MATH. SCAND. 1 (1953)

ON INFINITE DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS. I

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1. The theory of infinite differential equations with constant coefficients is quite complete as far as *analytic* solutions are concerned[1]. However, no general theory for non-analytic solutions seems to have been developped[2]. The first problem which arises in such a general theory is the following: what is to be meant by a solution of the given equation

$$Ly = \sum_{0}^{\infty} a_n y^{(n)}(x) = 0 \ ?$$

In order to avoid pathological solutions $y(x)$, it is necessary to require more than uniform convergence of the series $\Sigma a_n y^{(n)}(x)$. In section 2 we shall give a definition which seems to fulfil two important requirements: it allows us to develop a general theory which is simple and suitable for application to specific problems, and it is not so restricting that it excludes *interesting* problems. We cannot, for example, differentiate the series term by term, i. e. use the left translations

$$\sum a_{n-p} y^{(n)}(x), \quad p = 1, 2, \ldots ;$$

our main method of approach consists in replacing these translations by translations to the right (see the first formula of section 3). We shall prove that every solution has a formal development in fundamental solutions of the form $x^k e^{\lambda x}$; this development is unique, and operations on the function are reproduced in a natural way on the formal series.

Received March 16, 1953.

[1] For the literature on this subject, see e. g. Muggli, H., *Differentialgleichungen unendlich hoher Ordnung mit konstanten Koeffizienten*, Comment. Math. Helv. 11 (1938), 151–179.

[2] For a very general approach to the theory of infinite differential equations, see Arley, Niels and Borchsenius, Vibeke, *On the theory of infinite systems of differential equations and their application to the theory of stochastic processes and the perturbation theory of quantum mechanics*, Acta Math. 76 (1944), 261–322.

As an application of the above-mentioned theory, we then determine the class of differential equations for which it is true that every solution is analytic in a certain region. The solution of this problem is important since it exhibits the limits of the existing theory of infinite differential equations.

2. Let $F(z)$ be an integral function,

$$F(z) = \sum_0^\infty a_n z^n,$$

and assume that $F(0) = 1$. We shall also suppose that $F(z)$ is of at most order 1, convergence type, which means that $F(z)$ can be factorized

$$F(z) = \prod_1^\infty (1 - z/\lambda_\nu)^{\mu_\nu}, \qquad \sum_1^\infty \mu_\nu/|\lambda_\nu| < \infty,$$

where $\mu_\nu \geqq 1$ are integers. The equation which will be studied is then

$$Ly \equiv F(D)y \equiv \sum_0^\infty a_n y^{(n)}(x) = 0. \tag{1}$$

To define what we shall mean by a solution of the infinite differential equation (1), we introduce the sequence $\{A_n\}$ of Taylor coefficients of the integral function

$$F^*(z) = \prod_1^\infty (1 + z/|\lambda_\nu|)^{\mu_\nu} = \sum_0^\infty A_n z^n.$$

We then call $y(x)$ a solution of $Ly = 0$ in the interval (a, b) if (1) holds for $a < x < b$ and if

$$\sum_0^\infty A_n |y^{(n)}(x)| \tag{2}$$

converges uniformly in $a + \varepsilon < x < b - \varepsilon$ for every $\varepsilon > 0$. The uniform convergence of (2) defines a linear class of functions which we shall denote by $C = C(a, b)$. Let us observe here that the class C is in general *not* quasi-analytic.

Before we proceed further, we set down two simple but fundamental properties of the sequence $\{A_n\}$, which we formulate in a lemma.

LEMMA 1. *For the sequence* $\{A_n\}$ *it holds true that*

$$|a_n| \leqq A_n \tag{3}$$

and

$$A_{n+p} \leqq A_n \cdot A_p, \quad n, p \geqq 0. \tag{4}$$

3. Let us now assume that $y(x)$ is a solution of $Ly = 0$ on $a < x < b$ and to simplify the notations, suppose, that the origin belongs to this interval. For $a < \xi < b$ we define

$$b_p = b_p(\xi) = \sum_0^\infty a_{n+p} y^{(n)}(\xi)\ , \quad p = 1, 2, \ldots ,$$

and

$$G(z;\xi) = \sum_{p=1}^\infty b_p z^{p-1} .$$

If $\xi = 0$, we write simply $G(z)$. On account of (4) we have $|b_p(\xi)| \leqq M(\xi) A_p$ and $G(z;\xi)$ is an integral function of order less than 1 or of order 1, minimal type. The importance and significance of these functions G is clear from the following heuristic reasoning. If $y(x)$ is a formal solution of our equation (1), this is also true for every derivative $y^{(k)}(x)$. If we use this, it follows that

$$Y(z;\xi) = \sum_0^\infty y^{(n)}(\xi) z^{-n} = \frac{zG(z;\xi)}{F(z)} \qquad \text{(formally).}$$

In particular, if $y(x) = x^k e^{\lambda x}$ is a solution, then the series defining Y converges for $|z| > |\lambda|$ and we have

$$Y(z;0) = z \cdot \frac{k!}{(z-\lambda)^{k+1}} .$$

With a given solution $y(x)$ we now associate the formal development

$$y(x) \sim \sum_{\nu=1}^\infty \sum_{k=0}^{\mu_\nu - 1} c_{\nu k} x^k e^{\lambda_\nu x} = \sum_1^\infty P_\nu(x) e^{\lambda_\nu x} ; \tag{5}$$

the constants $c_{\nu k}$ are determined from the principal part of the meromorphic function $G(z)/F(z)$ at the pole λ_ν, which by assumption is of order μ_ν at most, i. e. such that

$$\frac{G(z)}{F(z)} - \sum_{k=0}^{\mu_\nu - 1} c_{\nu k} \frac{k!}{(z-\lambda_\nu)^{k+1}}$$

is regular at λ_ν. From what was said above, it is clear that for a finite sum $y(x)$ of fundamental solutions the formal development is finite and has the sum $y(x)$.

4. We shall now prove the following fundamental theorem.

THEOREM 1. *If $y(x)$ and $z(x)$ are solutions of the equation* (1) *on the interval* (a, b) *and if $P_\nu(x)$ and $Q_\nu(x)$ denote the polynomials which by* (5) *are associated with $y(x)$ and $z(x)$, then*

$$\alpha y(x) + \beta z(x) \sim \sum_1^\infty \left(\alpha P_\nu(x) + \beta Q_\nu(x)\right) e^{\lambda_\nu x} \tag{6}$$

and

$$y(x+\xi) \sim \sum_1^\infty e^{\lambda_\nu \xi} P_\nu(x+\xi) e^{\lambda_\nu x}, \quad a < \xi < b . \tag{7}$$

The relation (6) is an immediate consequence of the definition since $G(z)$ depends linearly on the function $y(x)$.

For the proof of (7), we need two lemmas.

LEMMA 2. *If $P_\nu(x)$ is of degree $< \mu_\nu - 1$, then $y(x)$ is a solution of*

$$F_1(D)y = 0$$

where

$$F_1(z) = \frac{\lambda_\nu}{\lambda_\nu - z} \cdot F(z) ,$$

and conversely.

We first observe that the class $C_1(a, b)$ associated with $F_1(z)$ contains $C(a, b)$. We form the function

$$\varphi(x) = F_1(D)y(x) .$$

This function can be differentiated and we get $(1 - D/\lambda_\nu)\varphi(x) = 0$, whence $\varphi(x) = ke^{\lambda_\nu x}$. Furthermore, the series being absolutely convergent by (2) and (4),

$$\varphi(0) = F_1(D)y(0) = -\lambda_\nu \sum_{n=0}^\infty a_n \sum_{m=0}^{n-1} \lambda_\nu^{n-m-1} y^{(m)}(0) =$$

$$-\lambda_\nu \sum_{m=0}^\infty \lambda_\nu^m \sum_{n=0}^\infty a_{n+m+1} y^{(n)}(0) = -\lambda_\nu G(\lambda_\nu) .$$

It follows from our definition that $P_\nu(x)$ is of degree $< \mu_\nu - 1$ if and only if $G(\lambda_\nu) = 0$. Our assumption therefore implies $G(\lambda_\nu) = \varphi(0) = k = 0$ and hence $\varphi(x) \equiv 0$. The function $y(x)$ is thus a solution of $F_1(D)y = 0$.

Conversely, if $F_1(D)y = 0$, then in particular $F_1(D)y(0) = -\lambda_\nu G(\lambda_\nu) = 0$, which proves the lemma.

LEMMA 3. *If $y(x)$ is a solution of $F_0(D)y = 0$ and belongs to the class C corresponding to $F(z) = (1 + z/\lambda)^p F_0(z)$, then the functions $G_0(z)$, $G(z)$, which are associated with $y(x)$ and F_0, F respectively, are related in the same way:*

$$G(z) = (1 + z/\lambda)^p G_0(z) .$$

For $p = 1$ the lemma is obvious. The general case then follows by induction.

Let us now turn to the proof of Theorem 1. We form the function

$$z(x) = y(x) - P_\nu(x) e^{\lambda_\nu x} .$$

By (6), the development of $z(x)$ does not contain $e^{\lambda_\nu x}$. The function $G(z)$ which is associated with $z(x)$ therefore satisfies

$$G(\lambda_\nu) = G'(\lambda_\nu) = \ldots = G^{(\mu_\nu - 1)}(\lambda_\nu) = 0 .$$

Furthermore, $z(x)$ is a solution of $F_1(D)z = 0$ according to Lemma 2. If $G_1(z)$ is associated with F_1 and $z(x)$, then by Lemma 3

$$G_1(\lambda_\nu) = G_1'(\lambda_\nu) = \ldots = G_1^{(\mu_\nu - 2)}(\lambda_\nu) = 0 .$$

We find that $z(x)$ is a solution of

$$F_{\mu_\nu}(D)z = \lambda_\nu^{\mu_\nu} \frac{F(D)}{(D-\lambda_\nu)^{\mu_\nu}} z(x) = 0 .$$

Hence $G(z; \xi) = (1 - z/\lambda_\nu)^{\mu_\nu} G_{\mu_\nu}(z; \xi)$ by Lemma 3, and the development of $z(x+\xi)$ does not contain $e^{\lambda_\nu x}$. This proves the desired relation (7).

5. We can now prove the following fundamental uniqueness theorem.

THEOREM 2. *If the development of a solution $y(x)$ vanishes identically, then $y(x) \equiv 0$.*

It follows from the last part of the proof of Theorem 1 that

$$y_k(x) = \prod_{\nu > k} (1 - D/\lambda_\nu)^{\mu_\nu} y(x)$$

vanishes identically. From assumption (2) we immediately infer that

$$y(x) = \lim_{k \to \infty} y_k(x) \equiv 0 .$$

THEOREM 3. *If, at some point x_0, $y^{(n)}(x_0) = 0$, $n \geqq 0$, then $y(x) \equiv 0$.*

Under this assumption, $G(z; x_0) \equiv 0$, and so the development vanishes identically. Theorem 3 can evidently be proved directly without any difficulty.

The above theorem can be formulated thus: the class of solutions of $Ly = 0$ is quasi-analytic. This is, however, in general not true of the class C which belongs to the equation. It is important to observe that the restriction (2) on the solutions is essential in the proof of Theorem 3, which probably is false, if we only require uniform convergence of the series (1). In this connection it may be of interest to observe the simple fact that a "solution" is always determined by the values of its derivatives at two points.

Let us assume that $\Sigma_0^\infty a_n y^{(n)}(x) = 0$, where the series converges uniformly for $a \leqq x \leqq b$, and that $y^{(n)}(a) = y^{(n)}(b) = 0$, $n \geqq 0$. If we form the function

$$f(s) = \int_a^b e^{-xs} y(x) dx ,$$

partial integrations show that

$$F(s)f(s) = \sum_{n=0}^{\infty} a_n \int_a^b e^{-xs} y^{(n)}(x) dx = \int_a^b Ly(x) e^{-xs} dx = 0$$

by uniform convergence. Hence $f(s) \equiv 0$ which implies $y(x) \equiv 0$, $a \leqq x \leqq b$.

6. As an application of the above theory, we shall now determine the class of equations $Ly = 0$ for which all solutions are analytic. For these equations, the classical theory applies to every solution.

THEOREM 4. *A necessary and sufficient condition that every solution of the equation $Ly = 0$, which is defined on an open interval containing the closed interval (a, b), be analytic in (a, b) is that*

$$\text{(9)} \qquad \varliminf_{\nu \to \infty} \left| \frac{\alpha_\nu}{\beta_\nu} \right| > 0 , \qquad \lambda_\nu = \alpha_\nu + i\beta_\nu .$$

We need the following formal identity in the proof.

LEMMA 4. *If*

$$H(z; \zeta) = \sum_{n,m=0}^{\infty} a_{n+m+1} z^n \zeta^m$$

then

$$(z-\zeta)H(z; \zeta) = F(z) - F(\zeta) .$$

The straightforward calculation

$$(z-\zeta)H(z; \zeta) = (z-\zeta) \sum_{n,m=0}^{\infty} a_{n+m+1} z^n \zeta^m$$

$$= (z-\zeta) \sum_{q=0}^{\infty} a_{q+1} \sum_{p=0}^{q} z^{q-p} \zeta^p$$

$$= \sum_{q=0}^{\infty} a_{q+1} (z^{q+1} - \zeta^{q+1}) = F(z) - F(\zeta)$$

yields the lemma.

Let us now first assume that (9) holds. Let $y(x)$ be a solution on $a-\delta < x < b+\delta$, $\delta > 0$, and suppose that $a < 0 < b$. There are two rays,

l_1 and l_2, issuing from the origin and lying in the right half-plane and two rays, l_3 and l_4, in the left half-plane such that all but a finite number of zeros of $F(z)$ lie in the sectors which do not contain the imaginary axis. We may assume that there are no exceptions since we may consider the function $y(x) - \Sigma_1^n P_\nu(x) e^{\lambda_\nu x}$ instead of $y(x)$. Furthermore we may assume that, along the rays l_ν,

$$\lim_{|z| \to \infty} |z|^{-1} \log |F(z)| = 0 .$$

Let us now form the following function, defined for $a < x < b$.

$$z(x) = z_1(x) + z_2(x) =$$
$$\frac{1}{2\pi i} \int_{l_1+l_2} \frac{e^{z(x-b)}}{F(z)} G(z; b) dz + \frac{1}{2\pi i} \int_{l_3+l_4} \frac{e^{z(x-a)}}{F(z)} G(z; a) dz .$$

Since the integral functions $G(z; \xi)$ are of minimal type, the integrals converge for $a < x < b$ and represent a function $z(x)$ which is holomorphic in a domain

$$(10) \qquad |y| < \eta |x-a|, \quad |y| < \eta |x-b|, \quad a < x < b,$$

where η only depends on $\{\lambda_\nu\}$. Since $Le^{zx} = F(z) e^{zx}$ and

$$\overline{\lim_{r \to \infty}}\, r^{-1} \min_{|z|=r} \log |F(z)| = 0 ,$$

it follows that $Lz_i(x) = 0$, $i = 1, 2$, and since $z_i(x)$ is analytic, it is a solution. The main difficulty in the proof lies in the identification of the functions $y(x)$ and $z(x)$. We shall prove that they have the same formal development; the result then follows from Theorem 2.

It follows from the converse of Lemma 2 together with Lemma 3 that the coefficient of $e^{\lambda_\nu x}$, $\alpha_\nu < 0$, in the development of $z_1(x)$ vanishes, and a corresponding result holds for $z_2(x)$. It is thus sufficient to prove that $z_1(x)$ and $y(x)$ have identical developments with $\alpha_\nu > 0$. It is also, again by Lemmas 2 and 3, sufficient to prove that the coefficients of $x^{\mu_\nu - 1} e^{\lambda_\nu x}$ are the same. If $G_1(z)$ is the G-function associated with $z_1(x)$, we have to prove that $G(\lambda_\nu) = G_1(\lambda_\nu)$. We have

$$z_1^{(n)}(0) = \frac{1}{2\pi i} \int_{l_1+l_2} \frac{z^n e^{-zb}}{F(z)} G(z; b) dz .$$

This gives

$$G_1(\zeta) = \sum_{p=1}^{\infty} \zeta^{p-1} \sum_{n=0}^{\infty} a_{n+p} z_1^{(n)}(0) = \frac{1}{2\pi i} \int_{l_1+l_2} \frac{H(z; \zeta) e^{-zb}}{F(z)} G(z; b) dz ,$$

since, for ζ fixed, $H(z;\zeta)$ is an integral function of minimal type. According to Lemma 4,

$$(z-\lambda_\nu)H(z;\lambda_\nu) = F(z)\,.$$

Hence, by the Residue Theorem, we obtain

$$G_1(\lambda_\nu) = \frac{1}{2\pi i}\int\limits_{l_1+l_2} \frac{e^{-zb}G(z;b)}{z-\lambda_\nu}\,dz = G(\lambda_\nu;b)e^{-\lambda_\nu b}\,,$$

which by (7) gives $G_1(\lambda_\nu) = G(\lambda_\nu)$.

On the other hand, let us now assume that there exists a sequence $\{n_\nu\}$ such that $\lim_{\nu\to\infty}(\alpha_{n_\nu}/\beta_{n_\nu}) = 0$. We set $\lambda_{n_\nu} = s_\nu = \sigma_\nu + i\tau_\nu$ and consider sums of the form

$$y(x) = \sum_1^\infty c_\nu e^{s_\nu x}.$$

Since

$$|y^{(p)}(x)| \leqq \sum_1^\infty |c_\nu|\,|s_\nu|^p e^{|\sigma_\nu|}, \quad -1 \leqq x \leqq 1\,,$$

$y(x)$ is a solution of $Ly = 0$ on $(-1, 1)$ if

$$\sum_1^\infty |c_\nu| e^{|\sigma_\nu|} F^*(|s_\nu|) < \infty\,.$$

Since $F^*(z)$ is of minimal type and $\sigma_\nu/\tau_\nu \to 0$, the sequence $\{c_\nu\}$ can be chosen so that the above condition is satisfied and that further

$$\lim_{\nu\to\infty} |s_\nu|^{-1}\log|c_\nu| = 0\,.$$

The desired result now follows from the following elementary lemma, the proof of which can be omitted.

LEMMA 5. *A necessary and sufficient condition on the sequence $\{c_\nu\}$ that $y(x) = \sum_1^\infty \varepsilon_\nu c_\nu e^{s_\nu x}$, $|\varepsilon_\nu| \leqq 1$, be analytic in $(-1, 1)$ for every choice of $\{\varepsilon_\nu\}$ is that*

$$\overline{\lim_{\nu\to\infty}}\, |\tau_\nu|^{-1}\log|c_\nu| < 0\,.$$

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