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A SHORT NOTE ON CUNTZ SPLICE FROM A VIEWPOINT OF CONTINUOUS ORBIT EQUIVALENCE OF TOPOLOGICAL MARKOV SHIFTS

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Abstract

Let A be an $N \times N$ irreducible matrix with entries in $\{0, 1\}$. We present an easy way to find an $(N+3)\times(N+3)$ irreducible matrix $\bar{A}$ with entries in $\{0, 1\}$ such that the associated Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{\bar{A}}$ are isomorphic and $\det(1 - A) = -\det(1 - \bar{A})$. As a consequence, we find that two Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic if and only if the one-sided topological Markov shift (X_A, σ_A) is continuously orbit equivalent to either (X_B, σ_B) or $(X_{\bar{B}}, \sigma_{\bar{B}})$.

For an $N \times N$ irreducible matrix A with entries in $\{0, 1\}$, let us denote by $G(A)$ the abelian group $\mathbb{Z}^N/(1 - A^t)\mathbb{Z}^N$ and by u_A the position of the class $[(1, \dots, 1)]$ of the vector $(1, \dots, 1)$ in the group $G(A)$. Throughout this short note, matrices are all assumed to be irreducible and not permutation matrices. J. Cuntz in [4] has shown that the pair $(K_0(\mathcal{O}_A), [1])$, consisting of the K_0-group $K_0(\mathcal{O}_A)$ of the Cuntz-Krieger algebra $\mathcal{O}_A$ and the class $[1]$ of the unit in $K_0(\mathcal{O}_A)$, is isomorphic to $(G(A), u_A)$. In [14], M. Rørdam has shown that $(G(A), u_A)$ is a complete invariant of the isomorphism class of $\mathcal{O}_A$ (see [8], for $N \le 3$). For an $N \times N$ irreducible matrix $A = [A(i, j)]_{i,j=1}^N$ with entries in $\{0, 1\}$, the $(N + 2) \times (N + 2)$ irreducible matrix A_- defined by

$$A_- = \begin{bmatrix} A(1,1) & \dots & A(1,N-1) & A(1,N) & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ A(N-1,1) & \dots & A(N-1,N-1) & A(N-1,N) & 0 & 0 \\ A(N,1) & \dots & A(N,N-1) & A(N,N) & 1 & 0 \\ 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix}$$

is called the *Cuntz splice* for A, this was first introduced in [5] by J. Cuntz and is related to classification problem for Cuntz-Krieger algebras. In [5], he had

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used the notation A^- instead of the above A_-. The crucial property of the Cuntz splice is that $G(A_-)$ is isomorphic to $G(A)$ and $\det(1 - A_-) = -\det(1 - A)$. The Cuntz splice

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

for the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is denoted by 2_-. In the proof of the above-mentioned result by Rørdam, [14, Theorem 6.5], a theorem of J. Cuntz, [14, Theorem 7.2], is used which says that if $\mathcal{O}_2 \cong \mathcal{O}_{2_-}$ then $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A_-} \otimes \mathbb{K}$ holds for all irreducible non-permutation matrices A. Since Rørdam has proved $\mathcal{O}_2 \cong \mathcal{O}_{2_-}$ ([14, Lemma 6.4]), the result $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A_-} \otimes \mathbb{K}$ holds for all irreducible non-permutation matrices A. By using this result, Rørdam has also obtained that the group $G(A)$ is a complete invariant of the stable isomorphism class of $\mathcal{O}_A$.

Let us denote by $\mathrm{BF}(A)$ the abelian group $G(A^t) = \mathbb{Z}^N/(1 - A)\mathbb{Z}^N$, which is called the Bowen-Franks group for $N \times N$ matrix A, [2]. Although $\mathrm{BF}(A)$ is isomorphic to $G(A)$ as a group, there is no canonical isomorphism between them. Related to classification theory of symbolic dynamical systems, J. Franks [9] has shown that the pair $(\mathrm{BF}(A), \mathrm{sgn}(\det(1-A)))$ is a complete invariant of the flow equivalence class of the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ by using Bown-Franks's result [2] for the group $\mathrm{BF}(A)$ and Parry-Sullivan's result [13] for the determinant $\det(1 - A)$. Combining this with Rørdam's result for the stable isomorphism classes of the Cuntz-Krieger algebras, $\mathcal{O}_A$ is stably isomorphic to $\mathcal{O}_B$ if and only if $(\bar{X}_A, \bar{\sigma}_A)$ is flow equivalent to either $(\bar{X}_B, \bar{\sigma}_B)$ or $(\bar{X}_{B_-}, \bar{\sigma}_{B_-})$. The operation of Cuntz splicing is now one of basic tools to analyze the structure of Cuntz-Krieger algebras and more general graph C^*-algebras as seen in recent developments of classification of graph algebras (cf. [1], [7], etc.).

In [11], the author has introduced a notion of continuous orbit equivalence in one-sided topological Markov shifts to classify Cuntz-Krieger algebras from a view point of topological dynamical systems. In [12], H. Matui and the author have shown that the triple $(G(A), u_A, \mathrm{sgn}(\det(1-A)))$ is a complete invariant of the continuous orbit equivalence class of the right one-sided topological Markov shift (X_A, σ_A). This result is rephrased by using the above-mentioned result by Rørdam for isomorphism classes of the Cuntz-Krieger algebras such that the pair $(\mathcal{O}_A, \mathrm{sgn}(\det(1 - A)))$ is a complete invariant of the continuous orbit equivalence class of the one-sided topological Markov shift (X_A, σ_A). The C^*-algebra $\mathcal{O}_{A_-}$ is not necessarily isomorphic to $\mathcal{O}_A$, whereas they are stably isomorphic, because the position u_{A_-} in $G(A_-)$ generally is different from the position u_A in $G(A)$. We note that the group $G(A)$ determines the absolute value $|\det(1-A)|$. If $G(A)$ is infinite, $\mathrm{Ker}(1-A)$ is not trivial so that $\det(1-A)=0$. If $G(A)$ is finite, it forms a finite direct sum $\mathbb{Z}/m_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/m_r\mathbb{Z}$ for some $m_1,\dots,m_r\in\mathbb{N}$ so that $|\det(1-A)|=m_1\cdots m_r$ (cf. [5], [6], [14]).

By [12, Lemma 3.7], we know that there is a matrix A' with entries in $\{0,1\}$ such that the triples $(G(A),u_A,\mathrm{sgn}(\det(1-A)))$ and $(G(A'),u_{A'},-\mathrm{sgn}(\det(1-A')))$ are isomorphic, which means that there exists an isomorphism $\Phi\colon G(A)\to G(A')$ such that $\Phi(u_A)=u_{A'}$ and $\mathrm{sgn}(\det(1-A))=-\mathrm{sgn}(\det(1-A'))$. Following the given proof of [12, Lemma 3.7], the construction of the matrix A' seems to be slightly complicated and the matrix size of A' becomes much bigger than that of A. It is not an easy task to present the matrix A' for the given matrix A in a concrete way.

In this short note, we directly present an $(N+3)\times(N+3)$ matrix $\bar{A}$ with entries in $\{0,1\}$ such that $(G(A),u_A,\mathrm{sgn}(\det(1-A)))$ is isomorphic to $(G(\bar{A}),u_{\bar{A}},-\mathrm{sgn}(\det(1-\bar{A})))$. The matrix $\bar{A}$ is constructed such that if A is an irreducible non-permutation matrix, so is $\bar{A}$.

We define

$$A^\circ=\begin{bmatrix} A(1,1) & \dots & A(1,N-1) & A(1,N) & 0\\ \vdots & & \vdots & \vdots & \vdots\\ A(N-1,1) & \dots & A(N-1,N-1) & A(N-1,N) & 0\\ 0 & \dots & 0 & 0 & 1\\ A(N,1) & \dots & A(N,N-1) & A(N,N) & 0 \end{bmatrix}$$

and

$$\begin{aligned}\bar{A}&=(A^\circ)_-\\ &=\begin{bmatrix} A(1,1) & \dots & A(1,N-1) & A(1,N) & 0 & 0 & 0\\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots\\ A(N-1,1) & \dots & A(N-1,N-1) & A(N-1,N) & 0 & 0 & 0\\ 0 & \dots & 0 & 0 & 1 & 0 & 0\\ A(N,1) & \dots & A(N,N-1) & A(N,N) & 0 & 1 & 0\\ 0 & \dots & 0 & 0 & 1 & 1 & 1\\ 0 & \dots & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.\end{aligned} \tag{1}$$

The operation $A\to A^\circ$ is nothing but an expansion defined by Parry-Sullivan in [13], and preserves their determinant: $\det(1-A)=\det(1-A^\circ)$. The

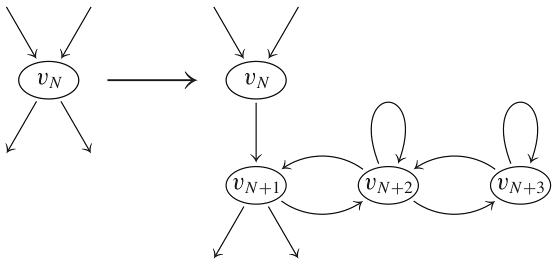

FIGURE 1

following figure is a graphical expression of the matrix $\bar{A}$ from A.

We provide two lemmas. The first one is seen in [2]. The second one is seen in [5] and [14] in a different form.

LEMMA 1 ([2, Theorem 1.3]). *The map*

$$\eta_A\colon (x_1, \dots, x_{N-1}, x_N, x_{N+1}) \in \mathbb{Z}^{N+1} \to (x_1, \dots, x_{N-1}, x_N + x_{N+1}) \in \mathbb{Z}^N$$

induces an isomorphism $\bar{\eta}_A$ *from* $G(A^\circ)$ *to* $G(A)$ *such that*

$$\bar{\eta}_A([(1, \dots, 1, 0)]) = u_A.$$

LEMMA 2 (cf. [5, Proposition 2], [14, Proposition 7.1]). *The map*

$$\xi_A\colon (x_1, \dots, x_N) \in \mathbb{Z}^N \to (x_1, \dots, x_N, 0, 0) \in \mathbb{Z}^{N+2}$$

induces an isomorphism $\bar{\xi}_A$ *from* $G(A)$ *to* $G(A_-)$ *such that*

$$\bar{\xi}_A([(1, \dots, 1, 0)]) = u_{A_-}.$$

PROOF. For $y = (y_1, \dots, y_N) \in \mathbb{Z}^N$, put

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} = (1 - A^t) \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}.$$

We then have

$$\xi_A(z) = \begin{bmatrix} z_1 \\ \vdots \\ z_N \\ 0 \\ 0 \end{bmatrix} = (1 - A_-^t) \begin{bmatrix} y_1 \\ \vdots \\ y_N \\ 0 \\ -y_N \end{bmatrix}.$$

Hence we have $\xi_A((1-A^t)\mathbb{Z}^N) \subset (1-A_-^t)\mathbb{Z}^{N+2}$ so that $\xi_A\colon \mathbb{Z}^N \to \mathbb{Z}^{N+2}$ induces a homomorphism from $G(A)$ to $G(A_-)$ denoted by $\bar{\xi}_A$. Suppose that $[\xi_A(x_1,\dots,x_N)] = 0$ in $G(A_-)$ so that

$$\begin{bmatrix} x_1 \\ \vdots \\ x_N \\ 0 \\ 0 \end{bmatrix} = (1-A_-^t)\begin{bmatrix} z_1 \\ \vdots \\ z_N \\ z_{N+1} \\ z_{N+2} \end{bmatrix}$$

for some $(z_1,\dots,z_{N+2}) \in \mathbb{Z}^{N+2}$. It then follows that $z_{N+1} = 0$, $z_{N+2} = -z_N$ so that

$$\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = (1-A^t)\begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}.$$

This implies $[(x_1,\dots,x_N)] = 0$ in $G(A)$ and hence $\bar{\xi}_A$ is injective.

For $(x_1,\dots,x_N,x_{N+1},x_{N+2}) \in \mathbb{Z}^{N+2}$, we have

$$\begin{aligned}
\begin{bmatrix} x_1 \\ \vdots \\ x_N \\ x_{N+1} \\ x_{N+2} \end{bmatrix} &= \begin{bmatrix} x_1 \\ \vdots \\ x_{N-1} \\ x_N - x_{N+2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{N+2} \\ x_{N+1} \\ x_{N+2} \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ \vdots \\ x_{N-1} \\ x_N - x_{N+2} \\ 0 \\ 0 \end{bmatrix} + (1-A_-^t)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -x_{N+2} \\ -x_{N+1} \end{bmatrix}.
\end{aligned}$$

This implies that

$$[(x_1,\dots,x_N,x_{N+1},x_{N+2})] = \bar{\xi}_A([(x_1,\dots,x_{N-1},x_N - x_{N+2})])$$

in $G(A_-)$. Therefore $\bar{\xi}_A\colon G(A) \to G(A_-)$ is surjective and hence an isomorphism. In particular, we see that $[(1,\dots,1,1,1)] = \bar{\xi}_A([(1,\dots,1,0)])$ in $G(A_-)$.

We have the following theorem by the preceding two lemmas.

THEOREM 3. *For an $N \times N$ matrix A with entries in $\{0, 1\}$, let $\bar{A}$ be the $(N+3) \times (N+3)$ matrix with entries in $\{0, 1\}$ defined in (1). Then there exists an isomorphism $\Phi: G(A) \to G(\bar{A})$ such that $\Phi(u_A) = u_{\bar{A}}$ and the matrices A, $\bar{A}$ satisfy $\det(1-A) = -\det(1-\bar{A})$. If A is an irreducible non-permutation matrix, so is $\bar{A}$.*

PROOF. Define $\Phi: G(A) \to G(\bar{A})$ by $\Phi = \bar{\xi}_{A^\circ} \circ \bar{\eta}_A^{-1}$ so that $\Phi(u_A) = \bar{\xi}_{A^\circ}([(1, \dots, 1, 0)]) = u_{\bar{A}}$. Since $\det(1-\bar{A}) = -\det(1-A^\circ) = -\det(1-A)$, we see the desired assertion.

Let P be an $N \times N$ permutation matrix coming from a permutation of the set $\{1, 2, \dots, N\}$. Since there exists a natural isomorphism $\Phi_P: G(A) \longrightarrow G(PAP^{-1})$ such that $\Phi_P(u_A) = u_{PAP^{-1}}$ and $\det(1-A) = \det(1-PAP^{-1})$, the triplet $(G(A), u_A, \det(1-A))$ does not depend on the choice of the vertex v_N in the directed graph of the matrix A.

We have some corollaries.

COROLLARY 4. *Let A be an irreducible non-permutation matrix with entries in $\{0, 1\}$. Then $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{\bar{A}}$ and $\det(1-A) = -\det(1-\bar{A})$.*

Let $\bar{1}$ denote the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which is the matrix $\bar{A}$ for the 1×1 matrix $A = [1]$. By the above theorem, we have $(K_0(\mathcal{O}_{\bar{1}}), u_{\bar{1}}) = (\mathbb{Z}, 1)$. Hence the simple purely infinite C^*-algebra $\mathcal{O}_{\bar{1}}$ has the same K-theory as the C^*-algebra $\mathcal{O}_1 = C(S^1)$ of the continuous functions on the unit circle S^1 with the positions of their units, whereas $(K_0(\mathcal{O}_{1_-}), u_{1_-}) = (\mathbb{Z}, 0)$ for the matrix $1_- = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ by [8] (cf. [5, p. 150]).

The following corollary has been shown in [12]. Its proof is now easy by using [14].

COROLLARY 5 ([12, Lemma 3.7]). *Let F be a finitely generated abelian group and u an element of F. Let $s = 0$ when F is infinite and $s = -1$ or 1 when F is finite. Then there exists an irreducible non-permutation matrix A such that*

$$(F, u, s) = (G(A), u_A, \operatorname{sgn}(\det(1-A))).$$

PROOF. By [14, Proposition 6.7 (i)], we know that there exists an irreducible non-permutation matrix A such that $(F, u) = (G(A), u_A)$. If $s = \operatorname{sgn}(\det(1-A))$, the matrix A is the desired one, otherwise $\bar{A}$ is the desired one.

Let A and B be two irreducible non-permutation matrices with entries in $\{0, 1\}$. The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *flip* continuously orbit equivalent if (X_A, σ_A) is continuously orbit equivalent to either (X_B, σ_B) or $(X_{\bar{B}}, \sigma_{\bar{B}})$. Similarly two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are said to be *flip* flow equivalent if $(\bar{X}_A, \bar{\sigma}_A)$ is flow equivalent to either $(\bar{X}_B, \bar{\sigma}_B)$, or $(\bar{X}_{\bar{B}}, \bar{\sigma}_{\bar{B}})$. We thus have the following corollaries.

COROLLARY 6. *Let A, B be irreducible matrices with entries in* $\{0, 1\}$ *that are not permutation matrices.*

(i) *$\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$ if and only if the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are flip continuously orbit equivalent.*

(ii) *$\mathcal{O}_A$ is stably isomorphic to $\mathcal{O}_B$ if and only if the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flip flow equivalent.*

Let us denote by $[\mathcal{O}_A]$ the isomorphism class of the Cuntz-Krieger algebra $\mathcal{O}_A$ as a C^*-algebra. Since $(G(A), u_A)$ is isomorphic to $(G(\bar{A}), u_{\bar{A}})$, we have $[\mathcal{O}_A] = [\mathcal{O}_{\bar{A}}]$. We regard the sign $\operatorname{sgn}(\det(1 - A))$ of $\det(1 - A)$ as the orientation of the class $[\mathcal{O}_A]$. Then we can say that the pair $([\mathcal{O}_A], \operatorname{sgn}(\det(1 - A)))$ is a complete invariant of the continuous orbit equivalence class of the one-sided topological Markov shift (X_A, σ_A).

In the rest of this short note, we present another square matrix $\tilde{A}$ of size $N + 3$ from a square matrix $A = [A(i, j)]_{i,j=1}^N$ of size N such that $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{\tilde{A}}$ and $\det(1 - A) = -\det(1 - \tilde{A})$. Define $(N + 3) \times (N + 3)$ matrix $\tilde{A}$ by setting

$$\tilde{A} = \begin{bmatrix} A(1,1) & \dots & A(1,N-1) & A(1,N) & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ A(N-1,1) & \dots & A(N-1,N-1) & A(N-1,N) & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ A(N,1) & \dots & A(N,N-1) & A(N,N) & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The difference between the previous matrix $\bar{A}$ in (1) and the above matrix $\tilde{A}$ is only the $(N + 2, N + 2)$-component. Its graphical expression of the matrix $\tilde{A}$ from A is the following figure.

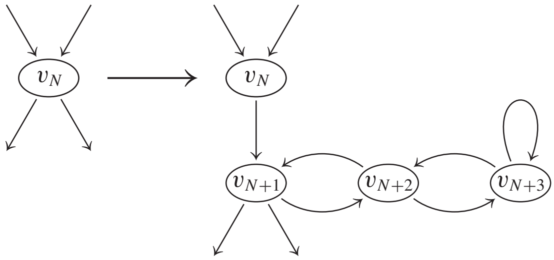

FIGURE 2

By virtue of [8], we know the following proposition.

PROPOSITION 7. *The Cuntz-Krieger algebras* $\mathcal{O}_{\bar{A}}$ *and* $\mathcal{O}_{\tilde{A}}$ *are isomorphic, and* $\det(1-\bar{A}) = \det(1-\tilde{A})$.

PROOF. Let us denote by $\bar{A}_i$ the ith row vector of the matrix $\bar{A}$ of size $N+3$. We put E_i the row vector of size $N+3$ such that $E_i = (0,\dots,0,\overset{i}{1},0,\dots,0)$ where the ith component is one, and the other components are zero. Then we have $\bar{A}_{N+2} = E_{N+1} + \bar{A}_{N+3}$. Since the $(N+2)$th row $\tilde{A}_{N+2}$ of $\tilde{A}$ is $\tilde{A}_{N+2} = E_{N+1} + E_{N+3}$, and the other rows of $\tilde{A}$ are the same as those of $\bar{A}$, the matrix $\tilde{A}$ is obtained from $\bar{A}$ by the primitive transfer

$$\bar{A} \underset{E_{N+1}+\bar{A}_{N+3}\to\tilde{A}_{N+2}}{\Longrightarrow} \tilde{A}$$

in the sense of [8, Definition 3.5]. We obtain that $\mathcal{O}_{\bar{A}}$ is isomorphic to $\mathcal{O}_{\tilde{A}}$ by [8, Theorem 3.7], and $\det(1-\bar{A}) = \det(1-\tilde{A})$ by [8, Theorem 8.4].

Before ending this short note, we refer to differences among the three matrices A_-, $\bar{A}$ and $\tilde{A}$ from a view point of dynamical system. As $(G(A_-), \det(1-A_-)) = (G(\bar{A}), \det(1-\bar{A})) = (G(\tilde{A}), \det(1-\tilde{A}))$, there is a possibility that their two-sided topological Markov shifts $(\bar{X}_{A_-}, \bar{\sigma}_{A_-})$, $(\bar{X}_{\bar{A}}, \bar{\sigma}_{\bar{A}})$, $(\bar{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ are topologically conjugate. We however know that they are not topologically conjugate to each other in general by the following example. Denote by $p_n(\bar{\sigma}_A)$ the cardinal number of the n-periodic points $\{x \in \bar{X}_A \mid \bar{\sigma}_A^n(x) = x\}$ of the topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$. The zeta function $\zeta_A(z)$ for $(\bar{X}_A, \bar{\sigma}_A)$ is defined by

$$\zeta_A(z) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(\bar{\sigma}_A)}{n} z^n\right)$$

(cf. [10]). It is well-known that the formula $\zeta_A(z) = \frac{1}{\det(1-zA)}$ holds [3]. Let us denote by 2_-, $\bar{2}$, $\tilde{2}$ the matrices A_-, $\bar{A}$, $\tilde{A}$ for $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ respectively. It is straightforward to see that

$$\zeta_{2_-}(z) = \frac{1}{1 - 4z + 3z^2 + 2z^3 - z^4},$$

$$\zeta_{\bar{2}}(z) = \frac{1}{1 - 3z + 4z^3 - z^4},$$

$$\zeta_{\tilde{2}}(z) = \frac{1}{1 - 2z - 2z^2 + 4z^3}.$$

The zeta function is invariant under topological conjugacy so that the topological Markov shifts $(\bar{X}_{2_-}, \bar{\sigma}_{2_-})$, $(\bar{X}_{\bar{2}}, \bar{\sigma}_{\bar{2}})$, $(\bar{X}_{\tilde{2}}, \bar{\sigma}_{\tilde{2}})$ are not topologically conjugate to each other.

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