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A THEOREM ON FUNCTIONS DEFINED ON A SEMI-GROUP

ARNE BEURLING

If G is the additive group of real numbers and $f(x)$ a continuous bounded function on G, it is known that the linear manifold spanned, in a suitable topology, by the set $\{T_\xi f = f(x+\xi),\ \xi \in G\}$ will always contain a continuous bounded group character provided f does not vanish identically. It has been shown by Godement that this property remains true for all locally compact abelian groups. There exists now a rather extensive litterature on this and on other aspects of the problem, while, on the other hand, very little is known concerning analogous properties of semi-groups. In an earlier paper [1] the author has considered the following case as being for semi-groups the simplest typical problem of its kind. Let S be the additive semi-group of non-negative integers x and let $f(x)$ belong to the space $L^2(S)$ referred to the measure taking the value 1 at each point $x \in S$. Is it true for an $f \not\equiv 0$ that the L^2-closure of the set $\{f(x+\xi),\ \xi \in S\}$ always contains a character, i.e. in this case a function of the form λ^x where λ is a complex number less than 1 in modulus? The answer is in the negative, and the same holds for L^2 over the non-negative reals as shown by Nyman in his thesis [2]. In the two cited papers a complete characterisation was given of the closed linear subsets $C \subset L^2$ having the property $T_\xi C \subset C$, $\xi \in S$, and it turned out that there exist non-empty sets of this kind that do not contain any character. This settles the stated problem, but leaves open the question whether the required property would hold for the manifold spanned by $\{f(x+\xi),\ \xi \in S\}$ in some *weaker* topology. The purpose of this paper is to show that even elementary function theory yields a positive answer to this question.

In the sequel we will consider the semi-group S formed by the positive reals $\leqq 1$ under multiplication. L^p will denote the space of measurable functions on the unit interval with the norm

$$\|f\|_p = \left\{ \int_0^1 |f(x)|^p dx \right\}^{1/p}.$$

To an $f \in L^p$, $p > 1$, we assign the closed linear manifolds C_f^r, $1 \leqq r \leqq p$,

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spanned by the set $\{T_\xi f = f(x\xi),\ \xi \in S\}$ in the topology of L^r, and our weak closure Γ_f^p will be defined as

$$\Gamma_f^p = \bigcap_{1 \leqq r < p} C_f^r \cap L^p .$$

We recall that each continuous character φ of S belonging to L^p and normalized by the condition $\varphi(1) = 1$ has the form $x^{-\lambda}$ where λ is a complex number in the half-plane $\mathrm{Re}(\lambda) < 1/p$. We also observe that if $f(x)$ vanishes almost everywhere on an interval $0 < x \leqq a$, then the same is true of $T_\xi f$ and the sets C_f^r cannot contain any function which does not vanish on $(0, a)$.

THEOREM I. *Let $f(x)$ belong to a space L^p, $1 < p < \infty$, and let it not vanish almost everywhere on any interval $0 < x \leqq a$. Then Γ_f^p contains at least one function of the form $x^{-\lambda}$, $\mathrm{Re}(\lambda) < 1/p$.*

According to a theorem of F. Riesz and Banach the function $x^{-\lambda}$ will belong to the set C_f^r if and only if for any $k \in L^{r'}$, $r' = r/(r-1)$, the condition

(1) $$\int_0^1 k(x) f(\xi x) dx = 0 , \qquad 0 < \xi \leqq 1 ,$$

implies

(2) $$\int_0^1 k(x) x^{-\lambda} dx = 0 .$$

On defining $f(x)$ and $k(x)$ as 0 for $x > 1$ and on setting

(3) $$g(\xi) = \int_0^\infty k(x) f(\xi x) dx , \qquad \xi > 0 ,$$

we obtain for $\xi > 1$ by an inequality of Jensen

$$|g(\xi)| \leqq \left\{\int_0^{1/\xi} |k(x)|^{r'} dx\right\}^{1/r'} \left\{\int_0^{1/\xi} |f(\xi x)|^p dx\right\}^{1/p} \left\{\int_0^{1/\xi} dx\right\}^{1-1/p-1/r'} \leqq \|k\|_{r'} \|f\|_p \xi^{-1/r} .$$

We next observe that the integrals

$$G(s) = \int_1^\infty g(\xi) \xi^{s-1} d\xi , \qquad \sigma < 1/r ,$$

$$K(s) = \int_0^1 k(x) x^{-s} dx , \qquad \sigma < 1/r ,$$

$$F(s) = \int_0^1 f(x) x^{s-1} dx , \qquad \sigma > 1/p ,$$

converge absolutely for $s = \sigma + it$ lying in the half-planes indicated.

On multiplying (3) by ξ^{s-1} where s is a complex number in the strip $1/p < \sigma < 1/r$ and on integrating with respect to ξ over $(0, \infty)$ we obtain $G(s)$ on the left side. The double integral on the right converges absolutely and a change of the order of integration yields the relation

$$G(s) = K(s)F(s) \, , \qquad 1/p < \sigma < 1/r \, . \tag{4}$$

We may now assume that $k(x)$ can be chosen so that $K(s) \not\equiv 0$, since the opposite assumption would imply $C_f^r = L^r$ for $r < p$, which is a stronger property than the one stated in the theorem. We already know that $F(s)$ is holomorphic for $\sigma > 1/p$ and from (4) we see that $F(s)$ coincides with $G(s)/K(s)$ in the strip $1/p < \sigma < 1/r$. Since the latter function can have no singularities other than isolated poles in $\sigma \leqq 1/p$, it follows that $F(s)$ can be continued analytically across the line $\sigma = 1/p$ and that it is meromorphic for $|s| < \infty$. Due to the inequality

$$|F(\sigma+it)| < \frac{\text{const.}}{(\sigma-1/p)^{1-1/p}}, \qquad \sigma > 1/p,$$

$F(s)$ cannot have any pole on the line $\sigma = 1/p$, and we have thus only two alternatives to consider: Firstly, $F(s)$ has at least one pole λ in the halfplane $\sigma < 1/p$; secondly, $F(s)$ is entire. In the first alternative we will have $K(\lambda) = 0$ not only for the particular $k(x)$ considered, but for any k satisfying (1) and belonging to some space L^r for $1 \leqq r < p$. Thus (1) implies (2) and consequently $x^{-\lambda}$ will belong to each C_f^r for $1 \leqq r < p$, which was our statement.

To finish the proof we have to show that the second alternative, $F(s)$ entire, cannot occur. Let α be a number in the open interval $(1/p, 1/r)$ and let M be a constant such that $F(s)$, $G(s)$, and $K(s)$ are in modulus $\leqq M$ on the line $\sigma = \alpha$. Thus $K(s)$ and $G(s)$ are not only bounded and holomorphic in the half-plane $\sigma < \alpha$ but also regular at each finite boundary point. Under these conditions we may apply the following expansion due to F. and R. Nevanlinna [3],

$$\log |K| = U_1 + U_2 + U_3$$

where $U_1 = -\Sigma G(s, a_n)$, G being the Green's function of the half-plane $H: \sigma < \alpha$, and a_n the zeros of $K(s)$ belonging to H. The second term U_2 stands for the Poisson integral of $\log |K(\alpha+it)|$, while U_3 is a non-positive harmonic function in H vanishing at each finite boundary point. Similarly,

$$\log |G| = V_1 + V_2 + V_3 \, .$$

Since each zero of $K(s)$ in $\sigma < \alpha$ will also be a zero of $G(s)$ we conclude that $V_1 - U_1 \leqq 0$. The difference $V_2 - U_2$ is the Poisson integral of

$\log|G(\alpha+it)|-\log|K(\alpha+it)| \leqq \log M$ and must, therefore, be $\leqq \log M$ throughout $\sigma < \alpha$. Finally, V_3 and U_3 have the form $c_1(\sigma-\alpha)$, $c_2(\sigma-\alpha)$ respectively, where c_1, c_2 are constants $\geqq 0$. Setting $c = c_1-c_2$ we will have

$$\begin{cases} |F(\sigma+it)| \leqq M\,, & \sigma \geqq \alpha\,, \\ |F(\sigma+it)| \leqq Me^{c(\sigma-\alpha)}, & \sigma \leqq \alpha\,. \end{cases} \tag{5}$$

If $c \geqq 0$, $F(s)$ will be bounded. Since $F(\sigma) \to 0$ for $\sigma \to \infty$, it follows by Liouville's theorem that $F(s)$ vanishes identically. This implies that $f(x) = 0$ almost everywhere, which is contradictory to our assumption. If $c < 0$ we set $a = e^c$ and define

$$F_1(s) = \int_0^1 f(ax)x^{s-1}dx\,. \tag{6}$$

On combining (5), (6), and the relation

$$F_1(s) = a^{-s}\Big(F(s)-\int_\alpha^1 f(x)x^{s-1}dx\Big)$$

we find that $F_1(s)$ is bounded in both half-planes $\sigma > \alpha$ and $\sigma < \alpha$. The conclusion $F_1(s) \equiv 0$ follows as in the previous case and leads now to the contradiction that $f(x)$ vanishes almost everywhere on the interval $(0, a)$.

In Theorem I the spaces L^p did not refer to the invariant measure dx/x of S, a fact which made it possible to use the metric of L^r, $r < p$, to define a suitable weak topology. Let now S be the additive semi-group of reals $x \geqq 0$ and let $L^p = L^p(S)$ be defined with respect to the invariant measure dx, while L_ε^p, $\varepsilon > 0$, is referred to the measure $e^{-\varepsilon x}dx$. By minor modifications of the previous proof we find that if $f(x)$ belongs to some L^p, $1 \leqq p < \infty$, and does not vanish almost everywhere on any interval (a, ∞) then there is at least one function of the form $e^{-\lambda x}$, $\mathrm{Re}(\lambda) > 0$, contained in each L_ε^p-closure of the linear combinations of the set $\{f(x+\xi),\ \xi \geqq 0\}$. We finally point out that the corresponding statement will remain true if we take the closure in the L^p-metric referring to the measure $e^{-x^\alpha}dx$ for $\alpha = 1/2$, but will be false if $\alpha < 1/2$.

REFERENCES

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INSTITUTE FOR ADVANCED STUDY, PRINCETON, N.J., U.S.A.