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A MAJORANT PRINCIPLE IN THE THEORY OF FUNCTIONS

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Introduction

1. This paper is devoted to the establishment of a majorant principle which illustrates the interrelation between multivalency and set of values of a meromorphic function. In addition to the information contained in the inequality obtained, the principle yields a simple and unified treatment of various problems in the theory of distribution of values of meromorphic functions.

In order to elucidate the underlying idea of the principle let us consider a function $w = f(z)$, meromorphic in the unit circle $|z| < 1$ of the complex z-plane. Denoting by $n(r, a)$ the number of zeros of $f(z)-a$ in the circular disk $|z| \leqq r < 1$, multiple zeros being counted with their multiplicity, we introduce the well-known function (see Nevanlinna [3], p. 157)

$$N(r, a) = \int_0^r \frac{n(t, a)}{t}\, dt$$

which characterizes the average density of the roots of the equation $f(z) = a$. By an easy reasoning, we can prove that this function $N(r, a)$ is subharmonic in a.

Starting from this property of $N(r, a)$ we can establish the desired majorant principle. The result is expressed as follows: Let $w = f(z)$ be a meromorphic function in the unit circle whose values lie in a plane domain G with at least three boundary points. Let the w-plane be covered with a non-negative mass μ, and let $\Omega(r)$ denote the total mass lying on the image of $|z| \leqq r$ by $f(z)$. Then the integral

$$\int_0^r \frac{\Omega(t)}{t}\, dt$$

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is at most equal to the corresponding integral of a function $x(z)$ which maps the unit circle onto the universal covering surface of G and satisfies $x(0) = f(0)$.

By virtue of the subharmonicity of $N(r, a)$ it can be readily shown that, as a special case, this majorant principle yields a sharpened version of the classical principle of Lindelöf.

As for earlier investigations related to this paper, we refer above all to certain results of Littlewood [1], [2] closely related to the above principle and to some other results given below. A paper of F. Riesz [4] must be mentioned here too. In this connection I also wish to express my acknowledgement to Dr. K. I. Virtanen, whose suggestions constitute an essential contribution to this paper.

2. Owing to the great arbitrariness of the mass distribution μ numerous special results follow from the above general theorem.

If the boundary of the domain G which contains the set of values of $f(z)$ is of positive capacity and if the mass is concentrated at a single point we obtain the improvement of Lindelöf's principle referred to above. In case G is the unit circle the result sharpens the classical lemma of Schwarz.

If the mass μ is defined as some Euclidean measure, e. g. as the Euclidean area or as the Euclidean length of some plane curve, the results obtained frequently have an intuitive geometric interpretation. This is especially true if the comparison domain G is simply connected.

An important application arises if $d\mu$ equals the spherical element of area. By Shimizu-Ahlfors's theorem, the majorant principle then gives an estimate of Nevanlinna's characteristic function $T(r)$. A very simple proof is obtained of the fact that a function is of bounded characteristic if it is meromorphic in the unit circle and omits a set of values of positive capacity.

Interesting conclusions may also be drawn if $d\mu$ is a hyperbolic element of area defined with respect to the domain G. In this way it is possible to establish e. g. the classical Picard-Landau theorem and also to obtain more general information about the value distribution of meromorphic functions.

The majorant principle

3. Let $w = f(z)$ be meromorphic in the unit circle $|z| < 1$, and let $n(r, a)$ denote, as above, the number of zeros of $f(z)-a$ in the disk $|z| \leqq r < 1$, multiple zeros being counted with their multiplicity. We begin this section by studying the function

$$N(r, a) = \int_0^r \frac{n(t,a)}{t}\, dt \,. \tag{1}$$

By partial integration it follows that

$$N(r, a) = \int_0^r \log \frac{r}{t}\, dn(t, a) \,.$$

Hence, $N(r, a)$ can also be represented in the form

$$N(r, a) = \sum_{i=1}^{k} \log \frac{r}{|z_i|} \,, \tag{2}$$

where $z_1, z_2, \ldots, z_k$ denote all points in $|z| \leqq r$ at which $f(z)$ takes the value a, counted according to their multiplicity.

The function $\log(r/|z|)$ is the Green's function $g(z, 0)$ of the circle $|z| \leqq r$ whose pole lies at $z = 0$. Since Green's function remains invariant under conformal transformations we can also write

$$N(r, a) = \sum_{i=1}^{k} g\big(P_i, f(0), F_r\big) \,, \tag{3}$$

where F_r is the image of $|z| \leqq r$ by $w = f(z)$, and $P_1, P_2, \ldots, P_k$ denote all points of the surface F_r having the point a as their projection in the w-plane. It follows from the original definition (1) that we must complete (2) and (3) by defining $N(r, a) = 0$ if $f(z) \neq a$ in $|z| \leqq r$.

The representation formula (3) is important because it reveals the subharmonicity of $N(r, a)$. For convenience of later reference we formulate this statement as a

LEMMA. *The function $N(r, a)$ is subharmonic in a, except for the logarithmic singularity at $a = f(0)$.*

In order to establish this lemma we make use of the following two well-known facts: Firstly, a function satisfying certain regularity conditions obviously fulfilled here is subharmonic if its value at every point is at most equal to its mean value on a sufficiently small circular disk with the point in question as center. Secondly, for a harmonic function the value at every point coincides with this mean value. With these two propositions in mind, the validity of the lemma is immediately seen from the representation (3) of $N(r, a)$.

4. We now apply the lemma to a simple special case which will be important for later considerations. Let $w = f(z)$ be regular in the unit circle

$|z| < 1$ and satisfy the conditions $f(0) = 0$, $|f(z)| < 1$ there. By Schwarz's lemma, the set of values taken by $f(z)$ in $|z| \leqq r$ lie in the disk $|w| \leqq r$. Hence, in $|w| \leqq r$ the function $N(r, w)$ has boundary values zero. Moreover, at $w = 0$ the function $N(r, w)$ becomes infinite like the Green's function of $|w| \leqq r$. By the lemma we thus have for $|w| \leqq r$

$$N(r, w) \leqq \log \frac{r}{|w|}. \tag{4}$$

Let us now suppose that in $|z| \leqq r$ the function $f(z)$ takes a given value a $(\neq 0)$ at the points $z_1, z_2, \ldots, z_n$. Then, by (2) and (4),

$$\sum_{i=1}^{n} \log \frac{r}{|z_i|} \leqq \log \frac{r}{|a|}. \tag{5}$$

Letting here $r \to 1$ we obtain the following result, which is actually Jensen's inequality in a slightly modified form. (Cf. also Littlewood [1], p. 487.)

Let $f(z)$ be regular in the unit circle, $|f(z)| < 1$ and $f(0) = 0$. Then, if $f(z)$ takes the value a at the points $z_1, z_2, \ldots,$

$$|a| \leqq \Pi |z_i|, \tag{6}$$

where each z_i appears with its multiplicity.

The inequality (6) is a direct improvement of Schwarz's lemma according to which

$$|a| \leqq |z_i|$$

for each separate z_i. From (6) we also immediately infer the well-known theorem that if $a \neq 0$, then the product $\Pi |z_i|$ cannot diverge to zero.

Since the inequality (5) holds for every value a taken by $f(z)$ we immediately get the following corollary which will be used below.

Let $f(z)$ be regular in the unit circle, $|f(z)| < 1$, and $f(0) = 0$. Further, let $a_1, a_2, \ldots, a_m$ be an arbitrary set of m complex numbers each of modulus at most equal to r (< 1). Then, if $z_1, z_2, \ldots, z_n$ denote points in $|z| \leqq r$ at each of which the value of $f(z)$ is one of the numbers $a_1, a_2, \ldots, a_m$, we have

$$\prod_{j=1}^{m} \frac{|a_j|}{r} \leqq \prod_{i=1}^{n} \frac{r}{|z_i|}. \tag{7}$$

5. The above results admit a far-reaching generalization which illustrates the value distribution of meromorphic functions and offers a convenient starting point for several applications.

Let us consider a meromorphic function $w = f(z)$. For the sake of simplicity, we again assume that $f(z)$ is defined in the unit circle. Instead of requiring boundedness, as in section 4, we now suppose that the values taken by $f(z)$ in the unit circle lie inside a given plane domain G. The only condition imposed on G is that its boundary must contain at least three points. As is well known, we can then perform a one-to-one conformal mapping of the unit circle onto the universal covering surface G^∞ of G. Let $x(z)$ denote such a mapping function normalized by the requirement $x(0) = f(0)$. By this normalization $x(z)$ is uniquely determined up to the value of the argument of $x'(0)$. For the following it is immaterial how this parameter is chosen. The inverse function of x will be denoted by x^{-1}.

We now form the function

$$x^{-1}(f(z))$$

and consider the uniquely determined branch which vanishes for $z = 0$. Since the values of $f(z)$ lie in G, this branch can be continued in the whole unit circle. By the monodromy theorem, the function $\varphi(z)$ thus obtained is single-valued. By definition, it satisfies the conditions $|\varphi(z)| < 1$ and $\varphi(0) = 0$.

Let

$$s_1, s_2, \ldots, s_m \tag{8}$$

and

$$t_1, t_2, \ldots, t_n \tag{9}$$

denote all points in $|z| \leqq r$ where $x(z)$ and $f(z)$, respectively, take a given value a $(\neq f(0))$. If t_j is one of the points (9) we have

$$\varphi(t_j) = x^{-1}(f(t_j)) = x^{-1}(a)\,.$$

By Schwarz's lemma,

$$|\varphi(t_j)| \leqq r\,,$$

so that $x^{-1}(a)$ must necessarily coincide with one of the points (8). In other words, all values that $\varphi(z)$ assumes at the points (9) are among the numbers (8). Thus the inequality (7) applies to $\varphi(z)$ and yields (cf. Littlewood [1], 487)

$$\sum_{j=1}^{n} \log \frac{r}{|t_j|} \leqq \sum_{i=1}^{m} \log \frac{r}{|s_i|}\,.$$

The left-hand side of this relation is the function $N(r, a)$ belonging to $f(z)$, whereas the right-hand term is the $N(r, a)$ of $x(z)$. Hence, *the function $N(r, a)$ $(a \neq f(0))$ belonging to $f(z)$ is at most equal to the $N(r, a)$ of the function $x(z)$ which maps the unit circle onto the universal covering surface of G and satisfies $x(0) = f(0)$.*

We shall now integrate the above inequality with respect to a. For this purpose, we consider a closed point set E in the plane and cover it with a non-negative mass μ, i. e., μ is a completely additive set function defined for every Lebesgue measurable subset of E. The mass distribution μ is assumed to be such that the integral

$$\int_E N(r, a) d\mu$$

is finite for every $r < 1$.

By the definition (1) of $N(r, a)$, the above integral can be written

$$\int_E N(r, a) d\mu = \int_0^r \frac{dt}{t} \int_E n(t, a) d\mu = \int_0^r \frac{\Omega(t)}{t} dt . \tag{10}$$

Here

$$\Omega(r) = \int_E n(r, a) d\mu$$

can evidently be interpreted as follows: $\Omega(r)$ *denotes the total mass on the Riemann surface* F_r *onto which* $f(z)$ *maps the disk* $|z| \leqq r$. Hereby every surface element e of F_r is furnished with the mass $\mu(e)$.

On the basis of formula (10) we can now summarize our results in the following theorem which contains all previous results as special cases.

THEOREM. *Let* $f(z)$ *be meromorphic in the unit circle and let its values lie inside a plane domain* G *with at least three boundary points. Then the integral*

$$\int_0^r \frac{\Omega(t)}{t} dt$$

belonging to $f(z)$ *is at most equal to the corresponding integral of the function* $x(z)$ *which maps the unit circle onto the universal covering surface of* G *and satisfies* $x(0) = f(0)$.

6. In order to compare the above majorant principle with classical principles in the theory of functions we return to the lemma of section 3. Let the values of $f(z)$ fall inside a domain G whose boundary is of positive capacity, i. e., which possesses a Green's function. As an immediate consequence of the lemma, we then obtain the inequality

$$N(r, a) \leqq g(a, f(0), G) , \tag{11}$$

which is true for every $r < 1$.

Using the representation formula (3) for $N(r, a)$, and letting r tend to 1, we get

(12)
$$\sum_i g(P_i, f(0), F) \leqq g(a, f(0), G).$$

Here F is the image of the unit circle by $f(z)$, and P_i denote all points of F over a. This inequality is sharp, since it is well known that equality occurs if $f(z)$ is the function mapping the unit circle onto G^∞.

From the inequality (12) the relation to the classical principle of Lindelöf is immediately seen; Lindelöf's principle yields the inequality

$$g(P_i, f(0), F) \leqq g(a, f(0), G)$$

for each separate point P_i.

7. It may be noted that the above theorem can be expressed in a slightly more general form. In fact, since the function

$$N(r, a) = \sum_i g(P_i, f(0), F_r)$$

is at most equal to the $N(r, a)$ of $x(z)$ for each a, it follows that for every $\varrho \geqq r$ the integral

$$\int_{F_r} g(P, f(0), F_\varrho) d\mu$$

cannot be greater than the corresponding integral belonging to $x(z)$. Now

$$\int_{F_r} g(P, f(0), F_\varrho) d\mu = \int_{|z| \leqq r} \log \frac{\varrho}{|z|} d\mu(w(z)) = \int_{|z| \leqq r} \log \frac{\varrho}{r} d\mu + \int_{|z| \leqq r} \log \frac{r}{|z|} d\mu$$

$$= \Omega(r) \log \frac{\varrho}{r} + \int_{F_r} g(P, f(0), F_r) d\mu = \Omega(r) \log \frac{\varrho}{r} + \int_0^r \frac{\Omega(t)}{t} dt .$$

Thus the above theorem holds if the integral

(13)
$$\int_0^r \frac{\Omega(t)}{t} dt$$

is replaced by the expression

(14)
$$\Omega(r) \log \frac{\varrho}{r} + \int_0^r \frac{\Omega(t)}{t} dt$$

where ϱ is any number satisfying the inequalities $r \leqq \varrho \leqq 1$.

If the integral (13) increases sufficiently slowly when $r \to 1$, then

$$\Omega(r) \log \frac{\varrho}{r} = o\left(\int_0^r \frac{\Omega(t)}{t} dt \right).$$

In such cases, therefore, no essential improvement is attained by replacing (13) by (14). If, however, (13) tends rapidly to infinity, the use of (14) instead of (13) can give a more accurate estimate.

Applications

8. In this section we briefly deal with certain applications of the above theorem. We begin by considering cases where the mass μ is connected with certain Euclidean measures.

A simple mass distribution is obtained if μ is of constant density 1, i. e. if $d\mu$ equals the element of area in the Euclidean plane metric. If the functions $f(z)$ considered are regular in the unit circle, in which case their values lie in a domain G not containing the point at infinity, we can take E to be the whole plane. $\Omega(r) = A(r)$ then denotes the Euclidean area of the image of $|z| \leqq r$ by $f(z)$.

We note incidentally that the results concerning the integral

$$\int_0^r \frac{A(t)}{t} dt \tag{15}$$

may be formulated so as to apply to the integral

$$\int_0^{2\pi} |f(re^{i\varphi})|^2 d\varphi \, . \tag{16}$$

This follows from the known formula

$$\int_0^r \frac{A(t)}{t} dt = \frac{1}{4} \int_0^{2\pi} |f(re^{i\varphi})|^2 d\varphi - \frac{\pi}{2} |f(0)|^2 \, ,$$

readily established, e. g. by means of the principle of the argument.

Let us suppose, in particular, that G is simply connected. Then the function $x(z)$, maximizing the integral (15), is schlicht and performs a one-to-one mapping of the unit circle onto G. This fact enables us to give simple criteria ensuring the boundedness of (15) (or (16)). For instance, since for a schlicht function

$$A(r) \leqq \pi M(r)^2$$

where $M(r)$ denotes the maximum modulus of the function on the circle $|z| = r$, we immediately get the following criterion: *The integral* (15) (*or* (16)) *is bounded for a regular $f(z)$ if the values of $f(z)$ lie in a simply connected domain G with more than one boundary point and if the function $x(z)$ mapping the unit circle onto G satisfies the condition*

(17) $$\int_0^1 M(t)^2\,dt < \infty .$$

We illustrate this criterion by an example which, although quite simple, leads to certain considerations of a more general nature.

Let G be the angular domain $|\arg w| < \omega \leqq \pi$. We then have, with the normalization $x(0) = 1$,

$$x(z) = \left(\frac{1+z}{1-z}\right)^{2\omega/\pi}.$$

Choosing $\omega = \pi/4-\delta$ where δ is an arbitrarily small positive number, we have for $x(z)$

$$M(r) = \left(\frac{1+r}{1-r}\right)^{1/2-2\delta/\pi}.$$

Hence, for this ω the condition (17) is fulfilled. Thus the criterion implies that if the values of $f(z)$ lie in the interior of an angle smaller than $\pi/2$, then (15) (and (16)) are bounded. This is no longer true if the angle equals $\pi/2$ as is shown by the counter-example

$$f(z) = \left(\frac{1+z}{1-z}\right)^{1/2}.$$

The criterion ensuring the boundedness of (15) was to be applied to the maximum modulus of the *majorant* function $x(z)$. This is essential, for we can construct a function $f(z)$ for which (15) is not bounded in spite of the fact that the maximum modulus of $f(z)$ satisfies the relation (17).

To prove this, consider the function

$$f(z) = \left(\frac{1+\pi(z)}{1-\pi(z)}\right)^{1/2}$$

where

$$\pi(z) = \prod_{\nu=1}^{\infty}\left(\frac{z-a_\nu}{1-\bar{a}_\nu z}\right) \qquad (a_1 = 0,\ |a_\nu| < 1)$$

is a convergent Blaschke product. This function maps the unit circle onto the angle $|\arg w| < \pi/4$.

First it is easily seen that the integral (15) is unbounded for this $f(z)$, irrespective of the special form of the Blaschke product. In fact, if (15) were bounded we should have a finite number M such that

(18) $$\lim_{r\to 1}\int_0^{2\pi} |f(re^{i\varphi})|^2 d\varphi = \int_0^{2\pi} |f(e^{i\varphi})|^2 d\varphi < M .$$

Now

$$\int_0^{2\pi} |f(e^{i\varphi})|^2 d\varphi \geqq \int_{|f|>1} |f(e^{i\varphi})|^2 d\varphi = \lim_{\eta\to 0} \int_0^{2\pi} |f(e^{i\varphi})|^{2-\eta} d\varphi + O(1) \ (\eta > 0)\,. \tag{19}$$

From the identity $\mathrm{Re}\{w\} = |w| \cos(\arg w)$ it follows, since $\pi(z)$ has boundary values of modulus 1 almost everywhere, that

$$\int_0^{2\pi} |f(e^{i\varphi})|^{2-\eta} d\varphi = 2\pi \frac{\mathrm{Re}\{f(0)^{2-\eta}\}}{\sin\dfrac{\eta\pi}{4}} = \frac{2\pi}{\sin\dfrac{\eta\pi}{4}}\,.$$

By (19), this contradicts (18).

We now choose the zeros a_ν such that there are n_ν zeros, equally spaced, on the circumference of the circle $|z| = 1-\nu^{-3}$, where $n_\nu \sim \nu^{3/2}$. This choice is legitimate since it implies the convergence of the Blaschke-product. On the other hand, an easy estimate shows that the maximum modulus of $f(z)$ then satisfies the relation

$$M(r) = O\big((1-r)^{-1/4}\big)\,. \tag{20}$$

Hence, this $f(z)$ furnishes a counterexample of the desired kind.

The zeros a_ν were chosen in the above manner in order to simplify the computations as much as possible. The order of magnitude in (20) could undoubtedly be lowered.

9. As a second application of the theorem of section 5, very similar to the above, let E be the imaginary axis and $d\mu$ equal to the Euclidean element of length. In this case too, we must assume that the functions considered are regular. Now $\Omega(r) = L(r)$ denotes the total length of the line segments belonging to the image of $|z| \leqq r$ by $f(z) = u+iv$ and lying above the imaginary axis.

By virtue of the formula

$$\int_0^r \frac{L(t)}{t} dt = \frac{1}{2}\int_0^{2\pi} |u(re^{i\varphi})| d\varphi - \pi\,|u(0)|\,,$$

resulting e. g. from the principle of the argument, we may formulate the results concerning the integral

$$\int_0^r \frac{L(t)}{t} dt \tag{21}$$

equally well in terms of the integral

$$\int_0^{2\pi} |u(re^{i\varphi})| d\varphi . \tag{22}$$

Since for every schlicht function we have

$$L(r) \leqq 2M(r) ,$$

the criterion corresponding to that of section 8 reads as follows: *The integral* (21) (*or* (22)) *is bounded for a regular* $f(z) = u+iv$ *if the values of* $f(z)$ *lie in a simply connected domain* G *with more than one boundary point and if the function* $x(z)$ *mapping the unit circle onto* G *satisfies the condition*

$$\int_0^1 M(t)dt < \infty .$$

Again, concrete examples can easily be given.

10. An important application of our theorem is obtained if E is the whole plane and $d\mu$ is defined as the spherical element of area divided by π, i. e.,

$$d\mu(a) = \frac{|a|d|a|d\arg a}{\pi(1+|a|^2)^2} . \tag{23}$$

This mass distribution is so regular that the theorem of section 5 can be applied to meromorphic $f(z)$ also. Now $\Omega(r) = S(r)$ denotes the area of the image of $|z| \leqq r$ by the mapping $f(z)$ onto the Riemann sphere.

By a well-known theorem of Shimizu and Ahlfors (see e. g. [3], p. 166), we have

$$\int_0^r \frac{S(t)}{t} dt = T(r), \tag{24}$$

where $T(r)$ is the characteristic function of $f(z)$. Thus in this case the theorem of section 5 enables us to majorize the important function $T(r)$.

A question often arising in the theory of the distribution of values of meromorphic functions is the following: Consider all functions $f(z)$ meromorphic in the unit circle and omitting a given set of values. How rapidly can the characteristic $T(r)$ increase for this class of functions? By our theorem, we immediately get a partial solution of this problem since we are able to determine the extremal function for which $T(r)$ is maximal. Whether the remaining part of the problem can be solved depends on the extent to which this extremal function is numerically mastered. We shall see in section 11 that our theorem also provides another means of obtaining information about the order of magnitude of $T(r)$.

In this connection we mention a simple consequence of the inequality (11). Suppose $f(z)$ is meromorphic in the unit circle and omits a set of values of positive capacity there. The complement of this set is a domain G which, by hypothesis, possesses a Green's function. At every point of G we have, by (11),

$$N(r, a) \leqq g(a, f(0), G)\,, \tag{25}$$

whereas $N(r, a) = 0$ if a is not in G.

By (10) and (24),

$$\int N(r, a) d\mu = T(r)$$

if $d\mu$ is defined by (23) and the integral is extended over the whole plane. Hence, by (25),

$$T(r) \leqq \int_G g(a, f(0), G) d\mu\ .$$

The right-hand integral being evidently finite and independent of r we have thus established in a simple manner the following well-known theorem: If a function $f(z)$ meromorphic in the unit circle omits a set of values of positive capacity, then $f(z)$ is of bounded characteristic.

By an easy modification of the above reasoning, this result can be extended to the case where $N(r, a)$, instead of being zero, is bounded in a set of positive capacity.

11. As a last example we take $E = G$ and put $d\mu = d\sigma$ equal to a hyperbolic element of area in G. This hyperbolic metric in G is introduced in the following well-known manner: If z denotes an image of the point w of G in the mapping $z = x^{-1}(w)$, where x^{-1} is a function (no matter which one) mapping the universal covering surface G^{∞} of G onto the unit circle, we define

$$d\sigma(w) = \frac{|dz|^2}{(1-|z|^2)^2}\,.$$

It is well-known that in this manner a metric is induced not only in G^{∞} but also in G. Now $\Omega(r) = H(r)$ denotes the area of the image of $|z| \leqq r$ by $f(z)$ in this hyperbolic metric.

From the point of view of our considerations this choice of μ represents a very simple mass distribution. In fact, by the theorem of section 5 we immediately get the following estimate for any function $f(z)$ which omits at least three values in the unit circle:

$$\int_0^r \frac{H(t)}{t}\, dt \leqq \int_0^r \frac{dt}{t} \int_0^t \int_0^{2\pi} \frac{t\, dt\, d\varphi}{(1-t^2)^2} = \frac{\pi}{2} \log \frac{1}{1-r^2}\,. \tag{26}$$

This estimate is sharp since equality occurs if $f(z)$ is a function which maps the unit circle conformally onto G^{∞}.

Without analyzing the relation (26) in more detail we remark that, e. g., Picard-Landau's theorem (See [3], p. 56) can easily be established by means of (26). Also, since it is possible ([3]) to estimate the integral

$$\int_0^r \frac{H(t)}{t}\, dt$$

in terms of $T(r)$, the relation (26) can give us information about the growth of $T(r)$.

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