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k-SMOOTHNESS: AN ANSWER TO AN OPEN PROBLEM

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Abstract

The aim of this paper is to characterize the k-smooth points of the closed unit ball of $\mathcal{K}(\mathcal{H}_1; \mathcal{H}_2)$. In this paper we also answer a question posed by A. Saleh Hamarsheh in 2015.

1. Introduction

For a unit vector x_o in a Banach space X, consider the set

$$J(x_o) := \{x^* \in X^* : \|x^*\| = 1,\ x^*(x_o) = \|x_o\|\}.$$

The point x is a *smooth point* if $J(x)$ consists exactly of one point. Let X be a real or complex Banach space. Let $S(X)$ denote the unit sphere. It is easy to see that the set $J(x)$ is convex and closed, and $J(x) \subset S(X^*)$. By the Hahn-Banach theorem we get $J(x) \neq \emptyset$ for all $x \in S(X)$. By $\operatorname{Ext} F$ we will denote the set of all extremal points of a given subset $F \subset X$. Let $B(X)$ denote the closed unit ball.

In [2], Khalil and Saleh generalize the notion of smoothness by calling a unit vector x in a Banach space X a *k-smooth point*, or a *multismooth point of order k* if $J(x)$ has exactly k linearly independent vectors, or equivalently, if $\dim(\operatorname{span} J(x)) = k$. For a natural number k, the set of k-smooth points in X is denoted by $\mathcal{N}^k_{sm}(X)$.

2. k-smoothness in the space $\mathcal{K}(\mathcal{H}_1; \mathcal{H}_2)$

For Banach spaces X and Y, $\mathcal{K}(X; Y)$ denotes the set of all compact operators from X into Y. In this paper, we will answer the question posed in [4, p. 2].

Open Problem 2.1 ([4, p. 2]). *For Banach spaces X and Y, let $T \in \mathcal{K}(X; Y)$ with $\|T\| = 1$. Is it true that T is a multismooth point of finite order k in $\mathcal{K}(X; Y)$ if and only if T^* attains its norm at only finitely many independent vectors, say at $y_1^*, y_2^*, \dots, y_r^* \in \operatorname{Ext} B(Y^*)$ such that each $Ty_1^*, Ty_2^*, \dots, Ty_r^*$*

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is a multismooth point of finite order, say m_i, in X^*, and where $k = m_1 + m_2 + \cdots + m_r$?

The answer is no, as this section demonstrates. Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert spaces over $\mathbb{K}$. Let $A \in \mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)$. We write $\mathscr{M}(A) := \{x \in S(\mathscr{H}_1) : \|Ax\| = \|A\|\}$. It is easy to check that $\mathscr{M}(A)$ is compact and $\dim \operatorname{span} \mathscr{M}(A) < \infty$. In particular, $\mathscr{M}(A) \neq \emptyset$. The following equality characterizes the extremal points of the closed unit ball in $\mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)^*$:

$$\begin{aligned}&\operatorname{Ext} B(\mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)^*)\\ &\qquad = \{x \otimes y \in \mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)^* : x \in S(\mathscr{H}_1), y \in S(\mathscr{H}_2)\}, \qquad (1)\end{aligned}$$

where $a \otimes b \colon \mathscr{K}(\mathscr{H}_1; \mathscr{H}_2) \to \mathbb{K}$, $(x \otimes y)(A) := \langle Ax|y\rangle$ for every $A \in \mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)$. This has been proved in [1] and [3]. The next lemma is quite useful.

LEMMA 2.2. *Let $A \in \mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)$. If $\|A\| = 1$, then*

$$\operatorname{Ext} J(A) = \{x \otimes Ax : x \in \mathscr{M}(A)\}\,.$$

PROOF. By computation, we see that $J(A)$ is an extremal subset of $B(\mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)^*)$. Thus we obtain

$$\operatorname{Ext} J(A) \subset \operatorname{Ext} B(\mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)^*). \qquad (2)$$

Combining (1) and (2), we immediately get

$$\operatorname{Ext} J(A) \subset \{x \otimes y : x \in S(\mathscr{H}_1),\ y \in S(\mathscr{H}_2),\ (x \otimes y)(A) = 1\}. \qquad (3)$$

It is a straightforward computation to by verify that

$$x \otimes y \in J(A) \quad \Longleftrightarrow \quad x \in \mathscr{M}(A),\ y = Ax. \qquad (4)$$

Next, from (3) and (4), it follows that

$$\operatorname{Ext} J(A) \subset \{x \otimes Ax : x \in \mathscr{M}(A)\}.$$

The reverse inclusion is clear by (1) and (4).

It is a straightforward computation to obtain the following lemma.

LEMMA 2.3. *Suppose that $x_1, \ldots, x_n$ are pairwise orthogonal vectors in $\mathscr{H}_1$. If $y_1, \ldots, y_n$ are pairwise orthogonal vectors in $\mathscr{H}_2$, then $\{x_j \otimes y_i : j, i = 1, \ldots, n\}$ is a linearly independent subset of $\mathscr{K}(\mathscr{H}_1; \mathscr{H}_2)^*$.*

The idea of the proof is rather simple. The next result is the main result of this section and it yields the characterization of $\mathcal{N}^k_{sm}(\mathcal{K}(\mathcal{H}_1;\mathcal{H}_2))$.

THEOREM 2.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces. Suppose that $A \in \mathcal{K}(\mathcal{H}_1;\mathcal{H}_2)$, $\|A\| = 1$. Then the following statements are equivalent:*

(a) $A \in \mathcal{N}^k_{sm}(\mathcal{K}(\mathcal{H}_1;\mathcal{H}_2))$,

(b) $k = (\dim \operatorname{span} \mathcal{M}(A))^2$.

For real Hilbert spaces a similar result holds with the value of k replaced by $\binom{n+1}{2}$, where $n = \dim \operatorname{span} \mathcal{M}(A)$. The lower value is due to the fact that $\alpha_j\overline{\alpha_i} = \alpha_i\overline{\alpha_j}$ for real α_i, α_j.

PROOF. It is not difficult to prove that a restriction $A|_{\operatorname{span}\mathcal{M}(A)}\colon \operatorname{span}\mathcal{M}(A) \to \mathcal{H}_2$ has to be a similarity (scalar multiple of an isometry). Namely, $\|Ax\| = \|A\| \cdot \|x\|$ for all $x \in \operatorname{span}\mathcal{M}(A)$. Since A is compact, $\dim \mathcal{M}(A) = n < \infty$ holds. So, there is a maximal orthonormal set $\{e_j \in \mathcal{M}(A) : j = 1, \dots, n\} \subset \mathcal{M}(A)$ such that the linear span of $\{e_j \in \mathcal{M}(A) : j = 1, \dots, n\}$ equals the linear span of $\mathcal{M}(A)$. The restriction $A|_{\operatorname{span}\mathcal{M}(A)}$ is a similarity, whence $Ae_j \perp Ae_i$ for $j \neq i$.

Note that $J(A)$ is a weak*-compact convex set and hence it is easy to see that $\dim \operatorname{span} J(A) = \dim \operatorname{span} \operatorname{Ext} J(A)$. Observe that

$$\begin{aligned}
k &= \dim \operatorname{span} J(A) = \dim \operatorname{span} \operatorname{Ext} J(A) \overset{\text{(Lemma 2.2)}}{=} \\
&= \dim \operatorname{span} \{x \otimes Ax : x \in \mathcal{M}(A)\} \\
&= \dim \operatorname{span} \left\{ \sum_{j=1}^{n} \alpha_j \cdot e_j \otimes A\Big(\sum_{j=1}^{n} \alpha_j \cdot e_j\Big) : \sum_{j=1}^{n} |\alpha_j|^2 = 1 \right\} \\
&= \dim \operatorname{span} \left\{ \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_j\overline{\alpha_i} \cdot e_j \otimes Ae_i : \sum_{j=1}^{n} |\alpha_j|^2 = 1 \right\} \\
&= \dim \operatorname{span}\{e_j \otimes Ae_i : j, i = 1, \dots, n\} \overset{\text{(Lemma 2.3)}}{=} n^2.
\end{aligned}$$

We may consider (a) $\Longleftrightarrow$ (b) as shown. The proof is complete.

Now we are able to answer the question posed in Open Problem 2.1.

EXAMPLE 2.5. Consider the Hilbert space $X := (\mathbb{C}^3, \langle \cdot \mid \cdot \rangle)$. We define an operator $T \in \mathcal{K}(X;X)$ by $T(x_1, x_2, x_3) := (x_1, x_2, 0)$. It is easy to see that $X^* = X$, $T^* = T$. From now on we may consider X and T instead of X^*, T^*. It is a straightforward verification to show that $\|T\| = 1$ and

$$\dim \operatorname{span} \mathcal{M}(T) = 2.$$

Therefore, T attains its norm at only two independent vectors, say at $y_1, y_2 \in S(X)$. Moreover, each y_1, y_2 is a multismooth point of finite order $m_1 = m_2 = 1$ in X (indeed, the Hilbert space X is smooth). It follows from Theorem 2.4 that T is a multismooth point of finite order 4 in $\mathscr{K}(X; X)$.

Summarizing, we obtain $k = 4 > 2 = 1 + 1 = m_1 + m_2$. The problem 2.1 is solved. Namely, the answer is no. The same example over the reals, has $k = 3 > 2 = m_1 + m_2$, so the answer is still no in this case.

3. Corrigendum to [2]

Unluckily, there is a mistake in [2] and we would like to correct it. Let us quote a result from [2, Theorem 2.2].

THEOREM 3.1 ([2, Theorem 2.2]). *Let $T \in S(\mathscr{K}(l^p))$, $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *T is a multi-smooth point of order k,*

(ii) *T attains its norm at exactly k-linearly independents elements, say $x_1, \dots, x_k$.*

Unfortunately, it follows from Theorem 2.4 and Example 2.5 that Theorem 3.1 does not hold. The proof of Theorem 3.1 contains a mistake. Namely, the authors in [2, Theorem 2.2] used the following implication:

if $x_1 \otimes y_1, \dots, x_n \otimes y_n$ are independent, then $x_1, \dots, x_n$ are independent.

In fact, the above implication is not true (see (5) and Example 3.2).

EXAMPLE 3.2. Consider the Hilbert space $X := (\mathbb{C}^3, \langle \cdot | \cdot \rangle)$. Consider the following vectors: $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c := (1/\sqrt{2}, 1/\sqrt{2}, 0)$. Thus $\|a\| = \|b\| = \|c\| = 1$. Define $f, g, h \in \mathscr{K}(X; X)^*$ by

$$f := a \otimes a, \quad g := b \otimes b, \quad h := c \otimes c. \tag{5}$$

It is a straightforward verification to show that the functionals $a \otimes a$, $b \otimes b$, $c \otimes c$ are linearly independent. On the other hand, the vectors a, b, c are not linearly independent.

4. *k*-smoothness related to exposed points

As an illustration of the application of Theorem 2.4 we prove a well-known result in a new way. Moreover, we will show that another theorem from [2] can be extended. Khalil and Saleh [2, Theorem 4.1] proved the following result.

THEOREM 4.1 ([2]). *Let X be a finite dimensional Banach space and $x \in S(X)$. If x is a smooth point of order $n = \dim X$, then x is an extreme point of the unit ball of X.*

Now, we generalize the above result.

THEOREM 4.2. *Let X be a finite dimensional Banach space and $x \in S(X)$. If x is a smooth point of order $n = \dim X$, then x is an exposed point of the unit ball of X.*

PROOF. Let x be a smooth point of order n of $S(X)$. It follows that there exist n independent unit functionals $a_1^*, \ldots, a_n^* \in S(X^*)$ such that $a_j^*(x) = 1$ for all $j = 1, \ldots, n$. We define a functional $b^* \in X^*$ by $b^* := \sum_{j=1}^{n} \frac{1}{n} a_j^*$. Since $\|b^*\| \leq 1$ and $b^*(x) = 1$, we get $\|b^*\| = 1$. Then we define a hyperplane $M := \{w \in X : b^*(w) = 1\}$. Clearly $x \in M \cap S(X)$. It is enough to show that $\{x\} = M \cap S(X)$. Assume, contrary to our claim, that there exists y in $M \cap S(X)$ such that $x \neq y$. It follows that

$$|a_j^*(y)| \leq 1 \quad \text{and} \quad 1 = b^*(y) := \sum_{j=1}^{n} \frac{1}{n} a_j^*(y). \tag{6}$$

It is easy to check that $1 \in \mathrm{Ext}[-1, 1]$ (or in complex case $1 \in \mathrm{Ext}\{z \in \mathbb{C} : |z| \leq 1\}$). So, by (6) we have $a_j^*(y) = 1$ for all $j = 1, \ldots, n$.

It is helpful to recall that $a_j^*(x) = 1$ for all $j = 1, \ldots, n$. It follows that $a_j^*(x) = a_j^*(y)$ for every $j = 1, \ldots, n$. Since $\{a_1^*, \ldots, a_n^*\}$ is total over X, we have $x = y$, which is a contradiction.

Let $\mathscr{H}$ be a finite-dimensional Hilbert space over $\mathbb{C}$. Suppose that $U \in \mathscr{L}(\mathscr{H})$ is an unitary operator. Although it is well known that U is an exposed point of the unit ball of $\mathscr{L}(\mathscr{H})$, we would like to give a simple proof of this using our main result, i.e., Theorem 2.4.

THEOREM 4.3. *Let $\mathscr{H}$ be a complex Hilbert space such that $\dim \mathscr{H} < \infty$. Then every unitary operator U in $\mathscr{L}(\mathscr{H})$ is an exposed point of the unit ball of $\mathscr{L}(\mathscr{H})$.*

PROOF. It is easy to see that $\mathscr{M}(U) = S(\mathscr{H})$, hence $\dim \operatorname{span} \mathscr{M}(U) = \dim \mathscr{H}$. By Theorem 2.4, U is a smooth point of order $(\dim \operatorname{span} \mathscr{M}(U))^2 = \dim \mathscr{L}(\mathscr{H})$. By Theorem 4.2, U is an exposed point of the closed unit ball of $\mathscr{L}(\mathscr{H})$.

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