PROJECTION OPERATORS ON MATRIX WEIGHTED L^p AND A SIMPLE SUFFICIENT MUCKENHOUPT CONDITION

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Abstract

Boundedness for a class of projection operators, which includes the coordinate projections, on matrix weighted L^p-spaces is completely characterised in terms of simple scalar conditions. Using the projection result, sufficient conditions, which are straightforward to verify, are obtained that ensure that a given matrix weight is contained in the Muckenhoupt matrix A_p class. Applications to singular integral operators with product kernels are considered.

1. Introduction

Singular integral operators form a natural generalisation of the classical Hilbert transform, and the action of such operators on $L^p(\mathbb{R})$ has been studied in great detail. The theory was extended in the '70s to include weighted L^p-spaces, with the seminal contribution being the paper by Hunt, Wheeden, and Muckenhoupt [7], where the Hilbert transform is shown to be bounded on weighted L^p, $1 < p < \infty$, if and only if the weight satisfy the so-called Muckenhoupt A_p-condition. Even though the A_p-conditions are quite involved, the results of Hunt, Wheeden, and Muckenhoupt are still very much operational since quite large classes of, e.g. polynomial, weights are known to satisfy the respective conditions, see [9].

A further generalisation of the Hilbert transform result to a vector valued setup is straightforward in the non-weighted case, but it posed a long-standing challenge to find a suitable generalisation in the (matrix-)weighted setup. A breakthrough came with the results [12], [11] of Treil and Volberg for $p = 2$. This lead to a correct definition of matrix A_p weights for $1 < p < \infty$, see [13]. The reader may consult [8] for an application of the Treil-Volberg result and the A_2-matrix condition to applied harmonic analysis.

The matrix A_p condition is considerably more complicated than the scalar condition, and there are no known straightforward sufficient conditions on a matrix weight to ensure membership in the A_p class except in very special cases (e.g., for diagonal weights and for weights with strong pointwise bounds on its spectrum). Bloom [1], [2] has considered sufficient conditions for the matrix A_2-condition in terms of certain weighted BMO-spaces.

It was first noticed in [8] that matrix A_2 weights on product domains can be used to study stability of multiply generated shift invariant systems. This provides a natural motivation to extend the study of matrix weights to the product domain setting. Furthermore, in the scalar valued case, A_p weights on product domains appear naturally in the study of singular integral operators with product type kernes, see [4].

In the present paper, we study matrix A_p-weights on general domains, with a particular focus on product domains. In Section 2 we study and characterise a family of projection operators on matrix weighted L^p on general domains. The family contains the coordinate projections as special cases. The characterisation is given in terms of simple scalar conditions.

Matrix A_p-weights on product domains are introduced in Section 3, and some fundamental properties of such weights are derived. The projection result from Section 2 is then applied in Section 4, in conjunction with the Treil-Volberg characterisation of matrix A_p-weights, to obtain a simple sufficient condition for a matrix to satisfy the product A_p-condition.

We show that the new sufficient condition covers many known examples of non-trivial matrix A_p weights, such as the ones considered by Bownik in [3]. However, we do provide an example of an A_2 matrix weight violating our condition so the condition is *not* exhaustive.

Finally, as an application of the theory, we consider a family of singular integral operators with product type kernels in the matrix weighted setup.

2. Projection operators

Consider a measure space, $(D, \mathfrak{S}, \mu)$ and recall that a scalar weight is a measurable function which is positive a.e. If $w \colon D \to \mathbb{C}$ is a scalar weight, we define the weighted space $L^p(w)$ as the set of measurable functions $f \colon D \to \mathbb{C}$ for which

$$\|f\|_{L^p(w)} := \left(\int_D |f|^p w \, \mathrm{d}\mu \right)^{1/p}$$

is finite, where μ is the measure on D. Likewise, if $W \colon D \to \mathbb{C}^{N \times N}$ is a matrix-valued function which is measurable and positive definite a.e., then the

space $L^p(W)$ is the set of measurable functions $f: D \to \mathbb{C}^N$ with

$$\|f\|_{L^p(W)} := \left(\int_D |W^{1/p} f|^p \, d\mu \right)^{1/p} < \infty.$$

Obviously, in order to turn $L^p(w)$ and $L^p(W)$ into Banach spaces, one has to factorize over $\{f: D \to \mathbb{C} : \|f\|_{L^p(w)} = 0\}$ and $\{f: D \to \mathbb{C}^N : \|f\|_{L^p(W)} = 0\}$, respectively. We can now state our main result giving a full characterization of a certain class of projections from $L^p(W)$ to $L^p(w)$.

THEOREM 2.1. *Let $W = [W_{ij}]_{i,j=1}^N : D \to \mathbb{C}^{N \times N}$ be a matrix-valued function which is measurable and positive definite a.e., let $w: D \to \mathbb{C}$ be a scalar weight and let $r: D \to \mathbb{C}^N$ be a unit vector valued function. Then the projection in the direction of r, $P_r: L^p(W) \to L^p(w)$ given by $P_r(f) = \langle f, r \rangle$, is bounded if and only if*

$$w^{1/p} \|W^{-1/p} r\| \in L^\infty. \tag{1}$$

In particular, if we denote the entries of powers of W by $W^s = [(W^s)_{ij}]$, where s is any real number, then $P_k = P_{e_k}: L^p(W) \to L^p(W_{kk})$ is bounded if and only if

$$(W_{kk})^{2/p}(W^{-2/p})_{kk} \in L^\infty, \tag{2}$$

and if $\lambda_i: D \to \mathbb{R}$ and $v_i: D \to \mathbb{C}^N$ are eigenvalues and eigenvectors, respectively, of W, then the projection $P_{v_i}: L^p(W) \to L^p(\lambda_i)$ is always bounded.

PROOF. We begin with necessity. Assume therefore that P_r is bounded. Then there exists a constant C such that $C\|f\|_{L^p(W)}^p \geq \|P_r f\|_{L^p(w)}^p$. Now let $\{f_\varepsilon\}_{\varepsilon>0}$, $f_\varepsilon: D \to \mathbb{C}$ be an approximate identity and T_k the translation operator, $T_k f = f(\cdot - k)$. Then

$$C = C \left\| T_k f_\varepsilon^{1/p} \frac{W^{-1/p} r}{\|W^{-1/p} r\|} \right\|_p^p = C \left\| T_k f_\varepsilon^{1/p} \frac{W^{-2/p} r}{\|W^{-1/p} r\|} \right\|_{L^p(W)}^p$$

$$\geq \left\| T_k f_\varepsilon^{1/p} \left\langle \frac{W^{-2/p} r}{\|W^{-1/p} r\|}, r \right\rangle \right\|_{L^p(w)}^p = \left\| T_k f_\varepsilon^{1/p} \|W^{-1/p} r\| \right\|_{L^p(w)}^p$$

$$= \left\| T_k f_\varepsilon^{1/p} w^{1/p} \|W^{-1/p} r\| \right\|_p^p.$$

Letting $\varepsilon \to 0$ we get that $w^{1/p}(k)\|W^{-1/p}(k)r(k)\| \leq C^{1/p}$ for a.e. k.

We now show that essential boundedness of $w^{1/p}\|W^{-1/p} r\|$ implies boundedness of P_r. Assume therefore that $w^{1/p}\|W^{-1/p} r\| \leq C$ a.e. Write

q for the Hölder conjugate of p. Then for every $\psi \in L^q(w)$:

$$C\|f\|_{L^p(W)}\|\psi\|_{L^q(w)} \geq \|f\|_{L^p(W)}\left(\int |\psi w^{1/q}|^q \left(|w|^{1/p}\|W^{-1/p}r\|\right)^q \, \mathrm{d}\mu\right)^{1/q}$$

$$= \|f\|_{L^p(W)}\left(\int |\psi w|^q \|W^{-1/p}r\|^q \, \mathrm{d}\mu\right)^{1/q}$$

$$\geq \int \|W^{1/p}f\| \|W^{-1/p}r\psi w\| \, \mathrm{d}\mu$$

$$\geq \int |\langle W^{1/p}f, W^{-1/p}r\psi w\rangle| \, \mathrm{d}\mu$$

$$= \int |P_r f \psi w| \, \mathrm{d}\mu,$$

so $P_r \colon L^p(W) \to L^p(w)$ is bounded.

Note that (1) reduces to (2) when $r \equiv e_k$, the constant function with all coordinates except the k'th being zero, and $w = w_{kk}$. Indeed,

$$\|W^{-1/p}e_k\|^2 = \langle W^{-1/p}e_k, W^{-1/p}e_k\rangle = \langle e_k, W^{-2/p}e_k\rangle = (W^{-2/p})_{kk}.$$

We finish the proof of the theorem by noting that

$$\lambda_i^{1/p}\|W^{-1/p}v_i\| = 1 \quad \text{a.e.,}$$

which is clearly in L^∞.

3. Muckenhoupt matrix weights

Let us first give a brief review of the A_p condition following [13]. We consider a domain $D \in \{\mathbb{R}^d, \mathbb{R}^m \times \mathbb{R}^n, \mathbb{T}^d, \mathbb{T}^m \times \mathbb{T}^n\}$, where $\mathbb{T}^d$ denotes the d-dimensional torus, and an associated measurable map $W \colon D \to \mathbb{C}^{N \times N}$, with values in the non-negative definite matrices.

We introduce a family $\mathscr{S}_D$ of subsets of D. For $D \in \{\mathbb{R}^d, \mathbb{T}^d\}$, $\mathscr{S}_D$ is the collection of all Euclidean balls in D, while in the product case, i.e. when $D \in \{\mathbb{R}^m \times \mathbb{R}^n, \mathbb{T}^m \times \mathbb{T}^n\}$, $\mathscr{S}_D$ is the collection of all product sets $B_r \times B_{r'}$, where B_r is a ball in $\mathbb{R}^m$ $[\mathbb{T}^m]$ and $B_{r'}$ is a ball in $\mathbb{R}^n$ $[\mathbb{T}^n]$. We define the following family of metrics:

$$\rho_t(x) = \|W^{1/p}(t)x\|, \qquad x \in \mathbb{C}^N, \quad t \in D,$$

with the dual metric given by

$$\rho_t^*(x) := \sup_{y \neq 0} \frac{|\langle x, y\rangle|}{\rho_t(y)} = \|W^{-1/p}(t)x\|, \qquad x \in \mathbb{C}^N, \quad t \in D.$$

Assuming suitable local integrability of ρ_t, we now average

$$\rho_{p,E}(x) := \left(\frac{1}{|E|} \int_E [\rho_t(x)]^p \ \mathrm{d}t \right)^{1/p},$$

and likewise for the dual metric

$$\rho_{p,E}^*(x) := \left(\frac{1}{|E|} \int_E [\rho_t^*(x)]^q \ \mathrm{d}t \right)^{1/q},$$

with q being p's Hölder conjugate, $1 = p^{-1} + q^{-1}$.

The A_p condition can then be stated as follows.

DEFINITION 3.1. For $1 < p < \infty$, we say that W is an $A_p(N, D, \mathscr{S}_D)$ *matrix weight* if $W : D \to \mathbb{C}^{N \times N}$ is measurable and positive definite a.e. such that W and $W^{-q/p}$ are locally integrable and there exists $C < \infty$ such that

$$\rho_{q,E}^* \leq C(\rho_{p,E})^*, \qquad E \in \mathscr{S}_D, \tag{3}$$

where the integrability assumptions ensure that both $\rho_{p,E}$ and $\rho_{p,E}^*$ are well-defined.

REMARK 3.2. Notice that $\rho_t^*(x) = \|(W^{-q/p})^{1/q}(t)x\|$, so

$$W \in A_p(N, D, \mathscr{S}_D) \quad \text{if and only if} \quad W^{-q/p} \in A_q(N, D, \mathscr{S}_D). \tag{4}$$

In the following, we will sometimes relax the notation $A_p(N, D, \mathscr{S}_D)$ by leaving out the N, the D, and/or the $\mathscr{S}_D$ if their values are clear from the context. Note that $A_p(1)$ is simply the set of scalar Muckenhoupt weights.

Roudenko introduced an equivalent condition to (3) in [10] which is often more straightforward to verify. In fact, Roudenko only considered the case $D = \mathbb{R}^d$, but the reader can easily verify that her proof in [10] works verbatim in the product and/or torus setup too. Condition (3) holds if and only if $W : D \to \mathbb{C}^{N \times N}$ is measurable and positive definite a.e. such that W and $W^{-q/p}$ are locally integrable and there exists $C' < \infty$ such that

$$\int_E \left(\int_E \left\| W^{1/p}(x) W^{-1/p}(t) \right\|^q \frac{\mathrm{d}t}{|E|} \right)^{p/q} \frac{\mathrm{d}x}{|E|} \leq C', \qquad E \in \mathscr{S}_D. \tag{5}$$

For scalar weights defined on $\mathbb{R}^m \times \mathbb{R}^n$, it is well-known that a product Muckenhoupt condition implies a uniform Muckenhoupt condition in each variable, see [4]. Condition (5) can be used to prove a similar result for product matrix weights. We have the following result.

PROPOSITION 3.3. *Suppose $W \in A_p(\mathbb{R}^m \times \mathbb{R}^n)$. Then the weight $x \mapsto W(x, y)$, obtained by fixing the variable $y \in \mathbb{R}^n$ is uniformly in $A_p(\mathbb{R}^m)$ for a.e. $y \in \mathbb{R}^n$.*

PROOF. Given a ball $B \subset \mathbb{R}^m$, we let $B_\varepsilon := B_\varepsilon(y) \subset \mathbb{R}^n$ be the ball of radius ε about $y \in \mathbb{R}^n$. First suppose $p \leq q$. Since $W \in A_p(\mathbb{R}^m \times \mathbb{R}^n)$ there exists a constant c_W independent of $B \times B_\varepsilon$ such that

$$\frac{1}{|B_\varepsilon|^2} \int_{B_\varepsilon} \int_{B_\varepsilon} \left[\int_B \left(\int_B \|W^{1/p}(x, y) W^{-1/p}(x', y')\|^q \frac{\mathrm{d}x'}{|B|} \right)^{p/q} \frac{\mathrm{d}x}{|B|} \right] \mathrm{d}y \, \mathrm{d}y'$$

$$\leq \int_{B_\varepsilon} \int_B \left(\int_{B_\varepsilon} \int_B \|W^{1/p}(x, y) W^{-1/p}(x', y')\|^q \frac{\mathrm{d}x' \, \mathrm{d}y'}{|B| \cdot |B_\varepsilon|} \right)^{p/q} \frac{\mathrm{d}x \, \mathrm{d}y}{|B| \cdot |B_\varepsilon|}$$

$$\leq c_W,$$

where we have used Hölder's inequality. Now we let $\varepsilon \to 0$ and use Lebesgue's differentiation theorem to conclude that for almost every $y \in \mathbb{R}^n$,

$$\int_B \left(\int_B \|W^{1/p}(x, y) W^{-1/p}(x', y)\|^q \frac{\mathrm{d}x'}{|B|} \right)^{p/q} \frac{\mathrm{d}x}{|B|} \leq c_W.$$

Since c_W is independent of B, it follows that $x \mapsto W(x, y)$ is uniformly in $A_p(\mathbb{R}^m)$ for a.e. $y \in \mathbb{R}^n$. In the case $q < p$, we use that $W^{-q/p} \in A_q(\mathbb{R}^m \times \mathbb{R}^n)$ by (4), which implies the following estimate

$$\int_{B_\varepsilon} \int_B \left(\int_{B_\varepsilon} \int_B \|W^{1/p}(x, y) W^{-1/p}(x', y')\|^p \frac{\mathrm{d}x' \, \mathrm{d}y'}{|B| \cdot |B_\varepsilon|} \right)^{q/p} \frac{\mathrm{d}x \, \mathrm{d}y}{|B| \cdot |B_\varepsilon|} \leq c_W.$$

By repeating the argument from the $p \leq q$ case, we conclude that the map $x \mapsto W^{-q/p}(x, y)$ is in $A_q(\mathbb{R}^m)$ for a.e. $y \in \mathbb{R}^n$ which again by (4) implies that $x \mapsto W(x, y)$ is in $A_p(\mathbb{R}^m)$ for a.e. $y \in \mathbb{R}^n$.

A similar result clearly holds true for the weight $y \mapsto W(x, y)$. The periodic case, i.e. the case $W \in A_p(\mathbb{T}^m \times \mathbb{T}^n)$, is also similar.

4. A sufficient matrix Muckenhoupt condition

Here we consider an application of the projection result to derive operational sufficient conditions for a matrix weight to be in the Muckenhoupt A_p class. The matrix A_p condition introduced in [13] is rather involved and it may be difficult to verify for a given matrix. The simpler A_2-case was settled in [12], [11]. An additional advantage of the projection approach is that it applies to both the regular matrix A_p condition and to the corresponding product setup.

We will need a characterization of the matrix condition A_p. Recall that the Riesz transform $R_j : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is given by

$$\mathscr{F}(R_j f)(\xi) = i \frac{\xi_j}{\xi} \mathscr{F} f(\xi), \qquad j = 1, 2, \dots, d.$$

The operator R_j lifts naturally to vector-valued functions by defining

$$R_j f := (R_j f_1, \dots, R_j f_N), \quad \text{for } f = (f_1, \dots, f_N) \in (L^p(\mathbb{R}^d))^N.$$

We have the following fundamental characterization of the matrix condition A_p based on Goldberg's results on vector valued singular integral operators, see [6, Theorems 5.1 and 5.2].

THEOREM 4.1. *Let $W = [W_{ij}] : \mathbb{R}^d \to \mathbb{C}^{N \times N}$ be a matrix-valued function which is measurable and positive definite a.e., and let $1 < p < \infty$. Suppose $W \in A_p(N, \mathbb{R}^d)$. Then the Riesz transforms $R_j : L^p(W) \to L^p(W)$ are bounded for all $j = 1, 2, \dots, d$. Conversely, if $R_j : L^p(W) \to L^p(W)$ is bounded for some $j \in \{1, 2, \dots, d\}$, then $W \in A_p(N, \mathbb{R}^d)$.*

PROOF. Suppose $W \in A_p(N, \mathbb{R}^d)$. It is well-known that the Riesz integral kernel associated with R_j, $j = 0, 1, \dots, d$, is given by the function

$$K_j(x) = c_d \frac{x_j}{|x|^{d+1}}, \qquad x \in \mathbb{R}^d \setminus \{0\},$$

which is homogeneous of degree $-d$. Using the homogeneity one easily checks that

$$|\nabla K_j(x)| \le c_d' |x|^{-d-1}, \qquad x \in \mathbb{R}^d \setminus \{0\}.$$

We can then call on [6, Theorems 5.1] to conclude that $R_j : L^p(W) \to L^p(W)$ is bounded.

Conversely, if $R_j : L^p(W) \to L^p(W)$ is bounded, one easily checks that the kernel K_j satisfies the non-degeneracy condition in [6, Theorems 5.2], and we may conclude that $W \in A_p(N, \mathbb{R}^d)$.

We now consider the product setup. For simplicity we focus on the case $D = \mathbb{R}^m \times \mathbb{R}^n$. The case $D = \mathbb{T}^m \times \mathbb{T}^n$ can be treated in a similar fashion. We write $z = (x, y) \in D$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Let $\tilde{R}_i^x$ denote the operator $R_i^x \otimes \mathrm{Id}_y$, where R_i^x is the Riesz transform acting on $\mathbb{R}^m$. Similarly, we let $\tilde{R}_j^y$ denote the operator $\mathrm{Id}_x \otimes R_j^y$, where R_j^y is the Riesz transform acting on $\mathbb{R}^n$. We have the following Corollary to Theorem 4.1.

COROLLARY 4.2. *Let $D = \mathbb{R}^m \times \mathbb{R}^n$ and let $W = [W_{ij}] : D \to \mathbb{C}^{N \times N}$ be a matrix-valued function which is measurable and positive definite a.e., and*

let $1 < p < \infty$. *Then* $W \in A_p(N, D, \mathscr{S}_D)$ *implies that the Riesz transforms* $\tilde{R}_i^x, \tilde{R}_j^y \colon L^p(W) \to L^p(W)$ *are bounded for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$. *Conversely, suppose* $\tilde{R}_i^x, \tilde{R}_j^y \colon L^p(W) \to L^p(W)$ *are bounded for some* $i \in \{1, 2, \ldots, m\}$ *and some* $j \in \{1, 2, \ldots, n\}$, *then* $W \in A_p(N, D, \mathscr{S}_D)$.

PROOF. Suppose $A_p(N, D, \mathscr{S}_D)$. Then Proposition 3.3 shows that $W(x, y)$ is uniformly A_p in each variable separately (a.e.). We can then use Theorem 4.1 together with a simple iteration argument to deduce that the operators $\tilde{R}_i^x, \tilde{R}_j^y \colon L^p(W) \to L^p(W)$ are bounded for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Conversely, suppose $\tilde{R}_i^x, \tilde{R}_j^y \colon L^p(W) \to L^p(W)$ are bounded for some $i \in \{1, 2, \ldots, m\}$ and some $j \in \{1, 2, \ldots, n\}$. Take any $f = (f_i)_{i=1}^N$ with $f_i \in C_c^\infty(\mathbb{R}^n)$, $i = 1, \ldots, N$, and fix $x_0 \in \mathbb{R}^m$. Let $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^m)$ be an approximation to the identity centered at $x_0 \in \mathbb{R}^m$. We let $R_j f$ denote the vector $(R_j f_i)_{i=1}^m$. Then using the boundedness of $\tilde{R}_j^y$,

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \varphi_\varepsilon(x) \big| W^{1/p}(x, y) R_j f(y) \big|^p \, \mathrm{d}x \, \mathrm{d}y$$
$$\leq C \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \varphi_\varepsilon(x) \big| W^{1/p}(x, y) f(y) \big|^p \, \mathrm{d}x \, \mathrm{d}y.$$

We let $\varepsilon \to 0$ to conclude that almost surely

$$\int_{\mathbb{R}^n} \big| W^{1/p}(x_0, y) R_j f(y) \big|^p \, \mathrm{d}y \leq C \int_{\mathbb{R}^n} \big| W^{1/p}(x_0, y) f(y) \big|^p \, \mathrm{d}y, \quad x_0 \in \mathbb{R}^m.$$

We now use Theorem 4.1 to conclude that $y \to W(x, y)$ is uniformly in A_p for a.e. x. A similar argument using $\tilde{R}_i^x$ shows that $x \to W(x, y)$ is uniformly in A_p for a.e. y. Using these uniform bounds it follows easily that $W \in A_p(N, D, \mathscr{S}_D)$.

We can now give a sufficient condition for membership in $A_p(N, D, \mathscr{S}_D)$.

THEOREM 4.3. *Let* $D = \mathbb{R}^m \times \mathbb{R}^n$ *and let* $W = [W_{ij}] \colon D \to \mathbb{C}^{N \times N}$ *be a matrix weight which is invertible a.e. Fix* $1 < p < \infty$ *and denote the entries of powers of* W *by* $W^s = [(W^s)_{ij}]$, *where* s *is any real number. Suppose that*

$$(W_{kk})^{2/p} (W^{-2/p})_{kk} \in L^\infty, \qquad k = 1, 2, \ldots, N$$

and that $(W^{2/p})_{kk}^{p/2} \in A_p(1)$ *for* $k = 1, 2, \ldots, N$. *Then* $W \in A_p(N, D, \mathscr{S}_D)$.

PROOF. Take $f \in L^p(W) \cap C_c^\infty(D)$. We write $f = \sum_{j=1}^N f_j e_j = \sum_{j=1}^N P_j(f) e_j$, and note that by definition the vector-valued operators $\tilde{R}_i^x$ and

$\tilde{R}_j^y$ act coordinate-wise on f, so it follows that for $K \in \{\tilde{R}_i^x\}_{i=1}^m \cup \{\tilde{R}_j^y\}_{j=1}^n$,

$$Kf := \sum_{j=1}^{N}(Kf_j)e_j.$$

According to Corollary 4.2, the scalar-valued transform K is bounded on $L^p\big((w_{kk}^{(2/p)})^{p/2}\big)$ for $k = 1, 2, \ldots, N$, so we obtain

$$\|Kf\|_{L^p(W)} \leq \sum_{j=1}^{N} \|(Kf_j)e_j\|_{L^p(W)} = \sum_{j=1}^{N} \|Kf_j\|_{L^p((W^{2/p})_{kk}^{p/2})}$$

$$\leq C \sum_{j=1}^{N} \|f_j\|_{L^p((W^{2/p})_{kk}^{p/2})} \leq C' \|f\|_{L^p(W)},$$

where we used that $P_j \colon L^p(W) \to L^p\big((W^{2/p})_{kk}^{p/2}\big)$ is bounded by Theorem 2.1. Now we use Corollary 4.2 to conclude that $W \in A_p(N, D, \mathscr{S}_D)$.

Note that Theorem 4.3 gives us an easily verifiable (at least for $p = 2$) sufficient condition for $W \in A_p(N)$. It is known that $W \in A_p(N)$ implies that $(W^{2/p})_{kk}^{p/2} \in A_p(1)$ for $k = 1, 2, \ldots, N$, indicating a possibility that the conditions of Theorem 4.3 in fact characterize $A_p(N)$. However, this is not the case as the following example illustrates.

EXAMPLE 4.4. Let W be given by

$$[0, 1] \ni x \mapsto W(x) = \begin{pmatrix} \sqrt{x} + \frac{1}{\sqrt{x}} & \frac{i}{\sqrt{x}} \\ -\frac{i}{\sqrt{x}} & \frac{1}{\sqrt{x}} \end{pmatrix}.$$

Then $W \in A_2(2)$ but (2) with $p = 2$ fails to hold. Indeed, $\det(W) \equiv 1$, so

$$W_{11}(W^{-1})_{11} = W_{22}(W^{-1})_{22} = 1 + \frac{1}{x} \notin L^\infty,$$

but

$$W_{a,b} = \int_a^b W(x)\, dx$$

$$= \begin{pmatrix} \frac{2}{3}(b^{3/2} - a^{3/2}) + 2(\sqrt{b} - \sqrt{a}) & 2i(\sqrt{b} - \sqrt{a}) \\ -2i(\sqrt{b} - \sqrt{a}) & 2(\sqrt{b} - \sqrt{a}) \end{pmatrix}$$

and

$$W_{a,b}^{(-1)} = \int_a^b W^{-1}(x)\,dx$$

$$= \begin{pmatrix} 2(\sqrt{b}-\sqrt{a}) & -2i(\sqrt{b}-\sqrt{a}) \\ 2i(\sqrt{b}-\sqrt{a}) & \frac{2}{3}(b^{3/2}-a^{3/2})+2(\sqrt{b}-\sqrt{a}) \end{pmatrix}$$

so

$$W_{a,b}W_{a,b}^{(-1)} = \tfrac{4}{3}\big((b-a)^2 - \sqrt{ab}\big(\sqrt{b}-\sqrt{a}\big)^2\big)I_2$$

and hence

$$\left\| \left(\frac{1}{b-a}W_{a,b}\right)^{1/2} \left(\frac{1}{b-a}W_{a,b}^{(-1)}\right)^{1/2} \right\|_F$$

$$= \frac{1}{b-a}\sqrt{\mathrm{tr}\Big(\big(W_{a,b}^{(-1)}\big)^{1/2}W_{a,b}\big(W_{a,b}^{(-1)}\big)^{1/2}\Big)}$$

$$= \frac{1}{b-a}\sqrt{\mathrm{tr}\Big(W_{a,b}W_{a,b}^{(-1)}\Big)}$$

$$= \frac{1}{b-a}\sqrt{\frac{8}{3}(b-a)^2 - \sqrt{ab}\big(\sqrt{b}-\sqrt{a}\big)^2} \le \frac{2\sqrt{2}}{\sqrt{3}},$$

where the Frobenius norm was used for convenience.

5. An application to vector valued singular integral operators

Let us consider singular integral operators on the Euclidean product space $\mathbb{R}^n \times \mathbb{R}^m$. Recall that a scalar weight $w(x, y)$ satisfies the (product) Muckenhoupt $A_p(1, \mathbb{R}^n \times \mathbb{R}^m)$-condition precisely when w is uniformly in $A_p(1)$ for each variable x and y separately. This makes it very easy to study singular integral operators with a corresponding product structure on $L^p(\mathbb{R}^n \times \mathbb{R}^m, w)$ using a simple iteration argument.

For example, the product Hilbert transform

$$f \to \mathrm{p.v.}\frac{1}{xy} * f$$

is bounded on $L^p(\mathbb{R} \times \mathbb{R}, w)$, $1 < p < \infty$, whenever $w \in A_p(\mathbb{R} \times \mathbb{R})$.

The case when the kernel is not separable but otherwise resemble a product Hilbert transform is much more complicated and has been studied in e.g. [5].

Suppose that K is locally integrable on $\mathbb{R}^n \times \mathbb{R}^m$ away from the cross $\{x = 0\} \cup \{y = 0\}$.

We let

$$\Delta_h^1 K(x, y) = K(x + h, y) - K(x, y),$$
$$\Delta_k^2 K(x, y) = K(x, y + k) - K(x, y),$$

and

$$\Delta_{h,k}^{1,2} K(x, y) = \Delta_h^1(\Delta_k^2(K)).$$

The following 5 technical conditions for some $A < \infty$ and $\eta > 0$ turn out to be important for establishing boundedness of the operator induced by K:

(C.1) $\left| \iint_{\alpha_1 < |x| < \alpha_2, \beta_1 < |y| < \beta_2} K(x, y) \, dx \, dy \right| \leq A$ for all $0 < \alpha_1 < \alpha_2$ and $0 < \beta_1 < \beta_2$,

(C.2) For K_1 given by $K_1(x) = \int_{\beta_1 < |y| < \beta_2} K(x, y) \, dy$ then $|K_1(x)| \leq A|x|^{-n}$ for all $0 < \beta_1 < \beta_2$, $|\Delta_h^1 K_1(x)| \leq A|h|^\eta |x|^{-n-\eta}$ for $|x| \geq 2|h|$, with a similar condition for $K_2(y) = \int_{\alpha_1 < |x| < \alpha_2} K(x, y) \, dx$,

(C.3) $|K(x, y)| \leq A|x|^{-n}|y|^{-m}$,

(C.4) $|\Delta_h^1 K(x, y)| \leq A|h|^\eta |x|^{-n-\eta}|y|^{-m}$ if $|x| \geq 2|h|$ with a similar condition on $\Delta_k^2 K(x, y)$,

(C.5) $|\Delta_{h,k}^{1,2} K(x, y)| \leq A(|h||k|)^\eta |x|^{-n-\eta}|y|^{-m-\eta}$ if $|x| \geq 2|h|$ and $|y| \geq 2|k|$.

THEOREM 5.1 ([5]). *Let $1 < p < \infty$. Suppose K is locally integrable on $\mathbb{R}^n \times \mathbb{R}^m$ away from the cross $\{x = 0\} \cup \{y = 0\}$. Assume that K satisfies (C.1)–(C.5). Then the truncated kernels*

$$K_\varepsilon^N(x, y) = K(x, y)\chi_{\varepsilon_1 < |x| < N_1}(x)\chi_{\varepsilon_2 < |y| < N_2}(y)$$

induce a uniformly bounded family of operators

$$T_\varepsilon^N(f) := f * K_\varepsilon^N$$

on $L^p(\mathbb{R}^n \times \mathbb{R}^m, w)$ whenever $w \in A_p(1, \mathbb{R}^n \times \mathbb{R}^m)$. Moreover, if

$$\iint_{\alpha_1 < |x| < \alpha_2, \beta_1 < |y| < \beta_2} K_\varepsilon^N(x, y) \, dx \, dy,$$

$$\int_{\alpha_1 < |y| < \alpha_2} K_\varepsilon^N(x, y) \, dy, \quad and \quad \int_{\beta_1 < |x| < \beta_2} K_\varepsilon^N(x, y) \, dx$$

converge a.e. to limits as $\varepsilon \to 0$ and $N \to \infty$ for all $0 < \alpha_1 < \alpha_2$, $0 < \beta_1 < \beta_2$, then for every $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m, w)$, $T_K(f) := \lim_{\varepsilon \to 0, N \to \infty} T_\varepsilon^N(f)$ defines a bounded operator on $L^p(\mathbb{R}^n \times \mathbb{R}^m, w)$ for $w \in A_p(1, \mathbb{R}^n \times \mathbb{R}^m)$.

A natural extension of Theorem 5.1 would be to lift the operator T_K to the matrix-weighted case. There are at present some technical obstacles that prevent us from carrying out this program in full generality, but we can use the results in the previous sections to obtain a partial result.

COROLLARY 5.2. *Let $1 < p < \infty$ and let $K: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ be a kernel of the type considered in Theorem 5.1. Suppose $W = [W_{ij}]: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}^{N \times N}$ be a matrix weight which is invertible a.e. Then the operator T_K lifted to the vector valued setting is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m, W)$ provided that*

$$(W_{kk})^{2/p}(W^{-2/p})_{kk} \in L^\infty, \qquad k = 1, 2, \ldots, N$$

and that $(W^{2/p})_{kk}^{p/2} \in A_p(1)$ for $k = 1, 2, \ldots, N$.

PROOF. Take $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m, W) \cap C_c^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, and write write the function as $f = \sum_{j=1}^N f_j e_j = \sum_{j=1}^N P_j(f) e_j$. It follows that

$$T_K f := \sum_{j=1}^N (T_K f_j) e_j,$$

so

$$\|T_K f\|_{L^p(W)} \le \sum_{j=1}^N \|(T_K f_j) e_j\|_{L^p(W)} = \sum_{j=1}^N \|T_K f_j\|_{L^p((W^{2/p})_{kk}^{p/2})}$$

$$\le C \sum_{j=1}^N \|f_j\|_{L^p((W^{2/p})_{kk}^{p/2})} \le C' \|f\|_{L^p(W)},$$

where we used Theorem 5.1, and the projection result, Theorem 2.1.

We conjecture that the conclusion of Corollary 5.2 holds true for any matrix weight $W = (w_{ij}): \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}^{N \times N}$ in the set of Muckenhoupt weights $A_p(N, \mathbb{R}^n \times \mathbb{R}^m)$, $1 < p < \infty$.

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