the toric hypersurface X_A equals*

$$\mathrm{gEDdegree}(X_A) = (2^{n-1} - 1) \cdot \deg(X_A)$$
$$- \sum_{\tau\subset\{1,\ldots,n\}} \min\left(\sum_{j\in\tau\cap\{1,\ldots,r\}} c_j, \sum_{j\in\tau\cap\{r+1,\ldots,n\}} c_j\right).$$

It is instructive to consider the case of surfaces in $\mathbb{P}^3$ and to compare with Corollary 3.2.

EXAMPLE 3.9. Let $n = 4$ and $r = 2$ and set $D = \deg(X_A)$. The polar degrees are $\delta_2 = D$, $\delta_1 = 3D - \min(c_1, c_3) - \min(c_1, c_4) - \min(c_2, c_3) - \min(c_2, c_4) = D + \max\left(|c_1 - c_2|, |c_3 - c_4|\right)$, and $\delta_0 = 3D - c_1 - c_2 - c_3 - c_4 = D$. Their sum gives us the simple formula

$$\mathrm{gEDdegree}(X_A) = 3D + \max\left(|c_1 - c_2|, |c_3 - c_4|\right).$$

Another class of toric surfaces arise for $n = 4$ and $r = 1$. In that case, $\delta_0 = \delta_2 = D$ and $\delta_1 = 2D$.

The results in this paper furnish exact formulas for the algebraic complexity of solving the optimization problems (3) and (4). We close this section with a numerical example.

EXAMPLE 3.10. Given a list $(u_1, u_2, u_3, u_4, u_5, u_6)$ of six real measurements, we seek to find the best approximation by a real vector $(x_1, x_2, x_3, x_4, x_5, x_6)$ that satisfies the model

$$x_1^{22} x_2^{23} x_3^{64} = x_4^{26} x_5^{14} x_6^{69}.$$

The general formula in [10, Corollary 2.10] for hypersurfaces of degree $d = 109$ says that

$$d \cdot \left(1 + (d-1)^1 + (d-1)^2 + (d-1)^3 + (d-1)^4 + (d-1)^5\right) = 1,616,535,525,241$$

is a bound for the algebraic degree of our optimization problem. Corollary 3.8 shows that the true answer is much smaller: $\mathrm{gEDdegree}(X_A) = 1348$. *Numerical Algebraic Geometry* [5] allows us to compute *all* complex critical points, and hence *all* local approximations.

4. Discriminants, tropicalization and hypersimplices

We computed the algebraic degree of the optimization problem (3) when the weight vector λ and the data vector u are generic. This generic behavior fails when these vectors are zeros of certain discriminants. In what follows we discuss those discriminants. Later in this section, we explore connections to tropical geometry: building on [8] and [9], we discuss the tropicalization of the conormal variety of a toric variety X_A. Thereafter, we conclude by returning to (1).

We begin by examining the genericity condition on the weight vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ that specifies the norm $\|x\|_\lambda = (\sum_{i=1}^n \lambda_i x_i^2)^{1/2}$. Following [24], we can define the ED degree of the toric variety X_A for any positive λ. However, it may be smaller than the generic one:

$$\mathrm{EDdegree}_\lambda(X_A) \leq \mathrm{gEDdegree}(X_A). \tag{26}$$

Such a drop occurred for $\lambda = (1, 1, \ldots, 1)$ in Example 3.1, but not in Example 1.3. Similar instances are featured in [10, Example 2.7, Corollary 8.7] and [24, Examples 1.1, Table 1, Proposition 4.1]. We now offer a characterization of the weights whose ED degree is generic.

As before, we write $X_A^\vee$ for the A-discriminant, that is, the projective variety dual to X_A. If the dual $X_A^\vee$ is a hypersurface in $\mathbb{P}^{n-1}$ then Δ_A denotes its defining polynomial. If $\mathrm{codim}(X_A^\vee) \geq 2$ then $\Delta_A = 1$. Following [14] but ignoring exponents, we define the *principal A-determinant* E_A to be the product of the polynomials Δ_α where α runs over all faces of A.

PROPOSITION 4.1. *Let $\lambda \in \mathbb{R}_{>0}^n$ be a weight vector such that the principal A-determinant E_A does not vanish at λ. Then equality holds in (26).*

PROOF. Theorem 5.4 in [10] states that the ED degree of a variety $X \subset \mathbb{P}^{n-1}$ agrees with the generic ED degree provided the conormal variety $\mathrm{Con}(X)$ is disjoint from the diagonal $\Delta(\mathbb{P}^{n-1})$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. This refers to the usual Euclidean norm $\| \cdot \|_1$ on $\mathbb{R}^n$. We apply this to the scaled toric variety $X =$

$\lambda^{1/2}X_A$ whose points are $\lambda^{1/2}x = (\lambda_1^{1/2}x_1 : \lambda_2^{1/2}x_2 : \cdots : \lambda_n^{1/2}x_n)$ where $x = (x_1 : x_2 : \cdots : x_n)$ runs over X_A. If x has non-zero coordinates then $x \in X_A$ means that $x = (t^{a_1} : t^{a_2} : \cdots : t^{a_n})$ for some $t \in (\mathbb{C}^*)^d$. The ED problem for X with respect to the norm $\| \cdot \|_1$ is identical to the ED problem for X_A with respect to $\| \cdot \|_\lambda$.

Proposition 4.1 claims that if the inequality in (26) is strict then $E_A(\lambda) = 0$. Suppose that the inequality in (26) is strict. By [10, Theorem 5.4], we know that $\mathrm{Con}(X) \cap \Delta(\mathbb{P}^{n-1})$ is non-empty. Then there exists a point $x \in X_A$ such that the hyperplane with normal vector $\lambda^{1/2}x$ is tangent to X at the point $\lambda^{1/2}x$. Let us first assume that x has non-zero coordinates. Then $x = (t^{a_1} : t^{a_2} : \cdots : t^{a_n})$ for some $t \in (\mathbb{C}^*)^d$. The tangency condition means that the hypersurface defined by the Laurent polynomial $\sum_{i=1}^n \lambda_i t^{2a_i}$ is singular at the point $t \in (\mathbb{C}^*)^d$. This implies that the hypersurface in the torus $(\mathbb{C}^*)^d$ defined by the Laurent polynomial $\sum_{i=1}^n \lambda_i t^{a_i}$ is singular. We conclude that λ lies in $X_A^\vee$, and hence $\Delta_A(\lambda) = 0$.

Suppose now that some of the coordinates x are zero. Then the support of x is a facial subset α of the columns of A. We now restrict to the torus orbit on X_A given by that subset. The hyperplane with normal vector $\lambda^{1/2}x|_\alpha$ is tangent to X_α at the point $\lambda^{1/2}x|_\alpha$ in that orbit. By the same argument as in the previous paragraph, we now find that $\Delta_\alpha(\lambda) = 0$.

Since the principal A-determinant E_A is the product of the α-discriminants Δ_α for all faces α of A, we conclude that $E_A(\lambda) = 0$ holds whenever the inequality in (26) is strict.

EXAMPLE 4.2. Let $d = 3$, $n = 6$, and

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

Then X_A is the Veronese surface in $\mathbb{P}^5$, with gEDdegree$(X_A) = 13$, and (3) is the problem of finding the best rank 1 approximation to a given symmetric 3×3-matrix. The principal A-determinant equals

$$E_A(\lambda) = \det \begin{pmatrix} 2\lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & 2\lambda_4 & \lambda_5 \\ \lambda_3 & \lambda_5 & 2\lambda_6 \end{pmatrix} \cdot \det \begin{pmatrix} 2\lambda_1 & \lambda_2 \\ \lambda_2 & 2\lambda_4 \end{pmatrix}$$

$$\cdot \det \begin{pmatrix} 2\lambda_1 & \lambda_3 \\ \lambda_3 & 2\lambda_6 \end{pmatrix} \cdot \det \begin{pmatrix} 2\lambda_4 & \lambda_5 \\ \lambda_5 & 2\lambda_6 \end{pmatrix} \cdot \lambda_1 \lambda_4 \lambda_6.$$

If EDdegree$_\lambda(X_A)$ drops below 13 then this product must be zero. We know from [10, Example 3.2] that EDdegree$_\lambda(X_A)$ drops down to 3 when $\lambda =$

$(1, 2, 2, 1, 2, 1)$. A computation reveals that $\mathrm{EDdegree}_\lambda(X_A) = 11$ when $\Delta_A(\lambda) \neq 0$ but one of the 2×2-determinants vanishes.

REMARK 4.3. If all proper faces α of A are affinely independent then E_A and Δ_A are equal up to a monomial factor, so they have the same vanishing locus in $\mathbb{R}^n_{>0}$. If this holds and if the hypersurface defined by $\sum_{i=1}^n x_i = 0$ inside X_A is non-singular then the usual Euclidean norm $\|\cdot\|_1$ exhibits the generic behavior, i.e. $\mathrm{EDdegree}_1(X_A) = \mathrm{gEDdegree}(X_A)$. This explains the generic behavior of $\|\cdot\|_1$ for rational normal curves in Example 1.3, and for the next example.

EXAMPLE 4.4. Consider the toric hypersurface (18). By [14, §9.1], its A-discriminant equals

$$\Delta_A = c_{r+1}^{c_{r+1}} \cdots c_n^{c_n} \cdot \lambda_1^{c_1} \cdots \lambda_r^{c_r} - (-1)^D \cdot c_1^{c_1} \cdots c_r^{c_r} \cdot \lambda_{r+1}^{c_{r+1}} \cdots \lambda_n^{c_n}.$$

Hence $\|\cdot\|_1$ is always ED generic when $D = \deg(X_A)$ is odd. If D is even then the hypothesis

$$c_1^{c_1} \cdots c_r^{c_r} \neq c_{r+1}^{c_{r+1}} \cdots c_n^{c_n}$$

ensures that Corollary 3.8 counts critical points correctly for the usual Euclidean norm.

Suppose now that $\lambda \in \mathbb{R}^n_{>0}$ with $E_A(\lambda) \neq 0$ has been fixed. The question arises which data vectors $u \in \mathbb{R}^n$ exhibit the generic behavior. There are three possible types of degeneracies:

- the *ED discriminant* [10] concerns collisions of critical points in the smooth locus of X_A;
- the *data singular locus* [17, §2.1] concerns critical points in the singular locus of X_A;
- the *data isotropic locus* [17, §2.2] concerns critical points that satisfy $\sum_{i=1}^n \lambda_i x_i^2 = 0$.

A careful study of all three for toric varieties X_A would be worthwhile. Generally none of these three loci are toric varieties themselves. We offer some preliminary observations:

- Example 7.2 in [10] shows that the ED discriminant is complicated and not toric even when X_A has codimension 1. It would be interesting to compute the degree of the ED discriminant for (18) and to compare it to Trifogli's formula in [10, Theorem 7.3].
- The data singular locus always contains the A-discriminant [17, Theorem 1].

- The data isotropic locus always contains the A-discriminant [17, Theorem 2].

The Matsui-Takeuchi formula for the degree of the *A*-discriminant given in Theorem 1.2 is an alternating sum of CM volumes of faces of *P*. A positive formula, as a sum of combinatorial numbers, was given independently by Dickenstein et al. [8]. In fact, Theorem 1.2 in [8] expresses every initial monomial of Δ_A explicitly in a positive manner. Such formulas are derived using Tropical Geometry [20]. Their advantage over [22] is that they furnish start systems for homotopy continuation in Numerical Algebraic Geometry [5].

In what follows we assume familiarity with basics of tropical geometry, especially on varieties given by monomials in linear forms [20, §5.5]. The *Horn uniformization* of the *A*-discriminant [8, §4] lifts to the following parametrization of the conormal variety of X_A.

PROPOSITION 4.5. *Let A be an integer d × n-matrix as above and X_A its projective toric variety in $\mathbb{P}^{n-1}$. The conormal variety $\mathrm{Con}(X_A)$ is the closure of the set of points (x, y) in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, where $x \in X_A$ and $x \cdot y \in \mathrm{kernel}(A)$. Its tropicalization is the set of points (u, v) in $(\mathbb{R}^n/\mathbb{R}\mathbf{1})^2$, where $u \in \mathrm{rowspace}(A)$ and $u + v$ is in the co-Bergman fan $\mathscr{B}^*(A)$.*

PROOF. The two statements are straightforward extensions of [8, Proposition 4.1] and [8, Corollary 4.3] respectively, obtained by keeping track of the tangent hyperplanes H_ξ at $\xi \in X_A$.

The tropical variety $\mathrm{trop}(\mathrm{Con}(X_A))$ is a balanced fan of dimension $n - 2$ in $(\mathbb{R}^n/\mathbb{R}\mathbf{1})^2$. The description above was used by Dickenstein and Tabera [9] to study singular hypersurfaces.

COROLLARY 4.6. *The polar degree $\delta_i(X_A)$ is the number of points in the intersection*

$$\mathrm{trop}(\mathrm{Con}(X_A)) \cap (L_{n-2-i} \times M_i) \subset (\mathbb{R}^n/\mathbb{R}\mathbf{1}) \times (\mathbb{R}^n/\mathbb{R}\mathbf{1}),$$

where L_{n-2-i} is a tropical $(n - 2 - i)$-plane and M_i is a tropical i-plane. These planes can be chosen as in [20, Corollary 3.6.16], and the count is with multiplicities as in [20, (3.6.5)].

In analogy to [8, Theorem 1.2], this corollary can be translated into an explicit positive formula for the polar degrees and hence for the ED degree of X_A. This should be useful for developing homotopy methods for solving the critical equations, which can now be written as

$$x + y = u, \quad x \in \tilde{X}_A \text{ and } x \cdot y \in \mathrm{kernel}(A), \qquad \text{for } \lambda = \mathbf{1}.$$

This formulation arises from [10, Theorem 5.2], where all varieties are regarded as affine cones.

We now return to the optimization problem (1). Here $n = \binom{d}{k}$ and A is the matrix whose columns are the vectors in $\{0, 1\}^d$ that have precisely k entries equal to 1. The $(d-1)$-dimensional polytope $P = \text{conv}(A)$ is the *hypersimplex* $\Delta_{d,k}$. The toric variety X_A represents generic torus orbits on the Grassmannian of k-dimensional linear subspaces in $\mathbb{C}^d$. The degree of X_A is the volume of $\Delta_{d,k}$. This is known (by [29]) to equal the *Eulerian number* $A(d-1, k-1)$. In what follows we determine the CM volumes, polar degrees and gED degree for the hyperpsimplex $\Delta_{d,k}$. Table 1 offers a summary of all values for $d \leq 8$. Here we may assume $2 \leq k \leq \lfloor d/2 \rfloor$ because the cases (d, k) and $(d, d-k)$ are isomorphic.

A couple of observations are in place. The last entry in the respective vectors is the Eulerian number $\text{Vol}(\Delta_{d,k}) = A(d-1, k-1)$. The ED degree is the sum of the polar degrees. The first polar degree δ_0 is the degree of the A-discriminant Δ_A. For $k = 2$ this is simply the determinant of the symmetric matrix with zero diagonal entries. For instance, for $d = 4$,

$$\Delta_A(\lambda) = \det \begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & 0 & \lambda_{34} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & 0 \end{pmatrix}.$$

A key point is that $\Delta_A(\lambda) \neq 0$ when $\lambda = (1, \ldots, 1)$. This ensures that the usual Euclidean metric is generic for (1). There is no degree drop due to the weights λ being special.

We close by presenting general formulas for the Chern-Mather volumes of hypersimplices:

PROPOSITION 4.7. *The Chern-Mather volumes for the hypersimplex $\Delta_{d,k}$ are*

$$V_0 = \binom{d}{k} \cdot \min(k, d - k)$$

and

$$V_\ell = \sum_{i=1}^{\min(k,\ell)} \binom{d}{\ell+1} \binom{d-\ell-1}{k-i} \cdot A(\ell, i-1), \qquad \text{for } \ell = 1, \ldots, d-1.$$

For $\ell = d-1$ this formula gives the Eulerian number $V_{d-1} = A(d-1, k-1) = \text{Vol}(\Delta_{d,k})$.

TABLE 1. Computing the generic ED degree for the toric variety of the hypersimplex $\Delta_{d,k}$.

d	k	Chern-Mather volumes	Polar degrees	gED degree
4	2	(12, 12, 8, 4)	(4, 12, 8, 4)	28
5	2	(20, 30, 30, 25, 11)	(5, 20, 40, 30, 11)	106
6	2	(30, 60, 80, 90, 72, 26)	(6, 30, 80, 120, 84, 26)	346
6	3	(60, 90, 120, 150, 132, 66)	(**96**, 300, 480, 480, 264, 66)	1686
7	2	(42, 105, 175, 245, 273, 189, 57)	(7, 42, 140, 280, 336, 210, 57)	1072
7	3	(105, 210, 350, 560, 714, 644, 302)	(315, 1302, 2940, 3920, 3192, 1470, 302)	13441
8	2	(56, 168, 336, 560, 784, 784, 464, 120)	(8, 56, 224, 560, 896, 896, 496, 120)	3256
8	3	(168, 420, 840, 1610, 2632, 3332, 2872, 1191)	(848, 4256, 12096, 21280, 23744, 16576, 6656, 1191)	86647
8	4	(280, 560, 1120, 2240, 3808, 5152, 4832, 2416)	(3816, 16016, 38976, 60480, 60928, 38976, 14496, 2416)	236104

PROOF. We apply the algorithm at the end of Section 2 to the face poset of $\Delta_{d,k}$. Since every face of the hypersimplex is a hypersimplex, it is convenient to proceed by induction. The base step is the subdiagram volume of a vertex of $\Delta_{d,k}$. Each vertex has $(d-k)k$ neighbors. These lie on a hyperplane in the ambient $(d-1)$-space. Their convex hull is a product of simplices $\Delta_{k-1} \times \Delta_{d-k-1}$. The normalized volume of such a product equals $\binom{d-2}{k-1}$. Hence the subdiagram volume of any vertex at $\Delta_{d,k}$ is $\binom{d-2}{k-1}$. The vertex figures of any positive-dimensional face at $\Delta_{d,k}$ is a simplex. In fact, the toric variety $X_{\Delta_{d,k}}$ has isolated singularities. Hence $\mu(\alpha/\beta) = 1$ for all subdiagram volumes at faces β with $\dim(\beta) \geq 1$.

From Proposition 4.7 one easily computes the polar degrees (6) and the ED degree (5). This solves an open problem, namely to determine the degree of the A-discriminant for $k \geq 3$. This was asked for $d = 6$ and $k = 3$ in [18, Problem 7]. Table 1 reveals that the answer is 96.

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