MATH. SCAND. 122 (2018), 84–90

UNBOUNDED SYMMETRIC ANALYTIC FUNCTIONS ON ℓ_1

IRYNA CHERNEGA and ANDRIY ZAGORODNYUK

Abstract

We show that each G-analytic symmetric function on an open set of ℓ_1 is analytic and construct an example of a symmetric analytic function on ℓ_1 which is not of bounded type.

Let $\mathscr{G}$ be the group of permutations on the set of positive integers $\mathbb{N}$. A function $f\colon \ell_1 \to \mathbb{C}$ is called *symmetric* if for every $\sigma \in \mathscr{G}$ and every $x = (x_1, \ldots, x_n, \ldots) \in \ell_1$, we have

$$f(x_1, \ldots, x_n, \ldots) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots).$$

Symmetric polynomials on ℓ_p (with respect to $\mathscr{G}$) and on $L_p[0, 1]$ (with respect to the group of measure-preserving permutations on $[0, 1]$) for $1 \leq p < \infty$ were first studied by Nemirovskii and Semenov in [10]. In [7], González, Gonzalo and Jaramillo investigated algebraic bases for various algebras of symmetric polynomials on rearrangement-invariant spaces. In particular, they proved that similarly to the classical finite-dimensional case, the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \qquad k = 1, 2, \ldots$$

form an algebraic basis for the algebra of all symmetric polynomials on ℓ_1.

We use the notation $\mathscr{P}(\ell_1)$ for the algebra of all polynomials on ℓ_1 and $\mathscr{P}_s(\ell_1)$ for the algebra of all symmetric polynomials on ℓ_1. The completion of $\mathscr{P}(\ell_1)$ in the metric of uniform convergence on bounded sets coincides with the algebra of entire analytic functions of bounded type $\mathscr{H}_b(\ell_1)$ on ℓ_1. We denote by $\mathscr{H}_{bs}(\ell_1)$ the subalgebra of all symmetric functions in $\mathscr{H}_b(\ell_1)$. The algebra $\mathscr{H}_{bs}(\ell_p)$ was investigated in [5], [4], [6] (see also the survey [3]).

It is known that there are entire functions on infinite-dimensional Banach spaces which are not bounded on some bounded subsets, that is, do not belong

Received 26 April 2016.
DOI: https://doi.org/10.7146/math.scand.a-102082

to $\mathscr{H}_b$ (see e.g. [9, p. 544]). Unbounded entire functions were investigated by many authors. For example, in [1] it was proved that there exists an entire function on a Banach space which is bounded on one of two disjoint balls and unbounded on the other. However, the question about existence of symmetric entire functions which are not of bounded type on infinite-dimensional Banach spaces has so far been open.

Let us recall that a function f on an open subset V of a Banach space X is *G-analytic* if it is analytic on every finite-dimensional subspace intersected with V. It is well known that in any infinite-dimensional Banach space there are discontinuous G-analytic functions (see [9, p. 58] for details).

In this paper we show that each symmetric G-analytic function on an open subset of ℓ_1 is analytic. Also we construct an example of a symmetric entire function which is not of bounded type. Finally we construct a simple example which demonstrates the known fact that the homomorphism

$$\Lambda_-\colon \mathscr{P}_s(\ell_1) \longrightarrow \mathscr{P}_s(\ell_1),$$
$$\Lambda_-\colon F_k \longmapsto -F_k$$

is discontinuous.

THEOREM 1. *Let U be an open subset of ℓ_1 and let $f\colon U \to \mathbb{C}$ be a G-analytic symmetric function. Then f is continuous and therefore analytic.*

PROOF. Let B be an open ball in U such that

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

for every $x \in B$, where P_n are n-homogeneous polynomials and the series converges pointwise. Such representation exists according to [9, Proposition 8.4].

Since f is symmetric, all polynomials P_n should be symmetric. Let us show that each symmetric polynomial P_n is continuous.

Let $P_n^{(m)}$ be the restriction of P_n to the m-dimensional space $V_m = \operatorname{span}(e_1, e_2, \dots, e_m) \subset \ell_1$, where $\{e_k\}_{k=1}^{\infty}$ is the standard basis of ℓ_1. Since $F_1^{(m)}, \dots, F_m^{(m)}$ form an algebraic basis in the space of all symmetric polynomials on V_m and P_n can be represented by an algebraic combination of polynomials $F_1^{(m)}, \dots, F_j^{(m)}$, where $j = \min(n, m)$, there is a polynomial $q_m(t_1, \dots, t_j)$ on $\mathbb{C}^j$ such that

$$P_n^{(m)}(x) = q_m(F_1^{(m)}(x), \dots, F_j^{(m)}(x)),$$

for all $x \in V_m$.

If $m \geq n = \deg P_n$, then $j = n$ and $q_m = q_n$. Indeed, otherwise $q_m - q_n$ would be a non-trivial polynomial on $\mathbb{C}^n$ with $(q_m - q_n)(F_1^{(n)}(x), \ldots, F_n^{(n)}(x)) \equiv 0$, which contradicts the algebraic independence of $F_1^{(n)}, \ldots, F_n^{(n)}$. So

$$P_n(x) = q_n(F_1(x), \ldots, F_n(x)),$$

$x \in \ell_1$ and consequently P_n is continuous. Therefore f is locally bounded and so analytic [9, p. 67].

Note that the same arguments work if we replace ℓ_1 by ℓ_p, for $1 \leq p < \infty$. The following lemma is a simple exercise from calculus.

LEMMA 2. *Let $\sum_{n=0}^{\infty} c_n t^n/n!$ be a power series whose radius of convergence equals* 1. *Then for every fixed k, the series $\sum_{n=0}^{\infty} c_{n+k} t^n/n!$ has the same radius of convergence.*

PROOF. Since $\gamma(t) = \sum_{n=0}^{\infty} c_n t^n/n!$ is analytic on the open unit disk $\mathbb{D}$, its kth derivative is analytic on $\mathbb{D}$ and its Taylor series expansion has the same radius of convergence. On the other hand

$$\frac{d^k \gamma(t)}{dt^k} = \sum_{n=k}^{\infty} \frac{n(n-1)\cdots(n-k+1)c_n t^{n-k}}{n!} = \sum_{n=k}^{\infty} \frac{c_n t^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{c_{n+k} t^n}{n!}$$

and the result follows.

For given $x, y \in \ell_1, x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, the *intertwining* (see e.g. [5]) of x and y, $x \bullet y \in \ell_1$, is defined by:

$$x \bullet y = (x_1, y_1, x_2, y_2, \ldots).$$

For a fixed sequence $\{c_n\}$ of complex numbers let us define, for each $x \in \ell_1$ such that series converges,

$$g(x) = \sum_{n=0}^{\infty} c_n G_n(x),$$

where

$$G_n(x) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1} \cdots x_{k_n}$$

and $G_0 = 1$. Then the following theorem holds:

THEOREM 3. *Suppose that g is well defined in the unit ball of ℓ_1 centered at zero. Then for every $x^0 \in \ell_1$, $\|x^0\| < 1$, g is well defined at $\widetilde{x} = x^{(m)} \bullet x^0$, where $x^{(m)} = (x_1, \ldots, x_m, 0, \ldots)$.*

PROOF. It is known [4] that

$$G_n(x \bullet y) = \sum_{k=0}^{n} G_k(x)G_{n-k}(y).$$

Taking into account that $G_k(x^{(m)}) = 0$ for $k > m$, we have

$$\begin{aligned}
&g(\widetilde{x}) \\
&= \sum_{n=0}^{\infty} c_n G_n(\widetilde{x}) = \sum_{n=0}^{\infty} c_n G_n(x^{(m)} \bullet x^0) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} G_k(x^{(m)})G_{n-k}(x^0) \\
&= \sum_{n=0}^{\infty} c_n G_n(x^0) + G_1(x^{(m)}) \sum_{n=1}^{\infty} c_n G_{n-1}(x^0) + G_2(x^{(m)}) \sum_{n=2}^{\infty} c_n G_{n-2}(x^0) + \\
&\qquad \cdots + G_{m-1}(x^{(m)}) \sum_{n=m-1}^{\infty} c_n G_{n+1-m}(x^0) + G_m(x^{(m)}) \sum_{n=m}^{\infty} c_n G_{n-m}(x^0).
\end{aligned}$$

The last equality has a finite number of terms

$$G_k(x^{(m)}) \sum_{n=k}^{\infty} c_n G_{n-k}(x^0).$$

Since $\|G_n\| = 1/n!$ (see [4, Lemma 3.1]), for each of the above terms we have

$$\begin{aligned}
\left|G_k(x^{(m)})\right| \left|\sum_{n=k}^{\infty} c_n G_{n-k}(x^0)\right| &\le \frac{\|x^{(m)}\|^k}{k!} \sum_{n=0}^{\infty} |c_{n+k}| \|G_{n-k}(x^0)\| \\
&\le \frac{\|x^{(m)}\|^k}{k!} \sum_{n=0}^{\infty} |c_{n+k}| \frac{\|x^0\|^n}{n!}.
\end{aligned}$$

By Lemma 2, the series

$$\sum_{n=0}^{\infty} |c_{n+k}| \frac{\|x^0\|^n}{n!}$$

converges. So g is well defined at $\widetilde{x}$.

EXAMPLE 4. Let (a_n) be a scalar sequence such that the radius of convergence of power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$ at zero is equal to 1. Let us define

$$f(x) = \sum_{n=0}^{\infty} a_n G_n(x).$$

Since $\|G_n\| = 1/n!$, the radius of convergence of f at zero,

$$\rho_0(f) = \frac{1}{\limsup\limits_{n\to\infty} \|a_n G_n\|^{1/n}},$$

is equal to the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$ at zero which is equal to 1. Consider $x \in \ell_1$. Then there is a positive integer m such that $x^{(m)} = (x_1, \dots, x_m, 0, \dots)$, $x^0 = (x_{m+1}, x_{m+2}, \dots)$ and $y = x^{(m)} \bullet x^0 \in \ell_1$ have $\|x^0\| < 1$ and $P(x) = P(y)$ for every symmetric polynomial P. Since $\rho_0(f) = 1$, the function f is defined at x^0. By Theorem 3, f is well defined at y and the equality $G_n(x) = G_n(y)$ for every n implies that f is well defined at x, with $f(x) = f(y)$. By Theorem 1, f is analytic. So f is an entire symmetric function, but it is not a function of bounded type, that is, $f \notin \mathscr{H}_{bs}(\ell_1)$.

For a given sequence of numbers $\mathbf{b} = \{b_k\}$ we consider a mapping $\Lambda_{\mathbf{b}}$ which is defined on the algebraic basis of symmetric polynomials F_k by

$$\Lambda_{\mathbf{b}}(F_k) = b_k F_k.$$

Obviously, this mapping can be extended by linearity and multiplicativity to a homomorphism in the algebra of all symmetric polynomials $\mathscr{P}_s(\ell_1)$. In [6] it is shown that $\Lambda_{\mathbf{b}}$ is continuous in the topology of $\mathscr{H}_{bs}(\ell_1)$ and can be extended to $\mathscr{H}_{bs}(\ell_1)$ if and only if $b_k = \varphi(F_k)$ for some continuous complex homomorphism φ on $\mathscr{H}_{bs}(\ell_1)$. In particular, $\Lambda_- = \Lambda_{\mathbf{b}}$ for $\mathbf{b} = (-1, -1, \dots, -1, \dots)$ is a discontinuous homomorphism. The reader can find more about discontinuous homomorphisms on $\mathscr{P}_s(\ell_p)$ in [2]. The following example gives us a function $h \in \mathscr{H}_{bs}(\ell_1)$ such that $\Lambda_-(h) \notin \mathscr{H}_{bs}(\ell_1)$.

EXAMPLE 5. Let us define

$$h(x) = \sum_{k=0}^{\infty} G_k(x).$$

By Waring's formula (see e.g. [8]), we can write

$$G_k = \sum_{\lambda_1+2\lambda_2+\dots+k\lambda_k=k} (-1)^{k-(\lambda_1+\lambda_2+\dots+\lambda_k)} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} F_1^{\lambda_1}\cdot\ldots\cdot F_k^{\lambda_k}.$$

On the other hand,

$$\sum_{\lambda_1+2\lambda_2+\ldots+k\lambda_k=k} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} = 1.$$

Therefore

$$\Lambda_-(G_k) = \sum_{\lambda_1+2\lambda_2+\cdots+k\lambda_k=k} (-1)^{k-(\lambda_1+\lambda_2+\ldots+\lambda_k)} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} \times (-1)^{\lambda_1+\lambda_2+\cdots+\lambda_k} F_1^{\lambda_1}\cdot\ldots\cdot F_k^{\lambda_k}$$
$$= (-1)^k \sum_{\lambda_1+2\lambda_2+\cdots+k\lambda_k=k} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} F_1^{\lambda_1}\cdot\ldots\cdot F_k^{\lambda_k}$$

and

$$\|\Lambda_-(G_k)\| = \left\| \sum_{\lambda_1+2\lambda_2+\cdots+k\lambda_k=k} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} F_1^{\lambda_1}\cdot\ldots\cdot F_k^{\lambda_k} \right\|$$
$$\leq \left\| \sum_{\lambda_1+2\lambda_2+\cdots+k\lambda_k=k} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} \right\| = 1.$$

So,

$$\rho_0(\Lambda_-(h)) = 1,$$

and thus $\Lambda_-(h) \notin \mathscr{H}_{bs}(\ell_1)$.

REFERENCES

1. Ansemil, J. M., Aron, R. M., and Ponte, S., *Behavior of entire functions on balls in a Banach space*, Indag. Math. (N.S.) 20 (2009), no. 4, 483–489.
2. Chernega, I., *Homomorphisms of the algebra of symmetric analytic functions on ℓ_1*, Carpathian Math. Publ. 6 (2014), no. 2, 394–398.
3. Chernega, I., *Symmetric polynomials and holomorphic functions on infinite dimensional spaces*, Journal of Vasyl Stefanyk Precarpathian National University 2 (2015), no. 4, 23–49.
4. Chernega, I., Galindo, P., and Zagorodnyuk, A., *The convolution operation on the spectra of algebras of symmetric analytic functions*, J. Math. Anal. Appl. 395 (2012), no. 2, 569–577.
5. Chernega, I., Galindo, P., and Zagorodnyuk, A., *Some algebras of symmetric analytic functions and their spectra*, Proc. Edinb. Math. Soc. (2) 55 (2012), no. 1, 125–142.
6. Chernega, I., Galindo, P., and Zagorodnyuk, A., *A multiplicative convolution on the spectra of algebras of symmetric analytic functions*, Rev. Mat. Complut. 27 (2014), no. 2, 575–585.
7. González, M., Gonzalo, R., and Jaramillo, J. A., *Symmetric polynomials on rearrangement-invariant function spaces*, J. London Math. Soc. (2) 59 (1999), no. 2, 681–697.
8. Gould, H. W., *The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences*, Fibonacci Quart. 37 (1999), no. 2, 135–140.
9. Mujica, J., *Complex analysis in Banach spaces*, North-Holland Mathematics Studies, vol. 120, North-Holland Publishing Co., Amsterdam, 1986.

10. Nemirovskiĭ, A. S., and Semenov, S. M., *The polynomial approximation of functions on Hilbert space*, Mat. Sb. (N.S.) 92(134) (1973), 257–281, English translation Mat. USSR Sbornik 21 (1973), no. 2, 255–277.

INSTITUTE FOR APPLIED PROBLEMS OF MECHANICS AND MATHEMATICS
UKRAINIAN ACADEMY OF SCIENCES
3 B, NAUKOVA STR.
LVIV 79060
UKRAINE
E-mail: icherneha@ukr.net

VASYL STEFANYK PRECARPATHIAN NATIONAL UNIVERSITY
57 SHEVCHENKA STR.
IVANO-FRANKIVSK 76018
UKRAINE
E-mail: andriyzag@yahoo.com