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k-SHELLABLE SIMPLICIAL COMPLEXES AND GRAPHS

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Abstract

In this paper we show that a k-shellable simplicial complex is the expansion of a shellable complex. We prove that the face ring of a pure k-shellable simplicial complex satisfies the Stanley conjecture. In this way, by applying an expansion functor to the face ring of a given pure shellable complex, we construct a large class of rings satisfying the Stanley conjecture.

Also, by presenting some characterizations of k-shellable graphs, we extend some results due to Castrillón-Cruz, Cruz-Estrada and Van Tuyl-Villareal.

Introduction

Let Δ be a simplicial complex on the vertex set $X := \{x_1, \dots, x_n\}$. Denote by $\langle F_1, \dots, F_r\rangle$ the simplicial complex Δ with facets $F_1, \dots, F_r$. Δ is called *shellable* if its facets can be given a linear order $F_1, \dots, F_r$, called a *shelling order*, such that for all $2 \leq j$, the subcomplex $\langle F_j\rangle \cap \langle F_1, \dots, F_{j-1}\rangle$ is pure of dimension $\dim(F_j) - 1$ (see [3] for probably the earliest definition of this term and [2] for a more recent exposition). Studying combinatorial properties of shellable simplicial complexes and algebraic constructions of their face rings and also the edge ideals associated to shellable graphs is a current trend in combinatorics and commutative algebra. See for example [2], [3], [4], [8], [11], [20].

In this paper, we recall from [15] the concept of k-shellability, and extend some results obtained previously by researchers. Actually, k is a positive integer and for $k = 1$, 1-shellability coincides with shellability.

Richard Stanley [18], in his famous article "Linear Diophantine equations and local cohomology", made a striking conjecture predicting an upper bound for the depth of a multigraded module. This conjecture is nowadays called the Stanley conjecture and the conjectured upper bound is called the Stanley depth of a module. The Stanley conjecture has become quite popular, with numerous publications dealing with different aspects of the Stanley depth. Although a counterexample has apparently recently been found to the Stanley conjecture

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(see [9]), this makes it perhaps even more interesting to explore the relationship between depth and Stanley depth.

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K. Dress proved in [8] that the simplicial complex Δ is shellable if and only if its face ring is clean. It is also known that cleanness implies pretty cleanness. Furthermore, Herzog and Popescu [11, Theorem 6.5] proved that, if $I \subset S$ is a monomial ideal, and S/I is a multigraded pretty clean ring, then the Stanley conjecture holds for S/I. It follows that, for a shellable simplicial complex Δ, the face ring $K[\Delta] = S/I_\Delta$ satisfies the Stanley conjecture where I_Δ denotes the Stanley-Reisner ideal of Δ. We extend this result, in the pure case, by showing that the face ring of a k-shellable simplicial complex satisfies the Stanley conjecture (see Theorem 3.2). We obtain this result by extending Proposition 8.2 of [11] and by presenting a filtration for the face ring of a k-shellable simplicial complex in Theorem 2.9.

A simple graph G is called shellable if its independence complex Δ_G is a shellable simplicial complex. Shellable graphs were studied by several researchers in recent years. For example, Van Tuyl and Villarreal in [20] classified all of shellable bipartite graphs. Also, Castrillón and Cruz characterized the shellable graphs and clutters by using the properties of simplicial vertices, shedding vertices and shedding faces [4].

Here, we present some characterizations of k-shellable graphs and extend some results of [4], [6] and [20] (see Theorems 4.5, 4.7 and 4.10). Our idea is to define a new notion, called a k-simplicial set, which is a generalization of the notion of simplicial vertex defined in [7] or [13].

1. Preliminaries

For basic definitions and general facts on simplicial complexes, we refer to Stanley's book [19].

A simplicial complex Δ is *pure* if all of its facets (maximal faces) are of the same dimension. The *link* and *deletion* of a face F in Δ are defined respectively by

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}$$

and

$$\mathrm{dl}_\Delta(F) = \{G \in \Delta : F \nsubseteq G\}.$$

Let G be a simple (no loops or multiple edges) undirected graph on the vertex set $V(G) = X$ and the edge set $E(G)$. The *independence complex* of G is denoted by Δ_G and F is a face of Δ_G if and only if there is no edge of G joining any two vertices of F. The *edge ideal* of G is defined a quadratic squarefree monomial ideal $I(G) = (x_i x_j : x_i x_j \in E(G))$. It is known that

$I(G) = I_{\Delta_G}$. We say G is a shellable graph if Δ_G is a shellable simplicial complex.

In the following we recall the concept of expansion functor in a combinatorial and an algebraic setting from [16] and [1], respectively.

Let $\alpha = (k_1, \ldots, k_n)$ be an n-tuple with positive integer entries in $\mathbb{N}^n$. For $F = \{x_{i_1}, \ldots, x_{i_r}\} \subseteq X$ define

$$F^\alpha = \{x_{i_1 1}, \ldots, x_{i_1 k_{i_1}}, \ldots, x_{i_r 1}, \ldots, x_{i_r k_{i_r}}\}$$

as a subset of $X^\alpha := \{x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}\}$. F^α is called *the expansion of F with respect to α*.

For a simplicial complex $\Delta = \langle F_1, \ldots, F_r \rangle$ on X, we define *the expansion of Δ with respect to α* as the simplicial complex $\Delta^\alpha = \langle F_1^\alpha, \ldots, F_r^\alpha \rangle$ (see [16]).

In [1], Bayati and Herzog defined the expansion functor in the category of finitely generated multigraded S-modules and studied some homological behaviors of this functor. We recall the expansion functor defined by them only in the category of monomial ideals and refer the reader to [1] for more general case in the category of finitely generated multigraded S-modules.

Set S^α a polynomial ring over K in the variables

$$x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}.$$

Whenever $I \subset S$ is a monomial ideal minimally generated by $u_1, \ldots, u_r$, the expansion of I with respect to α is defined by

$$I^\alpha = \sum_{i=1}^{r} P_1^{\nu_1(u_i)} \ldots P_n^{\nu_n(u_i)} \subset S^\alpha,$$

where $P_j = (x_{j1}, \ldots, x_{jk})$ is a prime ideal of S^α and $\nu_j(u_i)$ is the exponent of x_j in u_i.

EXAMPLE 1.1. Let $I \subset K[x_1, \ldots, x_3]$ be a monomial ideal minimally generated by $G(I) = \{x_1^2 x_2, x_1 x_3, x_2 x_3^2\}$ and let $\alpha = (2, 2, 1) \in \mathbb{N}^3$. Then

$$\begin{aligned} I^\alpha &= (x_{11}, x_{12})^2 (x_{21}, x_{22}) + (x_{11}, x_{12})(x_{31}) + (x_{21}, x_{22})(x_{31})^2 \\ &= (x_{11}^2 x_{21}, x_{11}x_{12}x_{21}, x_{12}^2 x_{21}, x_{11}^2 x_{22}, \\ &\quad x_{11}x_{12}x_{22}, x_{12}^2 x_{22}, x_{11}x_{31}, x_{12}x_{31}, x_{21}x_{31}^2, x_{22}x_{31}^2) \end{aligned}$$

It was shown in [1] that the expansion functor is exact and so $(S/I)^\alpha = S^\alpha / I^\alpha$. The following lemma implies that two above concepts of expansion functor are related.

LEMMA 1.2 ([16, Lemma 2.1]). *For a simplicial complex Δ and $\alpha \in \mathbb{N}^n$ we have $(I_\Delta)^\alpha = I_{\Delta^\alpha}$. In particular, $K[\Delta]^\alpha = K[\Delta^\alpha]$.*

In this paper we just study the functors $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ with $k_i = k_j$ for all i, j. For convenience, we set $\alpha = [k]$ when every component of α is equal to $k \in \mathbb{N}$. We call $I^{[k]}$ (resp. $\Delta^{[k]}$) the expansion of I (resp. Δ) with respect to k.

2. Some combinatorial and algebraic properties of k-shellable complexes

The notion of k-shellable simplicial complexes was first introduced by Emtander, Mohammadi and Moradi [10] to provide a natural generalization of shellability. It was shown in [10, Theorem 6.8] that a simplicial complex Δ is k-shellable if and only if the Stanley-Reisner ideal of its Alexander dual has k-quotients, i.e. there exists an ordering $u_1, \dots, u_r$ of the minimal generators of $I_{\Delta^\vee}$ such that if we for $s = 1, \dots, t$, put $I_s = (u_1, \dots, u_s)$, then for every s there are monomials v_{s_i} , $i = 1, \dots, r_s$, $\deg(v_{s_i}) = k$ for all i, such that $I_s : u_s = (v_{s_1}, \dots, v_{s_{r_s}})$.

In [15], we gave another definition of k-shellability and having k-quotients by adding a condition to Emtander, Mohammadi and Moradi's. In our definition the colon ideals $I_s : u_s$ were generated by regular sequences for all s and in this way, all of structural properties of monomial ideals with linear quotients were generalized. The reader is referred to [12] for the definition of monomial ideals with linear quotients.

DEFINITION 2.1 ([15]). Let Δ be a d-dimensional simplicial complex on X and let k be an integer with $1 \le k \le d+1$. Δ is called *k-shellable* if its facets can be ordered $F_1, \dots, F_r$, called *k-shelling order*, such that for all $j = 2, \dots, r$, the subcomplex $\Delta_j = \langle F_j \rangle \cap \langle F_1, \dots, F_{j-1} \rangle$ satisfies the following properties:

(i) it is generated by a nonempty set of maximal proper faces of $\langle F_j \rangle$ of dimension $|F_j| - k - 1$;

(ii) if Δ_j has more than one facet then for every two disjoint facets $\sigma, \tau \in \langle F_j \rangle \cap \langle F_1, \dots, F_{j-1} \rangle$ we have $F_j \subseteq \sigma \cup \tau$.

REMARK 2.2. It follows from the definition that two concepts 1-shellability and shellability coincide.

REMARK 2.3. Note that the notions of 1-shellability in our sense and Emtander, Mohammadi and Moradi's coincide. Although, for $k > 1$, a simplicial complex may be k-shellable in their concept and not in ours. For example, consider the complex $\Delta = \langle abc, aef, cdf\rangle$ on $\{a, b, \dots, f\}$. It is easy to check that Δ is 2-shellable in the sense of [10] but not in ours.

In the following proposition we describe some the combinatorial properties of k-shellable complexes.

PROPOSITION 2.4. *Let Δ be a d-dimensional (not necessarily pure) simplicial complex on X and let k be an integer with $1 \le k \le d+1$. Suppose that the facets of Δ can be ordered $F_1, \dots, F_r$. Then the following conditions are equivalent:*

(a) *$F_1, \dots, F_r$ is a k-shelling of Δ;*

(b) *for every $1 \le j \le r$ there exist the subsets $E_1, \dots, E_t$ of X such that the E_i are mutually disjoint and $|E_i| = k$ for all i and the set of the minimal elements of $\langle F_1, \dots, F_j\rangle \setminus \langle F_1, \dots, F_{j-1}\rangle$ is $\{\{a_1, \dots, a_t\} : a_i \in E_i$ for all $i\}$;*

(c) *for all i, j, $1 \le i < j \le r$, there exist $x_1, \dots, x_k \in F_j \setminus F_i$ and some $\ell \in \{1, \dots, j-1\}$ with $F_j \setminus F_\ell = \{x_1, \dots, x_k\}$.*

PROOF. (a) $\Rightarrow$ (b): let $\langle F_j\rangle \cap \langle F_1, \dots, F_{j-1}\rangle = \langle F_j \setminus \sigma_1, \dots, F_j \setminus \sigma_t\rangle$ where $|\sigma_i| = k$ for all i. Since for all $i \ne i'$, $F_j \subseteq (F_j \setminus \sigma_i) \cup (F_j \setminus \sigma_{i'})$, we have $\sigma_i \cap \sigma_{i'} = \emptyset$. Hence the minimal elements of $\langle F_1, \dots, F_j\rangle \setminus \langle F_1, \dots, F_{j-1}\rangle$ are in the form $\{a_1, \dots, a_t\}$ where $a_i \in \sigma_i$ for all i.

(b) $\Rightarrow$ (c): for all i, suppose that $E_i = \{x_{i1}, \dots, x_{ik}\}$. Let $1 \le i < j \le r$ and let $\{x_{1i_1}, \dots, x_{ti_t}\}$ be a minimal element of $\langle F_1, \dots, F_j\rangle \setminus \langle F_1, \dots, F_{j-1}\rangle$. Because $\{x_{1i_1}, \dots, x_{ti_t}\} \nsubseteq F_i$, we may assume that $x_{1i_1} \in F_j \setminus F_i$. We claim that $x_{11}, \dots, x_{1k} \in F_j \setminus F_i$. Suppose, on the contrary, that for some s, $x_{1s} \notin F_j \setminus F_i$ then $x_{1s} \in F_i$ and so $x_{1s} \notin \langle F_1, \dots, F_j\rangle \setminus \langle F_1, \dots, F_{j-1}\rangle$. It follows that $\{x_{1s}, x_{2i_2}, \dots, x_{ti_t}\}$ is not a minimal element of $\langle F_1, \dots, F_j\rangle \setminus \langle F_1, \dots, F_{j-1}\rangle$, a contradiction. Therefore $E_1 \subseteq F_j \setminus F_i$.

Now suppose that for all $\ell < j$, if E_1 is contained in $F_j \setminus F_\ell$ then $E_1 \subsetneqq F_j \setminus F_\ell$. Then there exists $y \in F_j \setminus E_1$ such that $\{y, x_{2i_2}, \dots, x_{ti_t}\}$ is a minimal element of $\langle F_1, \dots, F_j\rangle \setminus \langle F_1, \dots, F_{j-1}\rangle$ different from the elements of $\{\{a_1, \dots, a_t\} : a_i \in E_i$ for all $i\}$, a contradiction. Therefore there exists $\ell < j$ with $F_j \setminus F_\ell = E_1$.

(c) $\Rightarrow$ (a): let $F \in \langle F_j\rangle \cap \langle F_1, \dots, F_{j-1}\rangle$. Then $F \subseteq F_i$ for some $i < j$. By the condition (c), there exist $x_1, \dots, x_k \in F_j \setminus F_i$ and some $\ell \in \{1, \dots, j-1\}$ with $F_j \setminus F_\ell = \{x_1, \dots, x_k\}$. But $F_j \setminus \{x_1, \dots, x_k\}$ is a proper face of $\langle F_j\rangle \cap \langle F_1, \dots, F_{j-1}\rangle$, because $F_j \setminus \{x_1, \dots, x_k\} = F_j \cap F_\ell$. Moreover, $F_j \setminus \{x_1, \dots, x_k\}$ is a maximal face. Finally, since F is contained in $F_j \setminus \{x_1, \dots, x_k\}$, the assertion is completed.

EXAMPLE 2.5. The Figure 1 indicates the pure shellable and pure 2-shellable simplicial complexes of dimensions 1, 2 and 3 with 3 facets.

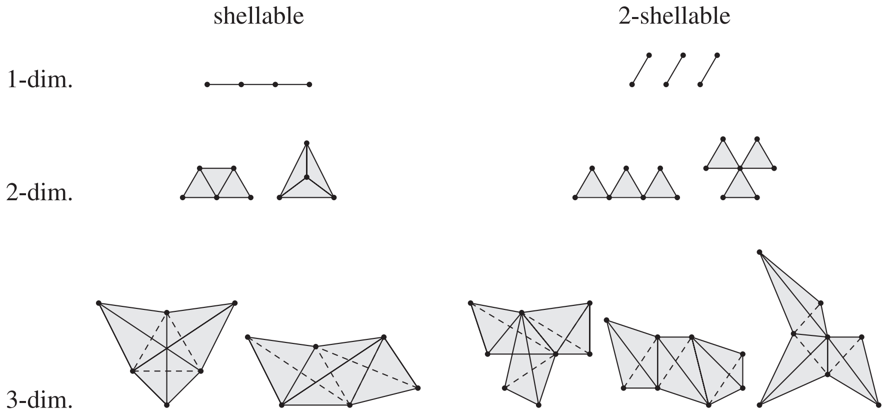

FIGURE 1

THEOREM 2.6. *Let Δ be a k-shellable complex and σ a face of Δ. Then $\mathrm{lk}_\Delta(\sigma)$ is again k-shellable.*

PROOF. Since Δ is k-shellable, so there exists an k-shelling order $F_1, \dots, F_r$ of facets of Δ. Let $F_{i_1}, \dots, F_{i_t}$ where $i_1 < \cdots < i_t$ be all of facets which contain σ. We claim that $F_{i_1} \setminus \sigma, \dots, F_{i_t} \setminus \sigma$ is a k-shelling order of $\mathrm{lk}_\Delta(\sigma)$. To this end we want to show that the condition (c) of Proposition 2.4 holds.

Set $G_j = F_{i_j} \setminus \sigma$. Consider ℓ, m with $1 \le \ell < m \le t$. By k-shellability of Δ, there are $x_1, \dots, x_k \in F_{i_m} \setminus F_{i_\ell} = (F_{i_m} \setminus \sigma) \setminus (F_{i_\ell} \setminus \sigma) = G_m \setminus G_\ell$ such that for some $s < i_m$ we have $\{x_1, \dots, x_k\} = F_{i_m} \setminus F_s$. It follows from $\sigma \subset F_{i_m}$ and $F_{i_m} \setminus F_s = \{x_1, \dots, x_k\}$ that $\sigma \subset F_s$. This implies that F_s is among the list $F_{i_1}, \dots, F_{i_m}$. Let $F_{i_{m'}} = F_s$. Hence $G_m \setminus G_{m'} = \{x_1, \dots, x_k\}$ and the assertion is completed.

For the simplicial complexes Δ_1 and Δ_2 defined on disjoint vertex sets, the *join* of Δ_1 and Δ_2 is $\Delta_1 \cdot \Delta_2 = \{\sigma \cup \tau : \sigma \in \Delta_1, \tau \in \Delta_2\}$.

THEOREM 2.7. *The simplicial complexes Δ_1 and Δ_2 are k-shellable if and only if $\Delta_1 \cdot \Delta_2$ is k-shellable.*

PROOF. Let Δ_1 and Δ_2 be k-shellable. Let $F_1, \dots, F_r$ and $G_1, \dots, G_s$ be, respectively, the k-shelling orders of Δ_1 and Δ_2. We claim that

$$F_1 \cup G_1, F_1 \cup G_2, \dots, F_1 \cup G_s, \dots, F_r \cup G_1, F_r \cup G_2, \dots, F_r \cup G_s$$

is a k-shelling order of $\Delta_1 \cdot \Delta_2$.

Let $F_i \cup G_j$ be a facet of $\Delta_1 \cdot \Delta_2$ which comes after $F_p \cup G_q$ in the above order. We have some cases:

Let $p < i$. Since Δ_1 is k-shellable, there exist $u_1, \ldots, u_k \in F_i \setminus F_p$ and some $\ell < i$ such that $F_i \setminus F_\ell = \{u_1, \ldots, u_k\}$. It follows that $u_1, \ldots, u_k \in (F_i \cup G_j) \setminus (F_p \cup G_q)$ and $(F_i \cup G_j) \setminus (F_\ell \cup G_j) = \{u_1, \ldots, u_k\}$.

Let $p = i$ and $q < j$. Since Δ_2 is k-shellable, there exist $v_1, \ldots, v_k \in G_j \setminus G_q$ and some $m < j$ such that $G_j \setminus G_m = \{v_1, \ldots, v_k\}$. Therefore we obtain $v_1, \ldots, v_k \in (F_i \cup G_j) \setminus (F_p \cup G_q)$ and $(F_i \cup G_j) \setminus (F_i \cup G_m) = \{v_1, \ldots, v_k\}$.

Conversely, suppose that $\Delta_1 \cdot \Delta_2$ is k-shellable with the k-shelling order $F_{i_1} \cup G_{j_1}, \ldots, F_{i_t} \cup G_{j_t}$. Let $F_{s_1}, \ldots, F_{s_r}$ be the ordering obtained from $F_{i_1} \cup G_{j_1}, \ldots, F_{i_t} \cup G_{j_t}$ after removing the repeated facets beginning on the left-hand. Then it is easy to check that $F_{s_1}, \ldots, F_{s_r}$ is a k-shelling order of Δ_1. In a similar way, it is shown that Δ_2 is k-shellable.

The following theorem, relates the expansion of a shellable complex to a k-shellable complex.

THEOREM 2.8. *Let Δ be a simplicial complex and $k \in \mathbb{N}$. Then Δ is shellable if and only if $\Delta^{[k]}$ is k-shellable.*

PROOF. Let $\Delta = \langle F_1, \ldots, F_r \rangle$ and let $\Delta_j = \langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ for $j = 2, \ldots, r$. Fix an integer j. If $\Delta_j = \langle F_j \setminus x_{i_1}, \ldots, F_j \setminus x_{i_t} \rangle$, then

$$\begin{aligned}\Delta_j^{[k]} &= \langle F_j^{[k]} \rangle \cap \langle F_1^{[k]}, \ldots, F_{j-1}^{[k]} \rangle \\ &= \langle F_j^{[k]} \setminus \{x_{i_1 1}, \ldots, x_{i_1 k}\}, \ldots, F_j^{[k]} \setminus \{x_{i_t 1}, \ldots, x_{i_t k}\} \rangle.\end{aligned}$$

Now by the Definition 2.1, if $F_1, \ldots, F_r$ is a shelling order of Δ then $F_1^{[k]}, \ldots, F_r^{[k]}$ is a k-shelling order of $\Delta^{[k]}$.

Conversely, suppose that $F_1^{[k]}, \ldots, F_r^{[k]}$ is a k-shelling order of $\Delta^{[k]}$ and set $\Delta_j^{[k]} = \langle F_j^{[k]} \rangle \cap \langle F_1^{[k]}, \ldots, F_{j-1}^{[k]} \rangle$ for $j = 2, \ldots, r$. Fix an index j. Hence $\Delta_j^{[k]} = \langle F_j^{[k]} \setminus \sigma_1, \ldots, F_j^{[k]} \setminus \sigma_t \rangle$ with $|\sigma_i| = k$ for all i. By Proposition 2.4(b), $\sigma_\ell \cap \sigma_m = \emptyset$ for all $\ell \neq m$. We claim that for every i, σ_i is the expansion of a singleton set. Suppose, on the contrary, that for some σ_s we have $x_{i_1 \ell}, x_{i_2 m} \in \sigma_s$ with $i_1 \neq i_2$ and let $F_j^{[k]} \cap F_{s'}^{[k]} = F_j^{[k]} \setminus \sigma_s$ for some s'. It follows from $|\sigma_s| = k$ that $x_{i_1 \ell'} \notin \sigma_s$ for some ℓ' with $1 \leq \ell' \leq k$. In particular, we conclude that $x_{i_1 \ell} \notin F_{s'}^{[k]}$ but $x_{i_1 \ell'} \in F_{s'}^{[k]}$. This is a contradiction, because $F_{s'}^{[k]}$ is the expansion of $F_{s'}$.

Therefore we conclude that for all $j = 2, \ldots, r$, the complex $\Delta_j^{[k]}$ is in the form

$$\Delta_j^{[k]} = \big\langle F_j^{[k]} \setminus \{x_{i_1}\}^{[k]}, \ldots, F_j^{[k]} \setminus \{x_{i_t}\}^{[k]} \big\rangle.$$

Finally, for all j, $\Delta_j = \langle F_j \rangle \cap \langle F_1, \dots, F_{j-1} \rangle$ will be in the form $\Delta_j = \langle F_j \setminus x_{i_1}, \dots, F_j \setminus x_{i_t} \rangle$. This implies that $\langle F_j \rangle \cap \langle F_1, \dots, F_{j-1} \rangle$ is pure of dimension $\dim(F_j) - 1$ for all $j \geq 2$, as desired.

Let R be a Noetherian ring and M be a finitely generated multigraded R-module. We call

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

a *multigraded finite filtration* of submodules of M if there exist the positive integers $a_1, \dots, a_r$ such that $M_i/M_{i-1} \cong \prod_{j=1}^{a_i} R/P_i(-\mathbf{a}_{ij})$ for some $P_i \in \operatorname{Supp}(M)$. $\mathcal{F}$ is called a *multigraded prime filtration* if $a_1 = \cdots = a_r = 1$. It is well known that every finitely generated multigraded R-module M has a multigraded prime filtration (see for example [14, Theorem 6.4]). In the following we present a multigraded finite filtration for the face ring of a k-shellable simplicial complex which we need in Section 3.

For $F \subset X$. We set $F^c = X \setminus F$ and $P_F = (x_i : x_i \in F)$.

THEOREM 2.9. *Let Δ be a simplicial complex and k a positive integer. If $F_1, \dots, F_r$ is a k-shelling order of Δ then there exists a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = S/I_\Delta$ with*

$$M_i = \bigcap_{\ell=1}^{r-i} P_{F_\ell^c}$$

and

$$M_i/M_{i-1} \cong \prod_{j=1}^{k^{a_i}} S/P_{F_{r-i+1}^c}(-\mathbf{a}_{ij}),$$

for all $i = 1, \dots, r$. Here $a_i = |\mathbf{a}_{ij}|$, for all $j = 1, \dots, k^{a_i}$.

PROOF. We set $a_1 = 0$ and for each $i > 2$ we denote by a_i the number of facets of $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$. If $F_1, \dots, F_r$ is a k-shelling of Δ, then for $i = 2, \dots, r$ we have

$$\bigcap_{j=1}^{i-1} P_{F_j^c} + P_{F_i^c} = P_{F_i^c} + P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}}, \tag{1}$$

where $\sigma_{i\ell} = F_i \setminus F_{i_\ell}$ and $|\sigma_{i\ell}| = k$ for $\ell = 1, \dots, a_i$. Actually, $F_{i_\ell} \cap F_i$'s are all facets of $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$. Since $\sigma_{i\ell} \cap \sigma_{i\ell'} = \emptyset$ for $1 \leq \ell < \ell' \leq a_i$, one can suppose that $P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} = (f_{ij} : j = 1, \dots, k^{a_i})$. Set $\mathbf{a}_{ij} = \deg(f_{ij})$

and it is clear that for all $j = 1, \dots, k^{a_i}$, $a_i = |\mathbf{a}_{ij}|$. We have the following isomorphisms:

$$\begin{aligned}\left(\bigcap_{j=1}^{i-1} P_{F_j^c}\right) \Big/ \left(\bigcap_{j=1}^{i} P_{F_j^c}\right) &\cong \bigcap_{j=1}^{i-1} P_{F_j^c} + P_{F_i^c} / P_{F_i^c} \\ &\cong P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} + P_{F_i^c} / P_{F_i^c} \\ &\cong P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} / (P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} \cap P_{F_i^c}) \\ &\cong P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} / P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} P_{F_i^c},\end{aligned}$$

where $a_i = |\mathbf{a}_{ij}|$, for $j = 1, \dots, k^{a_i}$. Now it is easy to check that the homomorphism

$$\begin{aligned}\theta\colon \prod^{k^{a_i}} S &\longrightarrow P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} / P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} P_{F_i^c}, \\ (r_1, \dots, r_{k^{a_i}}) &\longmapsto \sum_j^{k^{a_i}} r_j f_{ij} + P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} P_{F_i^c}\end{aligned}$$

is an epimorphism. In particular, it follows that

$$P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} / P_{\sigma_{i1}} \dots P_{\sigma_{ia_i}} P_{F_i^c} \cong \prod_{j=1}^{k^{a_i}} S / P_{F_i^c}(-\mathbf{a}_{ij}).$$

This completes the proof.

REMARK 2.10. In view of Theorem 2.9, let $\Delta = \langle F_1, \dots, F_r \rangle$ be a shellable simplicial complex and $\Delta_j = \langle F_1, \dots, F_j \rangle$. Then we have the prime filtration

$$(0) = I_\Delta \subset I_{\Delta_{r-1}} \subset \dots \subset I_{\Delta_1} \subset K[\Delta]$$

for $K[\Delta]$. In particular, it provides the following filtration for $K[\Delta^{[k]}]$:

$$(0) = I_{\Delta^{[k]}} \subset I_{\Delta^{[k]}_{r-1}} \subset \dots \subset I_{\Delta^{[k]}_1} \subset K[\Delta^{[k]}].$$

In other words, Theorem 2.9 gives a filtration for the face ring of the expansion of a shellable simplicial complex with respect to k.

REMARK 2.11. The filtration described in Theorem 2.9 in the case that $k > 1$ is not a prime filtration, i.e. the quotient of any two consecutive modules of the filtration is not cyclic. Consider the same notations of Theorem 2.9, we have the following prime filtration for $K[\Delta]$ when Δ has a k-shelling order:

$$\mathscr{F}: 0=\bigcap_{j=1}^{r} P_{F_j^c} \subset \cdots$$
$$\subset \bigcap_{j=1}^{i} P_{F_j^c} \subset \cdots \subset \sum_{j=1}^{k^{a_i}-1}(f_j)+\bigcap_{j=1}^{i} P_{F_j^c} \subset \sum_{j=1}^{k^{a_i}}(f_j)+\bigcap_{j=1}^{i} P_{F_j^c}=\bigcap_{j=1}^{i-1} P_{F_j^c}$$
$$\subset \cdots \subset K[\Delta],$$

where $(f_1, \ldots, f_{j-1}) : (f_j)$ is generated by linear forms for all $j = 2, \ldots, k^{a_i}$ and all $i = 1, \ldots, r$.

For all $2 \leq j \leq k^{a_i}$, suppose that $(f_1, \ldots, f_{j-1}) : (f_j) = P_{Q_j}$. Set $P_{Q_1} = (0)$. We have

$$\sum_{t=1}^{j}(f_t)+\bigcap_{t=1}^{i} P_{F_t^c} \Big/ \sum_{t=1}^{j-1}(f_t)+\bigcap_{t=1}^{i} P_{F_t^c}$$
$$\cong (f_j)/(f_j) \cap \left((f_1, \ldots, f_{j-1})+\bigcap_{t=1}^{i} P_{F_t^c}\right)$$
$$\cong (f_j)/f_j P_{L_{ij}},$$

where $L_{ij} = F_i^c \cup Q_j$, for $i = 1, \ldots, r$ and $j = 1, \ldots, k^{a_i}$. Therefore the set of prime ideals which defines the cyclic quotients of $\mathscr{F}$ is $\operatorname{Supp}(\mathscr{F}) = \{P_{L_{ij}} : i = 1, \ldots, r \text{ and } j = 1, \ldots, k^{a_i}\}$.

3. The Stanley conjecture

Consider a field K, and let R be a finitely generated $\mathbb{N}^n$-graded K-algebra, and let M be a finitely generated $\mathbb{Z}^n$-graded R-module. Stanley [18] conjectured that, in this case, there exist finitely many subalgebras $A_1, \ldots, A_r$ of R, each generated by algebraically independent $\mathbb{N}^n$-homogeneous elements of R, and there exist $\mathbb{Z}^n$-homogeneous elements $u_1, \ldots, u_r$ of M, such that $M = \bigoplus_{i=1}^{r} u_i A_i$, where $\dim(A_i) \geq \operatorname{depth}(M)$ for all i and where $u_i A_i$ is a free A_i-module of rank one.

Consider a finitely generated $\mathbb{Z}^n$-graded S-module M, a subset Z of $\{x_1, \ldots, x_n\}$, and a homogeneous element $u \in M$. The K-subspace $uK[Z]$ of M is called a *Stanley space of dimension* $|Z|$ if it is a free $K[Z]$-module of rank 1, i.e., the elements of the form uv, where v is a monomial in $K[Z]$, form a K-basis of $uK[Z]$. A *Stanley decomposition* of M is a decomposition $\mathscr{D}$ of M into a finite direct sum of Stanley spaces. The *Stanley depth* of $\mathscr{D}$, denoted sdepth($\mathscr{D}$), is the minimal dimension of a Stanley space in a decomposition $\mathscr{D}$. We set

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D}) : \mathscr{D} \text{ is a Stanley decomposition of } M\},$$

and we call this number the *Stanley depth of M*. The Stanley conjecture says that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ always holds.

The following lemma is needed in the proof of the main theorem of this section.

LEMMA 3.1. *Let $F_1, \dots, F_s \subset X$ with $F_i \cap F_j = \emptyset$, for all $i \neq j$ and $|F_i| = k > 1$. Let $f_1, \dots, f_{k^s}$ be a sequence of minimal generators of $P_{F_1} \dots P_{F_s}$ ordered with respect to lexicographical ordering $x_1 > x_2 > \dots > x_n$. Suppose that n_i is the minimal number of homogeneous generators of $(f_1, \dots, f_{i-1}) : (f_i)$ for $i = 2, \dots, k^s$. Then*

$$\max\{n_i : i = 2, \dots, k^s\} = n_{k^s} = (k-1)s.$$

Moreover, for all i, the colon ideal $(f_1, \dots, f_{i-1}) : (f_i)$ is generated by linear forms.

PROOF. By [5, Corollary 1.5], $P_{F_1} \dots P_{F_s}$ has linear quotients. To show equality, we use induction on s. If $s = 1$, the assertion is clear. Assume that $s > 1$. Let $P_{F_1} \dots P_{F_{s-1}} = (f_1, \dots, f_{k^{s-1}})$ and $F_s = \{x_1, \dots, x_k\}$. Then

$$P_{F_1} \dots P_{F_s} = x_1(f_1, \dots, f_{k^{s-1}}) + \dots + x_k(f_1, \dots, f_{k^{s-1}}).$$

Moreover,

$$(x_1 f_1, \dots, x_1 f_{k^{s-1}}, \dots, x_k f_1, \dots, x_k f_{k^{s-2}}) : x_k f_{k^{s-1}} = (x_1, \dots, x_{k-1}) + (f_1, \dots, f_{k^{s-2}}) : f_{k^{s-1}}.$$

Now, by the induction hypothesis, we have $n_{k^s} = (k-1) + (k-1)(s-1) = (k-1)s$, as desired.

Let

$$\mathscr{F}: 0 = M_0 \subset M_1 \subset \dots \subset M_r = M$$

be a prime filtration of M with $M_i/M_{i-1} \cong (S/P_i)(-\mathbf{a}_i)$. Then this filtration decomposes M as a multigraded K-vector space, that is, we have $M = \bigoplus_{i=1}^r u_i K[Z_i]$ and this is a Stanley decomposition of M where $u_i \in M_i$ is a homogeneous element of degree $\mathbf{a}_i$ and $Z_i = \{x_j : x_j \notin P_i\}$.

Now suppose that $\Delta = \langle G_1, \dots, G_r \rangle$ is a pure shellable simplicial complex on X and k is a positive integer. By Theorem 2.8, $\Delta^{[k]} = \langle G_1^{[k]}, \dots, G_r^{[k]} \rangle$ is pure k-shellable. For all i, set $F_i = G_i^{[k]}$. Consider the prime filtration $\mathscr{F}$ of $K[\Delta^{[k]}]$ described in Section 2. Then we have the Stanley decomposition

$$K[\Delta^{[k]}] = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{k^{a_i}} u_{ij} K[Z_{ij}],$$

where $Z_{ij} = \{x_\ell : \ell \notin L_{ij}\}$, $\deg(u_{ij}) = \mathbf{a}_{ij}$ and $|\mathbf{a}_{ij}| = a_i$ for all i, j. We claim that for all i, j, $|Z_{ij}| \geq \operatorname{depth}(K[\Delta^{[k]}])$.

By Corollaries 4.1 and 2.1 of [15], we have $\operatorname{depth}(K[\Delta^{[k]}]) = \dim(K[\Delta]) = |G_i|$. On the other hand, $|G_i| \geq a_i$. Now by combining all of these results with Lemma 3.1, we have

$$\begin{aligned} |Z_{ik^{a_i}}| &= kn - (|Q_{k^{a_i}}| + |F_i^c|) \\ &= kn - ((k-1)a_i + k\,\operatorname{ht}(P_{G_i^c})) \\ &= (k-1)(n - \operatorname{ht}(P_{G_i^c}) - a_i) + n - \operatorname{ht}(P_{G_i^c}) \\ &= (k-1)(|G_i| - a_i) + |G_i| \\ &\geq |G_i| = \dim(K[\Delta]). \end{aligned}$$

Thus we have shown the main result of this section:

THEOREM 3.2. *The expansion of the face ring of a pure shellable simplicial complex with respect to $k > 0$ satisfies the Stanley conjecture. In particular, the face ring of a pure k-shellable complex satisfies the Stanley conjecture.*

4. k-shellable graphs

Let G be a simple graph and let Δ_G the independence complex of G. We say that G is k-shellable if Δ_G has this property. The purpose of this section is to characterize k-shellable graphs.

Following Schrijver [17], the *duplication* of a vertex x_i of a graph G means extending its vertex set X by a new vertex $x_{i'}$ and replacing $E(G)$ by

$$E(G) \cup \{x_{i'}x_j : x_i x_j \in E(G)\}.$$

In other words, if $V(G) = \{x_1, \ldots, x_n\}$ then the graph G' obtained from G by duplicating $k_i - 1$ times the vertex x_i has the vertex set

$$V(G') = \{x_{ij} : i = 1, \ldots, n \text{ and } j = 1, \ldots .k_i\}$$

and the edge set

$$E(G') = \{x_{ir}x_{js} : x_i x_j \in E(G),\ r = 1, \ldots, k_i \text{ and } j = 1, \ldots, k_i\}.$$

EXAMPLE 4.1. Let G be a simple graph on the vertex set $V(G) = \{x_1, \dots, x_5\}$ and $E(G) = \{x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5, x_4x_5\}$. Let G' be obtained from G by duplicating 1 times the vertices x_1 and x_4 and 0 times the other vertices. Then G and G' are in the form

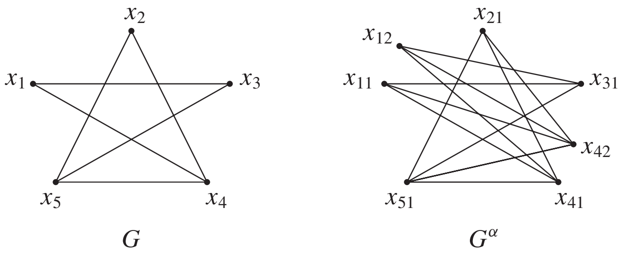

FIGURE 2

Also, the independence complexes of G and G' are, respectively, $\Delta_G = \langle x_1x_2, x_1x_5, x_2x_3, x_3x_4\rangle$ and $\Delta_{G'} = \langle x_{11}x_{12}x_{21}, x_{11}x_{12}x_{51}, x_{21}x_{31}, x_{31}x_{41}x_{42}\rangle$. Note that $\Delta_{G'} = \Delta_G^\alpha$, where $\alpha = (2, 1, 1, 2, 1)$.

In the following theorem we show that the simple graph obtained from duplicating $k-1$ times any vertex of a shellable graph is k-shellable.

THEOREM 4.2. *Let G be a simple graph on X and let G' be a new graph obtained from G by duplicating $k-1$ times any vertex of G. Then G is shellable if and only if G' is k-shellable.*

PROOF. It suffices to show that $\Delta_{G'} = \Delta_G^{[k]}$. Then Theorem 2.8 completes the assertion.

After relabeling of the vertices of G' one can assume that G' is a graph with the vertex set $X^{[k]} = \{x_{ij} : i = 1, \dots, n,\ \ j = 1, \dots, k\}$ and the edge set

$$\{x_{ir}x_{js} : x_ix_j \in G \text{ and } 1 \le r, s \le k\}.$$

Let F be an independent set of G' and let $\bar{F} = \{x_i : x_{ir} \in F \text{ for some } r\}$. If $|\bar{F}| = 1$ then $\bar{F}$ is an independent set in G. So assume that $|\bar{F}| > 1$. Suppose, on the contrary, that $x_i, x_j \in \bar{F}$ and $x_ix_j \in G$. By the construction of G', for all x_{ir} and x_{js} of $V(G')$, $x_{ir}x_{js} \in G'$. Therefore F contains an edge $x_{ir}x_{js}$ of G', a contradiction. This implies that $\bar{F}$ is an independent set in G. In particular, since $F \subset (\bar{F})^{[k]}$ we have $F \in \Delta_G^{[k]}$.

Conversely, suppose H is an independent set of G. Choose $x_{ir}, x_{js} \in H^{[k]}$. If $x_{ir}x_{js} \in G'$ then $x_ix_j \in G$, which is false since $x_i, x_j \in H$. Therefore $H^{[k]}$ is an independent set of G' and $H^{[k]} \in \Delta_{G'}$.

In the following we want to extend some results from [4], [6], [20]. Firstly, we present a generalization of the concept of simplicial vertex.

Let G be a simple graph. For $U \subset V(G)$ we define the *induced subgraph* of G on U to be the subgraph G_U on U consisting of those edges $x_ix_j \in E(G)$ with $x_i, x_j \in U$. For $x \in V(G)$, let $N_G(x)$ denote the open neighborhood of x, that is, all of vertices adjacent to x. We also denote by $N_G[x]$ the closed neighborhood of x, which is $N_G(x)$ together with x itself, so that $N_G[x] = N_G(x) \cup \{x\}$. Set $N_G(U) = \bigcup_{x\in U} N_G(x)$ and $N_G[U] = \bigcup_{x\in U} N_G[x]$.

Recall from [7] or [13] that a vertex $x \in V(G)$ is *simplicial* if the induced subgraph $G_{N_G[x]}$ is complete.

The simple graph G is a *complete r-partite* graph if there is a partition $V(G) = V_1 \cup \cdots \cup V_r$ of the vertex set, such that $uv \in E(G)$ if and only if u and v are in different parts of the partition. If $|V_i| = n_i$, then G is denoted by $K_{n_1,\dots,n_r}$.

Definition 4.3. The set S of pairwise non-adjacent vertices of G is a *k-simplicial set* if $G_{N_G[S]}$ is a r-partite complete graph with k-element parts $S_1, \dots, S_r$ having the following property:

$$\text{for every } S_\ell \text{ and every two vertices } x_i, x_j \in S_\ell,\ N_G(x_i) = N_G(x_j).$$

Note that every 1-simplicial set is a simplicial vertex.

Lemma 4.4. *Let $S \subset V(G)$ of pairwise non-adjacent vertices of G. Set $G' = G \setminus N_G[S]$ and $G'' = G \setminus S$. Then*

$$\text{(i)}\ \ \Delta_{G'} = \mathrm{lk}_{\Delta_G}(S) \qquad \textit{and} \qquad \text{(ii)}\ \ \Delta_{G''} = \bigcap_{x\in S} \mathrm{dl}_{\Delta_G}(x).$$

Proof. (i) "$\subseteq$": let $F \in \Delta_{G'}$. Since $F \subset V(G) \setminus N_G[S]$ we have $F \in \Delta_G$ and $F \cap S = \emptyset$. It remains to show that $F \cup S \in \Delta_G$. Let $F \cup S$ contain an edge $x_ix_j \in E(G)$. Then it should be $x_i \in S$ and $x_j \in F$. In particular, since x_i and x_j are adjacent it follows that $x_j \in N_G[S]$. This is impossible because $F \subset V(G) \setminus N_G[S]$. Therefore $F \cup S \in \Delta_G$. This implies that $F \in \mathrm{lk}_{\Delta_G}(S)$.

"$\supseteq$": let $F \in \mathrm{lk}_{\Delta_G}(S)$. Then $F \cup S \in \Delta_G$ and $F \cap S = \emptyset$. In order to prove that $F \in \Delta_{G'}$ it suffices to show that $F \cap N_G(S) = \emptyset$ and no two vertices of F are adjacent in G'. If $x \in F \cap N_G(S)$ then $S \cup \{x\}$ contains an edge of G and this contradicts $F \cup S \in \Delta_G$. Also, if $x_i, x_j \in F$ and $x_ix_j \in G'$ then $x_ix_j \in G$, which is again a contradiction.

(ii) "$\subseteq$": let $F \in \Delta_{G''}$. If for $x_i, x_j \in F$, x_ix_j is an edge of G, since $\{x_i, x_j\} \cap S = \emptyset$ we obtain that $x_ix_j \in G''$, which is not true. Hence $F \in \Delta_G$. In particular, it follows from $F \subset V(G) \setminus S$ that $F \cap S = \emptyset$ and so $F \in \bigcap_{x\in S} \mathrm{dl}_{\Delta_G}(x)$.

"$\supseteq$": let $F \in \bigcap_{x\in S} \mathrm{dl}_{\Delta_G}(x)$. Then for all $x \in S$ we have $F \in \mathrm{dl}_{\Delta_G}(x)$. Thus $F \in \Delta_G$ and $F \cap S = \emptyset$. Since F contains no edge of G it follows that F contains no edge of G'', either. This implies that $F \in \Delta_{G''}$.

Combining Theorem 2.6 with Lemma 4.4 we obtain the following corollary as an extension of Theorem 2.6 of [20].

COROLLARY 4.5. *Let $S \subset V(G)$ be a set of pairwise non-adjacent vertices. If G is k-shellable, then $G' = G \setminus N_G[S]$ is k-shellable, too.*

THEOREM 4.6. *If S is a k-simplicial set of G such that both $G \setminus S$ and $G \setminus N_G[S]$ are k-shellable, then G is k-shellable.*

PROOF. Let $G' = G \setminus N_G[S]$ and $G'' = G \setminus S$. Let $F_1, \ldots, F_r$ and $H_{r+1}, \ldots, H_s$ be, respectively, the k-shelling orders of $\Delta_{G''}$ and $\Delta_{G'}$. Set $F_i = H_i \cup S$, for $r+1 \le i \le s$. We show that $F_1, \ldots, F_s$ is a k-shelling order of Δ_G.

Note that $\Delta_{G''} = \mathrm{dl}_{\Delta_G}(S)$. This follows from the fact that S is a k-simplicial set. Hence $F_1, \ldots, F_s$ contains all of facets of Δ_G. Now let $1 \le j < i \le s$. If $i \le r$ or $j > r$, by k-shellability of $\Delta_{G'}$ and $\Delta_{G''}$, we are done. So suppose that $i > r$ and $j \le r$. Clearly, $S \subseteq F_i \setminus F_j$. On the other hand, $\Delta_{G'} \subset \Delta_{G''}$ and so there exists $\ell \le r$ such that $H_i \subset F_\ell$. This implies that $S = F_i \setminus F_\ell$ and therefore the assertion is completed.

The following theorem extends Theorem 2.1.13 of [6].

THEOREM 4.7. *Let S_1 be a k-simplicial set of G and let $S_1, \ldots, S_r$ be the parts of $G_{N_G[S_1]}$ and $G'_i = G \setminus N_G[S_i]$ for all $i = 1, \ldots, r$. Then G is k-shellable if and only if G'_i is k-shellable for all $i = 1, \ldots, r$.*

PROOF. Only if part follows from Theorem 4.5. Conversely, let G'_i be k-shellable for all $i = 1, \ldots, r$. Hence for every i, there exists a k-shelling order $F_{i1}, \ldots, F_{it_i}$ for $\Delta_{G'_i} = \mathrm{lk}_{\Delta_G}(S_i)$. We claim that

$$F_{11} \cup S_1, \ldots, F_{1t_1} \cup S_1, \ldots, F_{r1} \cup S_r, \ldots, F_{rt_r} \cup S_r$$

is a k-shelling order for Δ_G.

We first show that the above list is the complete list of facets of Δ_G. Let $F \in \Delta_G$. If for some $i \neq 1$, $S_i \subset F$ then F is in the above list. Otherwise, suppose that for all $i \neq 1$, $S_i \cap F = \emptyset$. Since the elements of S_1 are only adjacent to the elements of S_i's, for $i \neq 1$, it follows that $S_1 \subset F$. On the other hand, it is not possible that F contains only some of elements of a part, as S_i, but not all of elements of S_i. Because in this case, there are $x_j \in S_i \cap F$ and $x_{j'} \in S_i \setminus F$. This means that $N_G(x_j) \cap F = \emptyset$ but $N_G(x_{j'}) \cap F \neq \emptyset$ and so $N_G(x_j) \neq N_G(x_{j'})$, a contradiction.

Now consider $F_{ij} \cup S_i$ and $F_{\ell m} \cup S_\ell$. We have the following cases:

(i) $i < \ell$: then $S_\ell \subset (F_{\ell m} \cup S_\ell) \setminus (F_{ij} \cup S_i)$. Since $F_{\ell m} \cup S_1$ is an independence set in G, there exists a facet $F \in \Delta_G$ with $F_{\ell m} \cup S_1 \subset F$. In particular, F is in the form $F = F_{1p} \cup S_1$ for some $1 \le p \le t_1$. Thus $(F_{\ell m} \cup S_\ell) \setminus (F_{1p} \cup S_1) = S_\ell$.

(ii) $i = \ell$ and $j < m$: the assertion follows from k-shellability of $\Delta_{G'_i}$.

Therefore G is k-shellable.

LEMMA 4.8. *Let $S, T \subset V(G)$ and let T be an independent set of G. If $N_G[T] \subseteq N_G[S]$ then independent sets of $G \setminus N_G[S]$ are not maximal independent sets of $G \setminus S$.*

PROOF. If F is an independent set of $G \setminus N_G[S]$ then $F \cup T$ will be a larger independent set of $G \setminus S$.

COROLLARY 4.9. *Let S be a k-simplicial set of G and let $S_1, \dots, S_r$ be the parts of $G_{N_G[S]}$. Then for every S_i, independent sets of $G \setminus N_G[S_i]$ are not maximal independent sets of $G \setminus S_i$.*

PROOF. Since $N_G[S] \subseteq N_G[S_i]$ for all i, the assertion follows from Lemma 4.8.

In [4, Theorem 2] the authors proved that if x is a simplicial vertex of G and y adjacent to x, then G is shellable if and only if $G \setminus N_G[y]$ and $G \setminus y$ are shellable. The following theorem extends this result to k-shellable graphs.

THEOREM 4.10. *Let S_1 be a k-simplicial set of G and let $S_1, \dots, S_r$ be the parts of $G_{N_G[S_1]}$. Then G is k-shellable if and only if for each $i = 2, \dots, r$ the graphs $G'_i = G \setminus N_G[S_i]$ and $G''_i = G \setminus S_i$ are k-shellable.*

PROOF. "Only if part": let G be k-shellable. It follows from Theorem 4.7 that G'_i is k-shellable for all i. Fix an integer i. We want to show that G''_i is k-shellable. By again relabeling S_i's we can consider $i = r$.

Since each G'_i is k-shellable, so for every i, there exists a k-shelling order $F_{i1}, \dots, F_{it_i}$ for $\Delta_{G'_i} = \mathrm{lk}_{\Delta_G}(S_i)$. Moreover, by the proof of Theorem 4.7,

$$F_{11} \cup S_1, \dots, F_{1t_1} \cup S_1, \dots, F_{r1} \cup S_r, \dots, F_{rt_r} \cup S_r$$

is a k-shelling order for Δ_G. By the fact that $N_G(S_1) \subseteq N_G(S_r)$ we conclude that for every F_{rj} where $1 \le j \le t_r$, there exists $1 \le \ell \le t_1$ such that $F_{rj} \subseteq F_{1\ell}$. Therefore

$$F_{11} \cup S_1, \dots, F_{1t_1} \cup S_1, \dots, F_{r-1\ 1} \cup S_{r-1}, \dots, F_{r-1\ t_{r-1}} \cup S_{r-1}$$

will be a list of facets of $G \setminus S_r$. Furthermore, it is a k-shelling order of G''_r.

"If part": let for all $i = 2, \dots, r$ the graphs G'_i and G''_i are k-shellable. Fix an i and set $G' = G'_i$, $G'' = G''_i$, $S = S_i$. Let $F_1, \dots, F_r$ and $H_1, \dots, H_s$ be, respectively, the k-shelling orders of $\Delta_{G''}$ and $\Delta_{G'}$. We first show that

$$F_1, \dots, F_r, H_1 \cup S, \dots, H_s \cup S$$

is a list of facets of Δ_G, and furthermore, this list is a k-shelling order of Δ_G.

Let F be a facet of Δ_G. If $F \cap S = \emptyset$, then since F contains no edge of G, it contains no edge of G'' and so $F \in \Delta_{G''}$. Suppose $F \cap S \neq \emptyset$. Let $x \in F \cap S$ and let $y \in S$ with $y \neq x$. Since $N_G(y) = N_G(x)$, we have $y \in F$. Thus $S \subset F$. On the other F contains no edge of G'. Therefore $F \setminus S$ is a facet of $\Delta_{G'}$.

Now we show that above list is a k-shelling order. Set $F_{i+r} = H_i \cup S$ for all $i = 1, \dots, s$. Suppose F_i and F_j with $i < j$. If $j \leq r$ or $i \geq r + 1$ then by the k-shellability of $\Delta_{G''}$ and $\Delta_{G'}$, respectively, the assertion is completed. Let $i \leq r$ and $r < j$. Then $S \subset F_j \setminus F_i$. Since $H_{j-r} \in \Delta_{G''}$, there is F_ℓ with $\ell \leq r$ such that $H_{j-r} \subset F_\ell$. Therefore $F_j \setminus F_\ell = S$. This completes the assertion.

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