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ON THE ITERATION OF ANALYTIC FUNCTIONS

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The problem to be discussed and solved in the present note is the following: The theory of iteration developed by Fatou and Julia is concerned with rational and entire functions. What is the most general class of analytic functions to which the main results of this theory can be extended?

1. Introduction.

1.1. Let f be a function defined in a region (open connected set) D of the complex number sphere S and meromorphic in D. We call D the domain of f and $f(D)$ (which is also a region if f is not constant) the range of f. If $x \in D$ is such that also $f(x) \in D$, then $f^2(x) = f(f(x))$ exists. If also $f^2(x) \in D$, then this procedure can be iterated at least once more. We define $f^n(x)$ by $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$. For a given $x \in D$ two cases may occur: Either $f^n(x) \in D$ for all $n = 0, 1, 2, \ldots$, or there is a number N such that $f^n(x) \in D$ for $0 \leqq n \leqq N$ but $f^{N+1}(x) \notin D$. In the second case f^n is not defined at x for $n \geqq N+2$, and we say that the point x is of order $N+1$. If no such N exists, then x is said to be of infinite order. Denote by D_n the subset of D consisting of all points of order at least n and by D_ω the set of points of infinite order. Then $D = D_1 \supset D_2 \supset D_3 \supset \ldots \supset D_\omega = \bigcap D_n$, and it is also clear that D_n is the domain of f^n. We observe that although the sets D_n are open, this is not necessarily true for D_ω. We have $f(D_{n+1}) \subset D_n$ and $f(D_\omega) \subset D_\omega$.

1.2. In the sequel we require that D have the following property: *If $S-D$ contains an isolated point, then this point is an essential singularity of f.* On the other hand, we allow f to be continuable across other parts of the boundary of D.

This assumption is made for reasons of convenience only. If b is an isolated point of $S-D$ at which the analytic continuation of f is regular

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or has a pole, then we can always adjoin b to D and consider the continuation of f to this enlarged domain instead of considering f itself. This shows that our assumption is no essential restriction.

1.3. If D_ω is non-vacuous, then the problem of studying the behaviour of the infinite sequence f^n of functions on D_ω arises. If $D_\omega = S$, then the function f is obviously rational. This case was thoroughly studied by Fatou [1], [2], [3], and Julia [5]. If $D_\omega = S-\{\infty\}$, then f is entire, a case which was studied by Fatou [4].

The main problem in the theory of iteration is to decide whether the sequence f^n behaves "wildly" or regularly. It is therefore natural to ask for conditions under which the family $\{f^n\}$ is normal in the sense of Montel. In the two cases treated in the above mentioned papers by Fatou and Julia it is shown that D_ω always contains points where $\{f^n\}$ is not normal.

The purpose of this paper is twofold. Firstly we show that there is a third case in which $\{f^n\}$ is non-normal in D_ω, namely the case in which $D_\omega = S-\{a, b\}$ where a and b are different points of S. We also show that in all other cases except the three cases just mentioned, the family $\{f^n\}$ is normal in the interior of D_ω. Secondly we extend some of the fundamental theorems of Fatou and Julia to the third case mentioned above.

1.4. If $f^n(x) = y$, we say that x is a predecessor of y and y a successor of x, in both cases of order n.

If g is a one-to-one meromorphic mapping of $D \cup f(D)$, then the function h defined by $hg = gf$ with domain $g(D)$ and range $gf(D)$ is called the transformation of f by g. For g we shall use only functions of the form $g(x) = (ax+b)/(cx+d)$ (Möbius transformations).

2. The set D_ω.

2.1. LEMMA. *The set $S-D_\omega$ consists of the points in $S-D$ and all the predecessors of these points.*

PROOF. Let $x \in S-D_\omega$. Then either $x \in S-D$, in which case the lemma is proved, or $x \in D$, in which case there is a number n with $x \in D_n$, $x \notin D_{n+1}$. Thus $f^n(x)$ is defined and $\notin D$, which shows that x is a predecessor of order n of a point in $S-D$. Conversely, let $f^n(x) \in S-D$ for a value $n = 0, 1, 2, \ldots$. Then $f^{n+1}(x)$ is not defined at x and, therefore, the sequence of iterates of x is not infinite. Thus $x \in S-D_\omega$, and the lemma is proved.

2.2. Theorem. *If $S-D_\omega$ contains more than two points, then it contains no isolated point.*

Proof. If f is constant, then either D_ω is vacuous (if $f(D) \not\subset D$) or $D_\omega = D$ (if $f(D) \subset D$). In both cases the theorem is true, in the second case because of 1.2. We may therefore assume that f is non-constant. It follows that f and all its iterates are open mappings. Suppose that a is an isolated point of $S-D_\omega$. There are two cases.

Assume first that $a \notin D$. Then obviously a is an isolated point of $S-D$, and by the condition imposed on D in section 1.2 we see that a is an isolated essential singularity of f. By Picard's theorem and the fact that $S-D_\omega$ contains more than two points it follows that there is a point in $S-D_\omega$ whose predecessors of order one accumulate at a. But these predecessors are themselves $\in S-D_\omega$ by Lemma 2.1. This is a contradiction.

Secondly, assume that $a \in D$. Then there exists a number n such that $a \in D_n$, $a \notin D_{n+1}$. Thus f^n is defined at a but f^{n+1} is not. Put $b = f^n(a)$. Since $f^{n+1}(a)$ is not defined, we have $b \notin D$.

Now, the set $U = \{a\} \cup D_\omega$ is a neighborhood of a since a is isolated in $S-D_\omega$. We observe that f^n is defined at all points of U. Since f^n is an open mapping, it follows that $f^n(U)$ is a neighborhood of $b = f^n(a)$. But $f^n(U) = \{b\} \cup f^n(D_\omega)$ and the set $f^n(D_\omega) \subset D_\omega$. Thus $\{b\} \cup D_\omega$ is a neighborhood of b. If we combine this result with the result $b \notin D$, we see that b is an isolated point of $S-D_\omega$. We can now use the result obtained in the first case.

Thus the theorem is completely proved.

2.3. Using Theorem 2.2 we see that only the following cases can occur:

I: $S-D_\omega$ contains no point
II: $S-D_\omega$ contains one point
III: $S-D_\omega$ contains two points
IV: $S-D_\omega$ is dense in itself.

In the case I it follows that $D_\omega = S$. Thus $D = S$ and f has to be rational. Conversely, if f is rational and $D = S$, then f^n is defined everywhere and rational, and $D_\omega = S$.

In the case II we have $D_\omega = S-\{a\}$. Then either $D = S$ or $D = S-\{a\}$. The first case is impossible as f would then be rational and therefore also $D_\omega = S$. Thus $D = S-\{a\}$ and it follows from 1.2 that a is an essential singularity of f. It is also clear that a is not a value of f. Suppose namely that a has a predecessor b of order one. Then $b \neq a$ since a is not a predecessor. But by Lemma 2.1 the point b would be $\in S-D_\omega$ and this set would therefore contain two points.—By a Möbius transformation of f we can move a to ∞. The transformed function is defined in the entire plane and does not assume the value ∞. Thus it is entire *but not* a polynomial.

In the case III the set $D_\omega = S - \{a, b\}$. In the same way as in case II we see that $D = S$ is impossible. Thus there remain two possibilities:

III a: $S - D$ contains one point
III b: $S - D$ contains two points.

We treat first III a, and assume that $S - D = \{a\}$. By 1.2 the point a is an essential singularity of f. By Lemma 2.1 it follows that b is a predecessor of a, say $f^n(b) = a$. But then $f(b)$ is also a predecessor of a and, therefore, $f(b) \in S - D_\omega$. Thus either $f(b) = b$ or $f(b) = a$. In the first case $f^n(b) = b$ for all n, hence $b \in D_\omega$. Thus we see that $f(b) = a$. It also follows that b is not a successor. In fact, suppose that $f^n(c) = b$. Then $c \in S - D_\omega$ (Lemma 2.1 again) and, since a has no successors, we obtain $c = b$, which gives the contradiction $f^n(b) = b$, $b \in D_\omega$ as before. Finally we observe that b is the only root of the equation $f(x) = a$. By a Möbius transformation we can move a to infinity and b to the origin. The transformed function has an essential singularity at ∞, a pole at the origin, and is holomorphic at the rest of the sphere. It does not take the value 0. Thus it is of the form

$$x^{-n} e^{F(x)}$$

where n is a positive integer and F is entire and non-constant. Conversely, every Möbius transform of such a function belongs to the class III a.

We now turn to the case III b. Then both a and b are essential singularities of f, and none of them is a value of f. Suppose namely $f(c) = a$. Then $c \in S - D_\omega$ by Lemma 2.1, but this is impossible as c would then be a or b, i.e. singular. By a Möbius transformation we move a to infinity and b to the origin. The transformed function must have the form

$$x^n e^{F(x) + G\left(\frac{1}{x}\right)}$$

where n is an integer and F and G are entire non-constant functions. Conversely, every Möbius transform of such a function belongs to class III b.

We observe that if f belongs to III a, then f^n for $n > 1$ belongs to III b.

3. Normality of $\{f^n\}$.

3.1. In this section we review some known results of the theory of iteration, which we need in the proof of Theorem 3.2.

If $f^n(x) = x$, $n \neq 0$, we say that x is a fixed point. If $n > 0$ is the smallest number for which $f^n(x) = x$, then n is called the order of x and the set $\{x, f(x), f^2(x), \ldots, f^{n-1}(x)\}$ is called the cycle of x.

Let $c = f(c) \neq \infty$ be a fixed point of order one of f. Then c is called attractive, indifferent, or repulsive in the cases $|f'(c)| < 1$, $= 1$, or > 1 respectively. It is well known that if there exists a neighborhood of c in which the family $\{f^n\}$ is defined and normal, then c is either attractive or indifferent. Conversely, if c is attractive, then such a neighborhood exists. An indifferent fixed point c for which such a neighborhood exists is called a center.

If c is attractive and U is a connected neighborhood of c, then the sequence f^n converges to the constant value c in U uniformly on every compact subset of U.

If c is a center, then there exists a sequence $f^{n_\nu}(x)$ converging uniformly on every compact subset of U to the function x. (Cf. Fatou [2], pp. 55–56.)

3.2. Fatou and Julia proved that in the cases I and II the set D_ω contains infinitely many points where the family $\{f^n(x)\}$ is not normal (in case I, of course, with the exception of functions f of the form $f(x) = (ax+b)/(cx+d)$). The following theorem shows that this is true also in case III but not in case IV.

THEOREM. *In the case* III *the set* D_ω *contains infinitely many points at which the family* $\{f^n\}$ *is not normal, whereas in the case* IV *this family is normal at every interior point of* D_ω.

PROOF. Case III. Suppose that the number of points where $\{f^n\}$ is non-normal is finite. Then the subset U of D_ω in which $\{f^n\}$ is normal is open and connected.

Let the two points of $S - D_\omega$ be a and b. We know that at least one of them is an essential singularity of f, and that neither $f(x) = a$ nor $f(x) = b$ has infinitely many roots. In case III a, one of them has one root and the other none, and in case III b, they have no roots at all. It follows from Picard's theorem that the equation $f(x) = x$ has infinitely many roots which accumulate at one (case III a) or both (case III b) of the points a and b. Then all of these roots except a finite number are elements of U. Let c and d be two of these. Then c is either attractive or a center.

Suppose first that c is attractive. Then the sequence f^n converges to c everywhere in U. Thus $f^n(d) \to c$ in contradiction to the fact that $f^n(d) = d$.

Secondly, suppose that c is a center. Then there exists a sequence n_ν such that $f^{n_\nu}(x) \to x$ in U. Let $e \neq c$ be an element of U satisfying $f(e) = c$.

Such a point exists since $f(x) = c$ has infinitely many roots. We have $f^{n_\nu}(e) \to e$. But this contradicts the fact that $f^n(e) = c$ for $n \geqq 1$. This proves the theorem for the case III.

Case IV. We know that in this case the set $S - D_\omega$ consists of more than two points. Since $f^n(D_\omega) \subset D_\omega$, we see that the points of $S - D_\omega$ are not taken as values by f^n in D_ω. Thus $\{f^n\}$ is normal at every interior point of D_ω. This completes the proof of the theorem.

4. Some results for the case III.

4.1. We suppose in this section that f belongs to the class III, and we denote by F the set of points of D_ω at which the family $\{f^n\}$ is not normal. It is easy to see that F is completely invariant under f, i. e. if a point belongs to F, then all its successors and predecessors are also in F.

THEOREM. *In the case* III *let* a *and* b *be the points of* $S - D_\omega$, *let* $t \in F$, *and let* φ *be a function meromorphic at* t *but not one of the two exceptional functions* $\varphi(x) \equiv a$ *and* $\varphi(x) \equiv b$. *Then* t *is a point of accumulation of roots of the equations* $\varphi(x) = f^n(x)$.

Remark: We say that a function φ is exceptional in a region if the equations $f^n(x) = \varphi(x)$ have no roots in this region.

PROOF. 1. We assume that t is not a point of accumulation, and carry this assumption to a contradiction.

Since $\{f^n\}$ is non-normal at t and there exist two exceptional functions, there exists no third such function. Therefore φ is non-exceptional. It follows that to any neighborhood of t there exist infinitely many equations $f^n = \varphi$ which have roots in this neighbourhood. Choosing a fundamental system U_ν of neighbourhoods of t, we can therefore construct an increasing sequence n_ν of integers and a sequence t_ν of points so that $f^{n_\nu}(t_\nu) = \varphi(t_\nu)$ and $\lim t_\nu = t$. (In fact, let n_ν be choosen $> n_{\nu-1}$ and such that the equation $f^{n_\nu}(x) = \varphi(x)$ has a root t_ν in U_ν.) Thus t would certainly be a point of accumulation if infinitely many t_ν were different from t.

From the assumption made it follows therefore that $t_\nu = t$ for all $\nu \geqq N$. Put $n_N = k$ and $n_{N+1} = l$. Then $l > k$ and we have $f^k(t) = \varphi(t)$ and $f^l(t) = \varphi(t)$. It follows that

$$f^{l-k}(\varphi(t)) = f^{l-k}(f^k(t)) = f^l(t) = \varphi(t) ,$$

i.e. that $\varphi(t)$ is a fixed point. *Hence the theorem is proved under the added assumption that* $\varphi(t)$ *not is a fixed point.*

2. Our next step will be to use this preliminary result to prove the

theorem under another and weaker additional assumption, namely that F contains a point u which is not a fixed point.

From the result just proved it follows that the equations $f^n = u$ have roots which accumulate at t. (In fact let $\varphi(x) = \text{const.} = u$ in the result above. Then $\varphi(t) = u$ is not a fixed point.) Let $u_\nu \to t$ be a sequence of such roots with $u_\nu \neq t$. Since the points u_ν are predecessors of $u \in F$, they are also elements of F.

Returning now to the given function φ of the theorem we observe that φ is meromorphic in some neighbourhood of t, whence it follows that it is meromorphic at u_ν for all large ν. Consider now such a point u_ν instead of t and retrace the beginning of the argument in part 1 of this proof. It follows that $u_\nu = \lim_{\mu\to\infty} u_{\nu,\mu}$ where the numbers $u_{\nu,\mu}$ are roots of equations $f^n = \varphi$. It is immediate that t is then a point of accumulation of the set $\{u_{\nu,\mu}\}$.

Thus the theorem is proved under the assumption that F contains a point which is not a fixed point.

3. The assumption that the theorem is false thus leads us to conclude that F consists of fixed points only. Now let $x_1 \in F$. Then x_1 is a fixed point of a certain order n. Let x_2 be a predecessor of x_1. Then $x_2 \in F$ and therefore x_2 is also a fixed point. Now x_1 is a successor of x_2 and therefore contained in the cycle of x_2. It follows that this cycle coincides with that of x_1.

This shows that any predecessor of order one of x_1 is equal to $f^{n-1}(x_1)$, i.e. that the equation $f(x) = x_1$ has only one root, a fact which contradicts Picard's theorem since we already know that the two equations $f(x) = a$ and $f(x) = b$ have a finite number of roots.

Thus Theorem 4.1 is proved.

4.2. In this section we establish some results which follow easily from Theorem 4.1 and which are counterparts in the case III of fundamental results by Fatou and Julia in the cases I and II. We give them in the form of corollaries to Theorem 4.1.

Corollaries. *In the case* III *let the two points of $S - D_\omega$ be a and b. Then*

1° *If c is not equal to a or b, then every point of F is a point of accumulation of the predecessors of c.*

2° *Every point F is a point of accumulation of fixed points.*

3° *The set $F \cup \{a, b\}$ is perfect.*

Proofs. 1° follows directly from 4.1 by specializing φ to $\varphi(x) \equiv c$. Similarly 2° follows by choosing $\varphi(x) \equiv x$.

In order to prove 3° we observe that both a and b are in the closure of F. For the function f^2 has both these points as singularities. Thus if $c \in F$, the roots of $f^2 = c$ accumulate both at a and at b by Picard's theorem. Furthermore, by 1° every point of F is a point of accumulation of predecessors of c. But every such predecessor is in F. This shows that every point of $F \cup \{a, b\}$ is a point of accumulation of F. Since $F \cup \{a, b\}$ is obviously closed, 3° follows.

REFERENCES

1. P. Fatou, *Sur les équations fonctionelles (Premier Mémoire)*, Bull. Soc. Math. France 47 (1919), 161–271.
2. P. Fatou, *Sur les équations fonctionnelles (Deuxième Mémoire)*, Bull. Soc. Math. France 48 (1920), 33–94.
3. P. Fatou, *Sur les équations fonctionnelles (Troisième Mémoire)*, Bull. Soc. Math. France 48 (1920), 208–314.
4. P. Fatou, *Sur l'itération des fonctions transcendantes entières*, Acta Math. 47 (1926), 337–370.
5. G. Julia, *Mémoire sur l'itération des fonctions rationnelles*, J. Math. Pures Appl. (8) 1 (1918), 47–245.

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