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ON ANALYTIC FUNCTIONS OF SEVERAL VARIABLES. A THEOREM ON ANALYTIC CONTINUATION

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Introduction. Let $z_\nu = x_\nu + iy_\nu$, $\nu = 1, \ldots, m$, be complex variables. We shall use the vector notation $\boldsymbol{z} = (z_1, \ldots, z_m) = \boldsymbol{x} + i\boldsymbol{y}$. For real vectors we shall use the norm

$$\|\boldsymbol{x}\| = (x_1{}^2 + \ldots + x_m{}^2)^{1/2}.$$

We map the vector $\boldsymbol{z}$ on the point $(\boldsymbol{x}; \boldsymbol{y}) = (x_1, \ldots, x_m; y_1, \ldots, y_m)$ of a $2m$-dimensional space $\mathfrak{R}_{2m}$. If Ω is a point set in $\mathfrak{R}_{2m}$, the relation $\boldsymbol{z} \in \Omega$ will be used in place of $(\boldsymbol{x}; \boldsymbol{y}) \in \Omega$. In the present paper we shall study functions $f(\boldsymbol{z})$ analytic in a point set Ω defined by a condition

$$(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|) \in \omega$$

where ω is a connected point set in the quadrant $x \geqq 0, y \geqq 0$ of the (x, y)-plane. (A function is called analytic in a connected point set Ω if it is analytic in an open domain which contains Ω). The point set Ω is an open domain if and only if ω is an open domain relative to the quadrant $x \geqq 0, y \geqq 0$.

Corresponding to a point set ω in the quadrant $x \geqq 0, y \geqq 0$ we introduce another point set ω^* consisting of all points (x, y) which satisfy the conditions

$$x^2 - y^2 = x_0{}^2 - y_0{}^2;$$
$$0 \leqq x \leqq x_0,\ 0 \leqq y \leqq y_0$$

for a point (x_0, y_0) of ω. The point set ω^* will be called the *hyperbolic completion of* ω. (Fig. 1.)

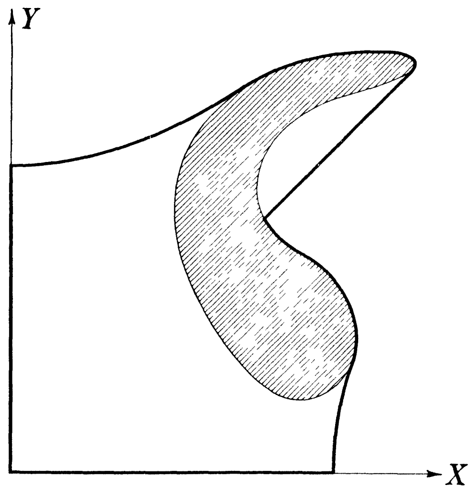

Fig. 1

The object of the present paper is to prove the following theorem.

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THEOREM 1. *Let ω be a connected point set in the quadrant $x \geqq 0$, $y \geqq 0$ of the (x, y)-plane and let ω^* be the hyperbolic completion of ω. Every function $f(\boldsymbol{z}) = f(z_1, \ldots, z_m)$ $(m \geqq 2)$ analytic in the point set $(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|) \in \omega$ then possesses an analytic continuation into a domain which contains the point set $(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|) \in \omega^*$.*

The point sets $(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|) \in \omega$ are characterized by the property that they are invariant with respect to every rotation

$$\begin{Bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{Bmatrix} \begin{Bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}^* \end{Bmatrix}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal matrices. These rotations, however, are not analytic if $\boldsymbol{A} \neq \boldsymbol{B}$ and it is not to be expected that every domain Ω of the form $(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|) \in \omega$ has an analytic completion (i. e. a regularity domain $\Omega' \supset \Omega$ with the property that every function analytic in Ω possesses an analytic continuation into the domain Ω') of the form $(\|\boldsymbol{x}\|, \|\boldsymbol{y}\|) \in \omega'$. (Compare [2].)

1. An auxiliary theorem. The following auxiliary theorem is very closely related to some classical results by F. Hartogs [1].

THEOREM 2. *Let Ω be a bounded, closed, and connected point set in $\mathfrak{R}_{2m}$ and let $R_1(\boldsymbol{z})$ and $R_2(\boldsymbol{z})$ be two positive functions, continuous in Ω and satisfying the condition $R_1(\boldsymbol{z}) \leqq R_2(\boldsymbol{z})$ where the sign of equality holds at least for one vector $\boldsymbol{z} = \boldsymbol{z}^* \in \Omega$. Every function $f(\boldsymbol{z}; w)$, analytic for all $(\boldsymbol{z}; w)$ satisfying*

$$\boldsymbol{z} \in \Omega; \qquad \big(|w| - R_1(\boldsymbol{z})\big)\big(|w| - R_2(\boldsymbol{z})\big) = 0\,,$$

possesses an analytic continuation into the point set

$$\boldsymbol{z} \in \Omega; \qquad R_1(\boldsymbol{z}) \leqq |w| \leqq R_2(\boldsymbol{z})\,.$$

PROOF. There exist a positive number h and an open domain $\Omega^* \supset \Omega$ such that $f(\boldsymbol{z}; w)$ is analytic and bounded in the domain

$$\text{(1)} \qquad \boldsymbol{z} \in \Omega^*; \qquad R_1(\boldsymbol{z}) - h < |w| < R_1(\boldsymbol{z}) + h$$

and in the domain

$$\text{(2)} \qquad \boldsymbol{z} \in \Omega^*; \qquad R_2(\boldsymbol{z}) - h < |w| < R_2(\boldsymbol{z}) + h\,.$$

In the domain (1) the function $f(\boldsymbol{z}; w)$ can be developed into a Laurent series

$$f(\boldsymbol{z}; w) = \sum_{n=-\infty}^{\infty} a_n(\boldsymbol{z})\, w^n,$$

where

$$a_n(z) = \frac{1}{2\pi i} \int_{|w|=\varrho(z)} f(z; w)\, w^{-n-1} dw\,. \tag{3}$$

The function $\varrho(z)$ has to be chosen such that $R_1(z)-h < \varrho(z) < R_1(z)+h$, but it can be chosen as a constant in the neighbourhood of any particular vector z. Hence $a_n(z)$ is an analytic function in Ω^*. Similarly we have in the domain (2)

$$f(z; w) = \sum_{n=-\infty}^{\infty} b_n(z)\, w^n,$$

where

$$b_n(z) = \frac{1}{2\pi i} \int_{|w|=\sigma(z)} f(z; w)\, w^{-n-1} dw\,. \tag{4}$$

The function $\sigma(z)$ must satisfy $R_2(z)-h < \sigma(z) < R_2(z)+h$. As in the preceding case it follows that $b_n(z)$ is analytic in Ω^*.

In the neighbourhood of z^* we can choose $\varrho(z) = \sigma(z)$, hence $a_n(z) = b_n(z)$ in a neighbourhood of z^* and therefore, by analytic continuation, in Ω^*. We have thus proved that the two Laurent series are identical and their common domain of convergence must contain the domain

$$z \in \Omega^*; \qquad R_1(z)-h < |w| < R_2(z)+h\,. \tag{5}$$

If K denotes the maximum of $|f(z; w)|$ in the domains (1) and (2), we get from (3)

$$|a_n(z)| \leqq K(R_1(z)-h)^{-n}$$

and from (4)

$$|b_n(z)| \leqq K(R_2(z)+h)^{-n}\,.$$

If we use the first estimate when $n < 0$ and the second estimate when $n > 0$, we find that the Laurent series converges uniformly in every closed subset of (5). Hence, it represents an analytic function in the domain (5) and this completes the proof of Theorem 2.

2. A fundamental lemma.

The proof of the following lemma is the most difficult part of our proof of Theorem 1.

THEOREM 3. *Let ω be a bounded, closed, and connected point set in $\mathfrak{R}_{2m}$ and let $a(z)$ and $b(z)$ denote functions continuous and positive in ω. A point set Ω in the $(z; w_1, w_2)$-space, where $w_1 = u_1+iv_1$, $w_2 = u_2+iv_2$, is defined by the conditions*

$$z \in \omega; \qquad {u_1}^2+{u_2}^2 = a(z),\ {v_1}^2+{v_2}^2 = b(z)\,.$$

Every function $f(z; w_1, w_2)$ analytic in Ω then possesses an analytic continuation into the point set

$$z \in \omega; \quad a(z)-u_1^2-u_2^2 = b(z)-v_1^2-v_2^2,\ u_1^2+u_2^2+v_1^2+v_2^2 \leqq a(z)+b(z)\,.$$

The last condition can obviously be replaced by $u_1^2+u_2^2 \leqq a(z)$ or by $v_1^2+v_2^2 \leqq b(z)$.

Proof. First we shall prove Theorem 3 in the particular case where $a(z) \neq b(z)$ for every vector $z \in \omega$. If h is a positive number satisfying

$$h < \tfrac{1}{2}\min a(z),\ h < \tfrac{1}{2}\min b(z),\ h < \tfrac{1}{2}\min |b(z)-a(z)|\,,$$

the point set Ω_h defined by the conditions

$$z \in \omega; \quad |u_1^2+u_2^2-a(z)| \leqq h,\ |v_1^2+v_2^2-b(z)| \leqq h$$

does not contain any point of the manifold $u_1^2+u_2^2 = v_1^2+v_2^2$. Let $f(z; w_1, w_2)$ be a given function analytic in Ω. We choose h so small that $f(z; w_1, w_2)$ is analytic in Ω_h.

We introduce new variables $z_1^*, \ldots, z_m^*$; $\zeta_1 = \xi_1+i\eta_1$, $\zeta_2 = \xi_2+i\eta_2$ by the transformation

$$\begin{aligned} z_\nu^* &= z_\nu,\ \nu = 1, \ldots, m; \\ \zeta_1 &= w_1^2+w_2^2,\ \zeta_2 = w_1+iw_2\,, \end{aligned} \tag{6}$$

which is non-singular when $\zeta_2 = w_1+iw_2 \neq 0$. The transformation is a one-to-one analytic mapping of the domain $w_1+iw_2 \neq 0$ on the domain $\zeta_2 \neq 0$.

The last two relations of (6) imply

$$\xi_1 = u_1^2+u_2^2-v_1^2-v_2^2 \tag{7}$$

and

$$4(u_1^2+u_2^2) = |2(u_1+iu_2)|^2 = \left|\zeta_2+\frac{\overline{\zeta_1}}{\zeta_2}\right|^2 = \frac{|\zeta_2|^4+2\xi_1|\zeta_2|^2+|\zeta_1|^2}{|\zeta_2|^2}\,. \tag{8}$$

It follows that the transformation (6) maps the point set Ω_h onto a point set which contains the point set Ω_h^* defined by the conditions

$$\begin{aligned} z^* \in \omega; \quad &\left|\xi_1-\big(a(z^*)-b(z^*)\big)\right| \leqq \tfrac{1}{2}h\,, \\ &\left||\zeta_2|^4+2\xi_1|\zeta_2|^2+|\zeta_1|^2-4a(z^*)|\zeta_2|^2\right| \leqq 2h|\zeta_2|^2\,. \end{aligned} \tag{9}$$

From the second condition follows that

$$2a(z^*)-\xi_1 \geqq 2a(z^*)-\big(a(z^*)-b(z^*)+\tfrac{1}{2}h\big) = a(z^*)+b(z^*)-\tfrac{1}{2}h > 0\,.$$

Hence the biquadratic equation

$$(10) \qquad |\zeta_2|^4 - 2\big(2a(z^*) - \xi_1 + h\big)|\zeta_2|^2 + |\zeta_1|^2 = 0$$

is satisfied by two positive values of $|\zeta_2|$ if it is satisfied by some real value of $|\zeta_2|^2$ (both the sum and the product of the solutions with respect to $|\zeta_2|^2$ are, in fact, positive). It follows that we may describe the point set $\Omega_h{}^*$ in the following way:

1) For a fixed $z^* \in \omega$ the variable ξ_1 runs through the interval

$$a(z^*) - b(z^*) - \tfrac{1}{2}h \leqq \xi_1 \leqq a(z^*) - b(z^*) + \tfrac{1}{2}h \, .$$

2) For fixed z^* and ξ_1 satisfying 1) the variable η_1 runs through an interval $|\eta_1| \leqq k(z^*; \xi_1)$ where $k(z^*; \xi_1)$ is the particular value of η_1 for which (10) has double roots.

3) For fixed z^* and ζ_1 satisfying 1) and 2) the variable ζ_2 runs through a point set consisting of two closed annular domains which are defined by the third condition in (9). They depend continuously on z^* and ζ_1 and they melt together into one annular domain when $|\eta_1|$ is close to its upper bound $k(z^*; \xi_1)$.

It follows from Theorem 2 that the function $g(z^*; \zeta_1, \zeta_2)$ corresponding to $f(z; w_1, w_2)$ by the transformation (6), has an analytic continuation into a domain $\Omega_h{}^{**}$ which we construct by adjoining to $\Omega_h{}^*$ all points $(z^*; \zeta_1, \zeta_2)$ where z^* and ζ_1 satisfy 1) and 2) while ζ_2 belongs to the domain between the annular domains in 3). The domain $\Omega_h{}^{**}$ is obviously defined by the conditions

$$z^* \in \omega; \quad \big|\xi_1 - \big(a(z^*) - b(z^*)\big)\big| \leqq \tfrac{1}{2}h \, ,$$
$$|\zeta_2|^4 + 2\xi_1|\zeta_2|^2 + |\zeta_1|^2 - 4a(z^*)|\zeta_2|^2 \leqq 2h|\zeta_2|^2 \, .$$

The corresponding conditions in the variables $z; w_1, w_2$ are, according to (7) and (8),

$$z \in \omega; \quad \big|u_1{}^2 + u_2{}^2 - v_1{}^2 - v_2{}^2 - \big(a(z) - b(z)\big)\big| \leqq \tfrac{1}{2}h, \; u_1{}^2 + u_2{}^2 \leqq a(z) + \tfrac{1}{2}h \, .$$

This completes the proof of Theorem 3 in the special case where $a(z) \neq b(z)$ for every $z \in \omega$.

Before we proceed with the proof of Theorem 3, we shall prove the following statement, which is a simple corollary of the special case that has already been proved.

Let ω be a connected, closed point set in $\mathfrak{R}_{2m}$ and Δ a closed domain in the quadrant $x \geqq 0, y \geqq 0$ of the (x, y)-plane. Further, we assume that Δ intersects every hyperbola $x^2 - y^2 = c$ in at most one arc, and that Δ does not intersect the straight line $y = x$. Let Δ^* denote the hyperbolic completion (defined in the introduction) *of Δ. Every function $f(z; w_1, w_2)$ analytic in the domain Ω defined by*

$$z \in \omega; \qquad \big((u_1^2+u_2^2)^{1/2}, (v_1^2+v_2^2)^{1/2}\big) \in \Delta$$

then possesses an analytic continuation into the domain Ω^* defined by

$$z \in \omega; \qquad \big((u_1^2+u_2^2)^{1/2}, (v_1^2+v_2^2)^{1/2}\big) \in \Delta^* .$$

We have, in fact, proved that $f(z; w_1, w_2)$ has an analytic continuation into the domain obtained from Ω by adjoining a hyperbolic arc like PQ in fig. 2 to Δ. But we may then even adjoin to Δ a strip bounded by hyperbolic arcs, indicated by dotted lines in fig. 2. According to Borel's covering theorem, a finite number of these strips will cover the hyperbolic completion. If the intersection of two strips is not empty, the corresponding continuations of $f(z; w_1, w_2)$ are identical in the extension of Ω corresponding to the intersection, because they are identical in the part of this extension which is contained in Ω. This completes the proof of the statement.

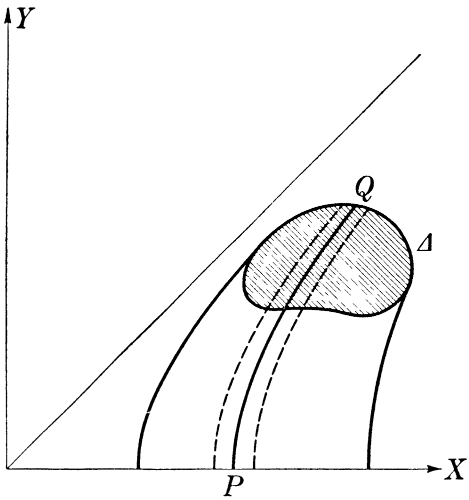

Fig. 2

We shall now pass to the proof of Theorem 3 in the general case. Let $z = z^0 \in \omega$ be a special vector, for which $a(z^0) = b(z^0) = a$. We have only to prove that a function $f(z; w_1, w_2)$ analytic in Ω has an analytic continuation into a domain which contains the point set

$$z = z^0; \qquad u_1^2+u_2^2 = v_1^2+v_2^2 \leqq a .$$

Without restricting the generality we may assume that $z^0 = \mathbf{0}$. The function $f(z; w_1, w_2)$ is then analytic in the domain

$$|z_\nu| < h, \nu = 1, \ldots, m; \quad |u_1^2+u_2^2-a| < h, \quad |v_1^2+v_2^2-a| < h$$

where h is a sufficiently small positive number. According to the above statement $f(z; w_1, w_2)$ possesses an analytic continuation into the part of the domain

$$|z_\nu| < h, \nu = 1,\ldots, m; \quad |u_1^2+u_2^2-v_1^2-v_2^2| < h,$$
$$u_1^2+u_2^2+v_1^2+v_2^2 < a+h$$

outside the point set

$$(11) \qquad |z_\nu| < h, \nu = 1,\ldots, m; \quad u_1^2+u_2^2 = v_1^2+v_2^2 \leqq a-h .$$

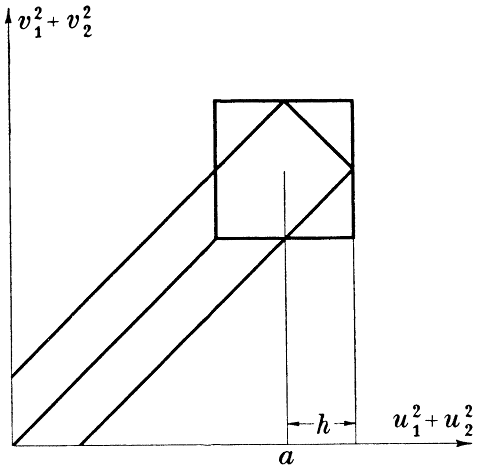

Fig. 3

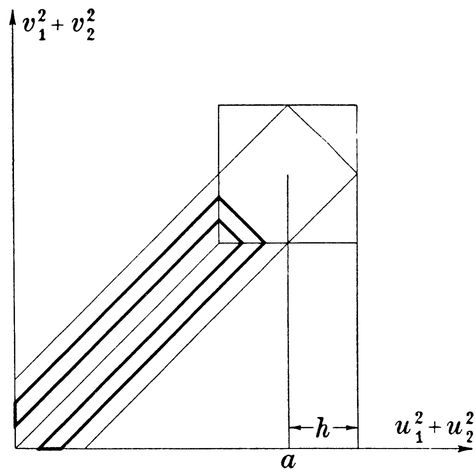

Fig. 4

We have indicated the domain in fig. 3, where we have used $u_1^2+u_2^2$ and $v_1^2+v_2^2$ as coordinates.

We replace u_1 by a new variable $u_1+\delta = u_1^*$ where δ is a real constant and we choose the absolute value of δ so small that

$$|u_1^{*2}-u_1^2| < \tfrac{1}{3}h$$

for every point of the domain. In the new variables the domain indicated in fig. 3 corresponds to a domain which contains the domain Δ drawn with heavy lines in fig. 4. But a function, analytic when $|z_\nu| < h$, $\nu = 1,\ldots, m$, while $(u_1^{*2}+u_2^2,\ v_1^2+v_2^2)$ belongs to Δ, possesses, according to the statement above, an analytic continuation into the hyperbolic completion of the domain in fig. 4. Thus, we find that the exceptional set (11) must be replaced by the set

(12) $$|z_\nu| < h, \nu = 1,\ldots, m; \quad (u_1-\delta)^2+u_2^2 = v_1^2+v_2^2 \leqq a\,.$$

But no point belongs to (12) for all small values of δ. Hence there exists no really exceptional point after all. This completes the proof of Theorem 3.

3. Proof of Theorem 1. We start by proving the following special case of Theorem 1.

THEOREM 4. *Let a and b denote two positive numbers. Every function $f(z) = f(z_1,\ldots, z_m)$ $(m \geqq 2)$ analytic when $\|x\| = a$, $\|y\| = b$ has an analytic continuation into a domain which contains the point set*

(13) $$\|x\|^2-\|y\|^2 = a^2-b^2, \quad \|x\|^2+\|y\|^2 \leqq a^2+b^2\,.$$

Proof. The conditions $\|\boldsymbol{x}\| = a$, $\|\boldsymbol{y}\| = b$ are equivalent to the following set of conditions

$$x_3^2+\ldots+x_m^2 \leqq a^2, \quad y_3^2+\ldots+y_m^2 \leqq b^2,$$
$$x_1^2+x_2^2 = a^2-(x_3^2+\ldots+x_m^2), \quad y_1^2+y_2^2 = b^2-(y_3^2+\ldots+y_m^2)$$

and, according to Theorem 3, a function, analytic in this point set, possesses an analytic continuation into the point set

$$x_3^2+\ldots+x_m^2 \leqq a^2, \quad y_3^2+\ldots+y_m^2 \leqq b^2,$$
$$x_1^2+x_2^2-y_1^2-y_2^2 = a^2-(x_3^2+\ldots+x_m^2)-b^2+(y_3^2+\ldots+y_m^2)\,,$$
$$x_1^2+x_2^2+y_1^2+y_2^2 = a^2-(x_3^2+\ldots+x_m^2)+b^2-(y_3^2+\ldots+y_m^2)\,,$$

but this set of conditions is obviously equivalent to the set (13).

Finally, we shall prove Theorem 1. If $(x_0, y_0) \in \omega$, the function $f(\boldsymbol{z})$ possesses, according to Theorem 4, an analytic continuation into a domain which contains the manifold

$$(14) \qquad \|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2 = x_0^2-y_0^2, \quad 0 \leqq \|\boldsymbol{x}\| \leqq x_0, \quad 0 \leqq \|\boldsymbol{y}\| \leqq y_0\,.$$

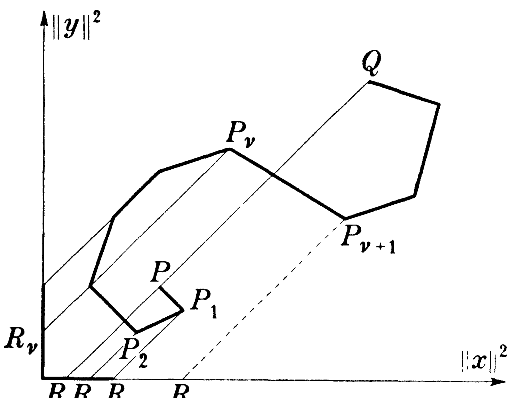

Fig. 5

To complete the proof we have to verify that the continuations of $f(\boldsymbol{z})$ along the manifolds (14) starting from two different points (x_1, y_1) and (x_2, y_2) where $x_1^2-y_1^2 = x_2^2-y_2^2$ are identical in the intersection of the manifolds. We have illustrated this in fig. 5, where we have used $\|\boldsymbol{x}\|^2$ and $\|\boldsymbol{y}\|^2$ as coordinates. Let $f(\boldsymbol{z})$ be a function, analytic when $(\|\boldsymbol{x}\|^2, \|\boldsymbol{y}\|^2) \in \omega$, and let P and Q be two points of ω situated so that PQ has the slope 1 in the diagram. We shall prove that the continuations of $f(\boldsymbol{z})$ along the segments PR and QR where R is the point of intersection between PQ and the boundary of the quadrant, are identical on the common part of the two segments. There exists a broken line $PP_1 \ldots P_{n-1}Q$ joining P and Q such that $f(\boldsymbol{z})$ is analytic when $(\|\boldsymbol{x}\|^2, \|\boldsymbol{y}\|^2)$ belongs to this line. Let $R, R_1, \ldots, R_{n-1}, R_n$ be the projections of the vertices $P, P_1, \ldots, P_{n-1}, Q$ in the direction of the straight line $\|\boldsymbol{x}\|^2 = \|\boldsymbol{y}\|^2$ on the boundary of the quadrant. Let us assume that we already have proved that $f(\boldsymbol{z})$ has an analytic continuation into the domain corresponding to the hyperbolic completion of $PP_1 \ldots P_\nu$. The continuation is then analytic in the domain corresponding to the segments of the coordinate axes drawn with heavy lines in fig. 5. Obviously, these form a connected set. But $f(\boldsymbol{z})$ has a continuation into the domain corresponding to the quadrangle $P_\nu P_{\nu+1} R_{\nu+1} R_\nu$, and this continuation is identical with the preceding one in a neighbourhood of $(\|\boldsymbol{x}\|^2, \|\boldsymbol{y}\|^2) = P_\nu$, hence, also when $(\|\boldsymbol{x}\|^2, \|\boldsymbol{y}\|^2)$ is on $P_\nu R_\nu$ and, finally, when $(\|\boldsymbol{x}\|^2, \|\boldsymbol{y}\|^2)$ is on the part of the segment $R_\nu R_{\nu+1}$ which belongs to the hyperbolic completion of $PP_1 \ldots P_\nu$. This implies that the two completions are identical everywhere, which completes the proof.

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