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ON THE REPRESENTATION OF LATTICES[1].

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Introduction. The concept of a modular lattice[2] arose from the study of normal subgroups of a group, and it has been shown that most known theorems on lattices of normal subgroups are actually valid for arbitrary modular lattices. It is therefore natural to ask whether every modular lattice is isomorphic to a lattice of normal subgroups of some group. As will be shown in Section 2 below, the answer to this question is negative. In Section 3 we obtain a different kind of representation applicable to arbitrary modular lattices. Modifying slightly the methods developed there we prove in Section 4 a somewhat stronger form of the fundamental representation theorem for arbitrary lattices.

1. Preliminaries. The symbols

$$\leq, \nleq, +, \cdot$$

will denote inclusion, non-inclusion, sum (least upper bound) and product (greatest lower bound) in an arbitrary lattice, while the symbols

$$\subseteq, \cup, \cap, \in, \notin$$

will refer to set-inclusion, union (set-sum), intersection (set-product), membership and non-membership. The following notations will also be used:

$\bigcup_{\varphi(x)} U_x =$ the union of all sets U_x for which the condition $\varphi(x)$ holds.

$\{x \mid \varphi(x)\} =$ the set of all elements x for which the condition $\varphi(x)$ holds.

$\{x_1, x_2, \ldots, x_n\} =$ the set whose elements are $x_1, x_2, \ldots, x_n$.

$\langle x_1, x_2, \ldots, x_n \rangle =$ the n-termed sequence whose first, second, ..., n-th terms are respectively $x_1, x_2, \ldots, x_n$.

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[2] For general information on lattices and modular lattices, see Birkhoff [1].

$X \times Y =$ the Cartesian product of X and Y, i.e., the set of all two-termed sequences, or ordered pairs, $\langle x, y\rangle$ with $x \in X$ and $y \in Y$.

As is well known, binary relations can be regarded as sets whose elements are ordered pairs. We can therefore apply to them the usual set-theoretic operations. We shall also need two other operations, relative multiplication and conversion. The relative product $R;S$ of two binary relations R and S is the set of all ordered pairs $\langle x, y\rangle$ such that $\langle x, z\rangle \in R$ and $\langle z, y\rangle \in S$ for some element z, while the converse R^{-1} of R is the set of all ordered pairs $\langle x, y\rangle$ such that $\langle y, x\rangle \in R$.

The domain of a binary relation R is defined by the formula

$$\operatorname{dmn} R = \{x \mid \langle x, y\rangle \in R \text{ for some } y\} .$$

We say that R is transitive if $R;R \subseteq R$, symmetric if $R^{-1} = R$. A binary relation that is both transitive and symmetric is called an equivalence relation; an equivalence relation whose domain is U is called an equivalence relation over U.

By a partitioning of a set U is meant a family $\mathcal{F}$ of mutually disjoint non-empty subsets of U such that

$$U = \bigcup_{X \in \mathcal{F}} X .$$

To each equivalence relation R over U there coresponds a unique partitioning $\mathcal{F}$ of U such that $\langle p, q\rangle \in R$ if and only if p and q belong to the same member of $\mathcal{F}$. Conversely, each partitioning of U corresponds in this manner to one and only one equivalence relation over U. The members of $\mathcal{F}$ will be called the equivalence classes of R or, more briefly, the R classes.

It is known that the family of all equivalence relations over a set U is a lattice. Here lattice-inclusion coincides with set-inclusion and lattice-multiplication with settheoretic intersection while the sum $R+S$ of two equivalence relations R and S over U is the smallest equivalence relation over U which contains both R and S. It is easy to see that an ordered pair $\langle x, y\rangle$ belongs to $R+S$ if and only if there exist finitely many elements $z_0, z_1, \ldots, z_n$ such that $z_0 = x, z_n = y$, and $\langle z_j, z_{j+1}\rangle \in R \cup S$ for $j = 0, 1, \ldots, n-1$. Hence $R+S$ is the union of the non-decreasing sequence

$$R;S, \quad R;S;R, \quad R;S;R;S, \quad R;S;R;S;R, \ldots . \tag{1.1}$$

The fundamental representation theorem for lattices (Whitman [5], Theorem 1) states that any lattice A is isomorphic to a sublattice $\mathcal{A}$ of the lattice of all equivalence relations over some set U. We shall be concerned with the possibility of choosing $\mathcal{A}$ in such a way that for $R, S \in \mathcal{A}$ the sequence (1.1) is constant from the first, second, or third term on, so that

$$R+S = R;S, \qquad \text{or} \qquad R+S = R;S;R, \qquad \text{or} \qquad R+S = R;S;R;S\,.$$

The following terminology therefore suggests itself:

Definition 1.1. *By a representation of a lattice A we mean an ordered pair $\langle F, U\rangle$ such that U is a set and F is a function which maps A isomorphically onto a sublattice of the lattice of all equivalence relations over U. We say that $\langle F, U\rangle$ is*

(i) *of type* 1 *if* $F(x)+F(y) = F(x);F(y)$ *for* $x, y \in A$,
(ii) *of type* 2 *if* $F(x)+F(y) = F(x);F(y);F(x)$ *for* $x, y \in A$,
(iii) *of type* 3 *if* $F(x)+F(y) = F(x);F(y);F(x);F(y)$ *for* $x, y \in A$.

The equation in (i) is equivalent to the condition that

$$F(x);F(y) = F(y);F(x)\,.$$

In other words, to say that A has a representation of type 1 means that A is isomorphic to a lattice of commuting equivalence relations. It is well known that the lattice of all normal subgroups of a group G has a representation of type 1; we let $U = G$, and for each normal subgroup H of G let $F(H)$ be the set of all ordered pairs $\langle u, v\rangle$ with $uv^{-1} \in H$. It is not known whether the converse of this result holds.

It is easy to see that a representation of type 1 is also of type 2, and a representation of type 2 is also of type 3. Somewhat less trivial is the following:

Theorem 1.2. *If a lattice A has a representation of type* 2, *then A is modular.*

Proof. Suppose $\langle F, U\rangle$ is a representation of A of type 2. Assuming that $a, b, c \in A$ and $a \leq c$, we shall show that

$$(a+b)\cdot c \leq a+b\cdot c\,.$$

If $p, q \in U$ and

$$\langle p, q\rangle \in F[(a+b)\cdot c]\,,$$

then

$$\langle p, q\rangle \in F(a+b) \qquad \text{and} \qquad \langle p, q\rangle \in F(c)\,.$$

Since

$$F(a+b) = F(a)+F(b) = F(a);F(b);F(a)\,,$$

there exist $r, s \in U$ with

$$\langle p, r\rangle \in F(a), \quad \langle r, s\rangle \in F(b), \quad \langle s, q\rangle \in F(a) .$$

Using the fact that $F(a) \subseteq F(c)$, together with the symmetry and transitivity of $F(c)$, we infer that

$$\langle r, p\rangle, \langle p, q\rangle, \langle q, s\rangle \in F(c), \qquad \langle r, s\rangle \in F(c) ,$$
$$\langle r, s\rangle \in F(b \cdot c), \qquad \langle p, q\rangle \in F(a);F(b \cdot c);F(a) = F(a+b \cdot c) .$$

Thus

$$F[(a+b) \cdot c] \subseteq F(a+b \cdot c), \qquad (a+b) \cdot c \leq a+b \cdot c .$$

2. Representations of type 1. The purpose of this section is to show that there exist modular lattices which do not have a representation of type 1. A preliminary result is needed.

LEMMA 2.1. *Every modular lattice A which has a representation of type* 1 *satisfies the following condition*:

$$(\alpha)\begin{cases} \textit{If } a_0, a_1, a_2, b_0, b_1, b_2 \in A \textit{ and if} \\ \quad x = (a_0+b_0) \cdot (a_1+b_1) \cdot (a_2+b_2) , \\ \quad y = (a_0+a_1) \cdot (b_0+b_1) \cdot [(a_0+a_2) \cdot (b_0+b_2)+(a_1+a_2) \cdot (b_1+b_2)] , \\ \textit{then} \\ \quad x \leq a_0 \cdot (a_1+y)+b_0 \cdot (b_1+y) . \end{cases}$$

PROOF. Suppose $\langle F, U\rangle$ is a representation of A of type 1. If $p, q \in U$ and

$$\langle p, q\rangle \in F(x) ,$$

then

$$\langle p, q\rangle \in F(a_j+b_j) = F(a_j);F(b_j) \qquad \text{for } j = 0, 1, 2 ,$$

and there exist elements $r_0, r_1, r_2 \in U$ such that

$$\langle p, r_j\rangle \in F(a_j), \qquad \langle r_j, q\rangle \in F(b_j) \qquad \text{for } j = 0, 1, 2 .$$

It follows that

$$\langle r_0, r_2\rangle \in [F(a_0);F(a_2)] \cap [F(b_0);F(b_2)] = F[(a_0+a_2) \cdot (b_0+b_2)] ,$$
$$\langle r_2, r_1\rangle \in [F(a_2);F(a_1)] \cap [F(b_2);F(b_1)] = F[(a_1+a_2) \cdot (b_1+b_2)] ,$$

and hence

$$\langle r_0, r_1\rangle \in F[(a_0+a_2) \cdot (b_0+b_2)+(a_1+a_2) \cdot (b_1+b_2)] .$$

Furthermore

$$\langle r_0, r_1\rangle \in F(a_0+a_1), \qquad \langle r_0, r_1\rangle \in F(b_0+b_1) .$$

Consequently

$$\langle r_0, r_1\rangle \in F(y) ,$$
$$\langle p, r_0\rangle \in F[a_0 \cdot (a_1+y)], \qquad \langle r_0, q\rangle \in F[b_0 \cdot (b_1+y)] ,$$
$$\langle p, q\rangle \in F[a_0 \cdot (a_1+y)+b_0 \cdot (b_1+y)] .$$

We conclude that

$$F(x) \subseteq F[a_0 \cdot (a_1+y)+b_0 \cdot (b_1+y)]$$

and hence

$$x \leq a_0 \cdot (a_1+y)+b_0 \cdot (b_1+y) .$$

In order to show that the condition (α) is not satisfied in every modular lattice we consider the lattice A of all subspaces of a projective plane P. The elements of A are the subspaces of P—the points and lines of P as well as the null space and P itself—while the product $x \cdot y$ and the sum $x+y$ of two subspaces x and y of P are respectively the largest subspace contained in both x and y and the smallest subspace containing both x and y. It is well known that this lattice is modular.

THEOREM 2.2. *If A is the lattice of all subspaces of a projective plane P, then the following conditions are equivalent*:

(i) *A has a representation of type* 1.
(ii) *A satisfies the condition* (α).
(iii) *P is Desarguesian.*
(iv) *A is isomorphic to a lattice of subgroups of some Abelian group G.*

PROOF. We know from Lemma 2.1 that (i) implies (ii), and from the remark following Definition 1.1 that (iv) implies (i). It is therefore sufficient to show that (ii) implies (iii) and (iii) implies (iv).

Assume that (ii) holds, and let $a_0, a_1, a_2, b_0, b_1, b_2$ be distinct points of P such that the lines

$$a_0+b_0,\ a_1+b_1,\ a_2+b_2$$

meet in a common point z. Defining x and y as in Lemma 2.1 we have $z \leq x$ (with equality holding unless the three lines coincide) and hence

$$z \leq a_0 \cdot (a_1+y)+b_0 \cdot (b_1+y) .$$

It follows that $y \neq 0$, for otherwise

$$a_0 \cdot (a_1+y)+b_0 \cdot (b_1+y) = a_0 \cdot a_1+b_0 \cdot b_1 = 0 .$$

Choose a point z_2 of P with $z_2 \leq y$. Then the lines a_0+a_1 and b_0+b_1 meet in z_2. Furthermore

$$z_2 \leq (a_0+a_2) \cdot (b_0+b_2)+(a_1+a_2) \cdot (b_1+b_2) ,$$

whence there exist points z_0 and z_1 such that

$$z_1 \leq (a_0+a_2) \cdot (b_0+b_2), \qquad z_0 \leq (a_1+a_2) \cdot (b_1+b_2), \qquad z_2 \leq z_0+z_1 .$$

Thus the three pairs of lines

$$a_1+a_2,\ b_1+b_2; \quad a_2+a_0,\ b_2+b_0; \quad a_0+a_1,\ b_0+b_1$$

meet in the three collinear points z_0, z_1 and z_2. Hence P is Desarguesian.

Finally suppose (iii) holds. It is well known that A is then isomorphic to the lattice of all subspaces of a three-dimensional vector space V over a division ring (skew-field). But V is an Abelian group under vector addition, and the subspaces of V are subgroups of V. Hence (iv) holds.

THEOREM 2.3. *A free modular lattice with four or more generators does not have a representation of type* 1.

PROOF. We shall show that such a lattice does not satisfy the condition (α). Since this condition can be expressed in the form of an equation, it is sufficient to show that there exists a modular lattice with exactly four generators in which (α) fails. For this purpose we make use of the fact that there exists a non-Desarguesian plane P generated by four points (see e.g. Hall [2], Theorem 4.6, p. 239). The lattice A of all subspaces of P is generated by the same four points, and it follows from Theorem 2.2 that the condition (α) fails in A.

THEOREM 2.4. *The lattice of all subspaces of a non-Desarguesian projective plane is not isomorphic to a lattice of normal subgroups of a group, and neither is a free modular lattice with four or more generators.*

PROOF. By Theorems 2.2 and 2.3 and the remark following Definition 1.1.

In the proof of Theorem 2.3 it was shown that a free modular lattice with four generators contains six elements a_0, a_1, a_2, b_0, b_1, b_2 for which the condition (α) fails. Hence there exists an equation in four variables which fails for the generators of this lattice but holds in every lattice of commuting equivalence relations. We can actually write down such an equation explicitly. Suppose z, a_0, a_1, a_2 are elements of a modular lattice A, and let

$$z' = (a_1+z)\cdot(a_2+z)\,,$$

$$b_0 = (a_0+z')\cdot(a_1+a_2),\quad b_1 = (a_1+z')\cdot(a_2+a_0),\quad b_2 = (a_2+z')\cdot(a_0+a_1)\,,$$

$$c_0 = (a_1+a_2)\cdot(b_1+b_2),\quad c_1 = (a_2+a_0)\cdot(b_2+b_0),\quad c_2 = (a_0+a_1)\cdot(b_0+b_1)\,.$$

Using the same ideas as in the proof of Lemma 2.1, we can show that if A has a representation of type 1, then

$$c_0 \leq c_1+c_2\,. \tag{2.1}$$

Applied to the lattice of all subspaces of a projective plane this inequality implies the special case of Desargues' theorem where the sides of one of the two triangles involved pass through the vertices of the other. (We let

a_0, a_1, a_2 and b_0, b_1, b_2 be the vertices of the two triangles and z the center of perspectivity.) It follows that the above inequality is not identically satisfied in every modular lattice, and must therefore fail if z, a_0, a_1, a_2 are distinct generators of a free modular lattice[3].

3. Representations of type 2. We shall now prove that every modular lattice has a representation of type 2. The following terminology will be useful:

DEFINITION 3.1. *By a weak representation of a lattice A we mean an ordered pair $\langle F, U\rangle$ where U is a set and F is a one-to-one function mapping A onto a set of equivalence relations over U in such a way that*

$$F(x \cdot y) = F(x) \cap F(y) \quad \text{whenever } x, y \in A .$$

DEFINITION 3.2. *Given two weak representations $\langle F, U\rangle$ and $\langle G, V\rangle$ of a lattice A, we say that $\langle G, V\rangle$ is an extension of $\langle F, U\rangle$ if $U \subseteq V$ and*

$$G(x) \cap (U \times U) = F(x) \quad \text{for } x \in A .$$

LEMMA 3.3. *Any lattice A has a weak representation $\langle F, U\rangle$ which satisfies the following condition:*

$$(\beta) \quad \begin{cases} \text{For any two distinct elements } u \text{ and } v \text{ of } U, \text{ the set} \\ \qquad \{x \mid x \in A \text{ and } \langle u, v\rangle \in F(x)\} \\ \text{is either empty or has a smallest element.} \end{cases}$$

PROOF. With each element z of A associate two distinct elements $\varphi(z)$ and $\psi(z)$ in such a way that the sets $\{\varphi(z), \psi(z)\}$, $\{\varphi(z'), \psi(z')\}$ are disjoint for $z \neq z'$. Let

$$U = \bigcup_{z \in A} \{\varphi(z), \psi(z)\} ,$$

and for each element x of A let $F(x)$ be the equivalence relation over U whose equivalence classes are the sets

$$\{\varphi(z), \psi(z)\}$$

with $z \leq x$ and the sets

$$\{\varphi(z)\}, \ \{\psi(z)\}$$

with $z \nleq x$.

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[3] This provides an affirmative answer to the question raised in Problem 27 of Birkhoff [1]. The above inequality and the condition (α) are based on similar ideas as the identities $\mathfrak{U}_l$ and $\mathfrak{U}_L$ used in Schützenberger [4] to characterize Desarguesian projective planes. See the footnote at the end of this paper.

It is a simple matter to check that $\langle F, U\rangle$ is a weak representation of A. Furthermore, if u and v are distinct elements of U and if the set

$$B = \{x \mid x \in A \text{ and } \langle u, v\rangle \in F(x)\}$$

is not empty, then

$$\langle u, v\rangle = \langle \varphi(z), \psi(z)\rangle \quad \text{or} \quad \langle u, v\rangle = \langle \psi(z), \varphi(z)\rangle$$

for some $z \in A$. Hence

$$B = \{x \mid x \in A \text{ and } z \leq x\},$$

so that z is the smallest element of B. Thus the condition (β) is satisfied.

LEMMA 3.4. *Suppose A is a modular lattice and $\langle F, U\rangle$ is a weak representation of A satisfying the condition (β). If $a, b \in A$ and*

$$\langle p, q\rangle \in F(a+b),$$

then there exists an extension $\langle G, V\rangle$ of $\langle F, U\rangle$ such that $\langle G, V\rangle$ satisfies the condition (β) and

$$\langle p, q\rangle \in G(a);G(b);G(a).$$

PROOF. Dismissing as trivial the case in which $p = q$, take two distinct elements r and s which do not belong to U, and let

$$V = U \cup \{r, s\}.$$

Let d be the smallest element of the set

$$\{x \mid x \in A \text{ and } \langle p, q\rangle \in F(x)\},$$

and let

$$a' = a+d, \quad b' = a' \cdot b.$$

Note that $d \leq a+b$ and hence $a \leq a' \leq a+b$. It follows by the modular law that

$$a' = a+b'.$$

In defining $G(x)$ we consider four cases:

If $a' \leq x$, then $\langle p, q\rangle \in F(x)$, so that p and q belong to the same $F(x)$ class. In this case the $G(x)$ classes shall be the $F(x)$ classes which do not contain p and q, and the set obtained by adding r and s to the $F(x)$ class containing p and q.

If $a \leq x$ and $b' \not\leq x$, then $d \not\leq x$ and hence p and q belong to different $F(x)$ classes. In this case the $G(x)$ classes shall be the $F(x)$ classes not containing p or q, the set obtained by adding r to the $F(x)$ class containing p, and the set obtained by adding s to the $F(x)$ class containing q.

If $a \not\leq x$ and $b' \leq x$, then the $G(x)$ classes shall be the $F(x)$ classes and the set $\{r, s\}$.

If $a \not\leq x$ and $b' \not\leq x$, then the $G(x)$ classes shall be the $F(x)$ classes and the sets $\{r\}$, $\{s\}$.

Clearly these conditions define a function G which maps A onto a set of equivalence relations over V. Furthermore, if $u, v \in U$ and $x \in A$ then the following four statements hold:

(1) $\langle u, v\rangle \in G(x)$ if and only if $\langle u, v\rangle \in F(x)$.

(2) $\langle u, r\rangle \in G(x)$ if and only if $\langle u, p\rangle \in F(x)$ and $a \leq x$.

(3) $\langle u, s\rangle \in G(x)$ if and only if $\langle u, q\rangle \in F(x)$ and $a \leq x$.

(4) $\langle r, s\rangle \in G(x)$ if and only if $b' \leq x$.

From (1) we infer that

$$G(x) \cap (U \times U) = F(x) \qquad \text{for } x \in A ,$$

and the fact that F is one-to-one implies that G is also one-to-one. Using (1) - (4) and the symmetry of the relations $G(x)$ we easily see that

$$G(x \cdot y) = G(x) \cap G(y) \qquad \text{for } x, y \in A .$$

Hence $\langle G, V\rangle$ is a weak representation of A and an extension of $\langle F, U\rangle$.

We next show that $\langle G, V\rangle$ satisfies the condition (β). If u and v are distinct members of U, then the set

$$\{x \mid x \in A \text{ and } \langle u, v\rangle \in G(x)\} = \{x \mid x \in A \text{ and } \langle u, v\rangle \in F(x)\}$$

is either empty or else has a smallest element. If $u \in U$, and if the set

$$B = \{x \mid x \in A \text{ and } \langle u, r\rangle \in F(x)\}$$

is non-empty, then it follows from (2) that the set

$$C = \{x \mid x \in A \text{ and } \langle u, p\rangle \in F(x)\}$$

is also non-empty. Hence C has a smallest element c, and we use (2) again to infer that $a+c$ is the smallest element of B. Similarly, if $u \in U$, then the set

$$\{x \mid x \in A \text{ and } \langle u, s\rangle \in G(x)\}$$

is either empty or else has a smallest element. The set

$$\{x \mid x \in A \text{ and } \langle r, s\rangle \in G(x)\} = \{x \mid x \in A \text{ and } b' \leq x\}$$

has the smallest element b'. Using the symmetry of the relations $G(x)$ we conclude that $\langle G, V\rangle$ satisfies the condition (β).

Finally we have

$$\langle p, r\rangle \in G(a), \qquad \langle r, s\rangle \in G(b), \qquad \langle s, q\rangle \in G(a) ,$$

so that

$$\langle p, q\rangle \in G(a);G(b);G(a) .$$

LEMMA 3.5. *Suppose A is a lattice and λ is a limiting ordinal, and suppose with each ordinal $\xi < \lambda$ there is associated a weak representation $\langle F_\xi, U_\xi\rangle$ of A such that if $\xi < \eta < \lambda$, then $\langle F_\eta, U_\eta\rangle$ is an extension of $\langle F_\xi, U_\xi\rangle$. If*

$$U_\lambda = \bigcup_{\xi<\lambda} U_\xi \quad \textit{and} \quad F_\lambda(x) = \bigcup_{\xi<\lambda} F_\xi(x) \quad \textit{for } x \in A ,$$

then $\langle F_\lambda, U_\lambda\rangle$ is a weak representation of A and an extension of $\langle F_\xi, U_\xi\rangle$ for $\xi < \lambda$. Furthermore, if the condition (β) is satisfied by each $\langle F_\xi, U_\xi\rangle$ with $\xi < \lambda$, then this condition is also satisfied by $\langle F_\lambda, U_\lambda\rangle$.

PROOF. The fact that

$$(3.1) \qquad F_0(x), F_1(x), \ldots, F_\xi(x), \ldots \qquad (\xi < \lambda)$$

is a (possibly transfinite) non-decreasing sequence of equivalence relations implies that $F_\lambda(x)$ is also an equivalence relation and that

$$\text{dmn}\, F_\lambda(x) = \bigcup_{\xi<\lambda} \text{dmn}\, F_\xi(x) = \bigcup_{\xi<\lambda} U_\xi = U_\lambda .$$

Furthermore, if $u, v \in U_\xi$ and $x \in A$, then

$$\langle u, v\rangle \in F_\lambda(x) \quad \text{if and only if} \quad \langle u, v\rangle \in F_\eta(x) \quad \text{for some } \eta < \lambda .$$

In view of the monotonic character of the sequence (3.1) we may assume that $\xi \leq \eta$. Then $\langle U_\eta, F_\eta\rangle$ is an extension of $\langle U_\xi, F_\xi\rangle$, so that

$$\langle u, v\rangle \in F_\eta(x) \quad \text{if and only if} \quad \langle u, v\rangle \in F_\xi(x) .$$

Consequently

$$(3.2) \qquad F_\lambda(x) \cap (U_\xi \times U_\xi) = F_\xi(x) \qquad \text{for } \xi < \lambda, x \in A .$$

This implies that the function F_λ is one-to-one.

If $x, y \in A$, then

$$F_\lambda(x \cdot y) = \bigcup_{\xi<\lambda} F_\xi(x \cdot y) = \bigcup_{\xi<\lambda} [F_\xi(x) \cap F_\xi(y)] .$$

Again using the fact that the sequence (3.1) is non-decreasing, we infer that

$$F_\lambda(x \cdot y) = \Big[\bigcup_{\xi<\lambda} F_\xi(x)\Big] \cap \Big[\bigcup_{\xi<\lambda} F_\xi(y)\Big] = F_\lambda(x) \cap F_\lambda(y) .$$

Thus $\langle F_\lambda, U_\lambda\rangle$ is a weak representation of A and, by (3.2), an extension of $\langle F_\xi, U_\xi\rangle$ for $\xi < \lambda$.

Now suppose $\langle F_\xi, U_\xi\rangle$ satisfies the condition (β) for $\xi < \lambda$. If u and v are distinct members of U_λ, then there exists an ordinal $\xi < \lambda$ such that $u, v \in U_\xi$. Hence, by (3.2),

$$\{x | x \in A \text{ and } \langle u, v\rangle \in F_\lambda(x)\} = \{x | x \in A \text{ and } \langle u, v\rangle \in F_\xi(x)\} ,$$

and we infer that this set is either empty or else has a smallest element. Thus $\langle F_\lambda, U_\lambda\rangle$ also satisfies the condition (β).

LEMMA 3.6. *Suppose A is a modular lattice and $\langle F, U\rangle$ is a weak representation of A satisfying the condition (β). Then there exists an extension $\langle G, V\rangle$ of $\langle F, U\rangle$ such that $\langle G, V\rangle$ satisfies the condition (β) and*

$$F(a+b) \subseteq G(a);G(b);G(a) \quad \textit{for } a, b \in A .$$

PROOF. Letting K be the set of all ordered quadruples $\langle p, q, a, b\rangle$ such that $a, b \in A$ and $\langle p, q\rangle \in F(a+b)$, we arrange the members of K into a (possibly transfinite) sequence

$$\langle p_0, q_0, a_0, b_0\rangle, \langle p_1, q_1, a_1, b_1\rangle, \ldots, \langle p_\xi, q_\xi, a_\xi, b_\xi\rangle, \ldots \qquad (\xi < \lambda) ,$$

and use Lemmas 3.4 and 3.5 to obtain a sequence

$$\langle F_0, U_0\rangle, \langle F_1, U_1\rangle, \ldots, \langle F_\xi, U_\xi\rangle, \ldots \qquad (\xi \leq \lambda)$$

of weak representations of A with the following properties:

(1) $F_0 = F$ and $U_0 = U$.

(2) If $\xi < \lambda$, then $\langle F_{\xi+1}, U_{\xi+1}\rangle$ is an extension of $\langle F_\xi, U_\xi\rangle$ and

$$\langle p_\xi, q_\xi\rangle \in F_{\xi+1}(a_\xi);F_{\xi+1}(b_\xi);F_{\xi+1}(a_\xi) .$$

(3) If $\xi \leq \lambda$ and ξ is a limiting ordinal, then

$$U_\xi = \bigcup_{\eta<\xi} U_\eta, \quad F_\xi(x) = \bigcup_{\eta<\xi} F_\eta(x) \quad \text{for } x \in A .$$

(4) $\langle F_\xi, U_\xi\rangle$ satisfies the condition (β) for $\xi \leq \lambda$.

We let $V = U_\lambda$ and $G = F_\lambda$, and note that if $a, b \in A$ and

$$\langle p, q\rangle \in F(a+b) ,$$

then $\langle p, q, a, b\rangle$ is one of the quadruples $\langle p_\xi, q_\xi, a_\xi, b_\xi\rangle$ and hence

$$\langle p, q\rangle \in F_{\xi+1}(a);F_{\xi+1}(b);F_{\xi+1}(a) \subseteq G(a);G(b);G(a) .$$

Thus

$$F(a+b) \subseteq G(a);G(b);G(a) \quad \text{for } a, b \in A .$$

THEOREM 3.7. *Every modular lattice has a representation of type* 2.

PROOF. It follows from Lemmas 3.3 and 3.6 that if A is any modular lattice, then there exists an infinite sequence

$$\langle F_0, U_0\rangle, \langle F_1, U_1\rangle, \ldots, \langle F_n, U_n\rangle, \ldots$$

of weak representations of A satisfying the condition (β) and such that $\langle F_{n+1}, U_{n+1}\rangle$ is an extension of $\langle F_n, U_n\rangle$ with

$$F_n(a+b) \subseteq F_{n+1}(a);F_{n+1}(b);F_{n+1}(a) \quad \text{for } a, b \in A .$$

Letting

$$V = \bigcup_{n<\infty} U_n, \quad G(x) = \bigcup_{n<\infty} F_n(x) \quad \text{for } x \in A ,$$

we infer from Lemma 3.5 that $\langle G, V\rangle$ is a weak representation of A. Furthermore, if $a, b \in A$ then

$$G(a+b) = \bigcup_{n<\infty} F_n(a+b) \subseteq \bigcup_{n<\infty} F_{n+1}(a);F_{n+1}(b);F_{n+1}(a)$$

$$\subseteq G(a);G(b);G(a) \subseteq G(a)+G(b) \subseteq G(a+b) .$$

Thus

$$G(a)+G(b) = G(a);G(b);G(a) \quad \text{for } a, b \in A ,$$

and $\langle G, V\rangle$ is a representation of A of type 2.

4. Representations of type 3. In the preceding section the modular law was used only once, in the proof of Lemma 3.4. That the modularity of the lattice is essential for the construction given there is clear from Theorem 1.2. However, replacing this lemma by a slightly different one but using otherwise the same reasoning as in Section 3, we shall be able to show that every lattice has a representation of type 3.

LEMMA 4.1. *Suppose A is a lattice and $\langle F, U\rangle$ is a weak representation of A. If $a, b \in A$ and*

$$\langle p, q\rangle \in F(a+b) ,$$

then there exists an extension $\langle G, V\rangle$ of $\langle F, U\rangle$ such that

$$\langle p, q\rangle \in F(a);F(b);F(a);F(b) .$$

PROOF. Take three distinct elements r, s, t which do not belong to U, and let

$$V = U \cup \{r, s, t\} .$$

In defining $G(x)$ we consider four cases.

If $a \leq x$ and $b \leq x$, then $\langle p, q\rangle \in F(x)$, so that p and q belong to the same $F(x)$ class. In this case the $G(x)$ classes shall be the $F(x)$ classes which do not contain p and q, and the set obtained by adding r, s and t to the $F(x)$ class containing p and q.

If $a \leq x$ and $b \nleq x$, then the $G(x)$ classes shall be the $F(x)$ classes which do not contain p, the set obtained by adding r to the $F(x)$ class containing p, and the set $\{s, t\}$.

If $a \not\leq x$ and $b \leq x$, then the $G(x)$ classes shall be the $F(x)$ classes which do not contain q, the set obtained by adding t to the $F(x)$ class containing q, and the set $\{r, s\}$.

If $a \not\leq x$ and $b \not\leq x$, then the $G(x)$ classes shall be the $F(x)$ classes and the sets $\{r\}$, $\{s\}$, $\{t\}$.

These conditions define a function G which maps A onto a set of equivalence relations over V. Furthermore, if $u, v \in U$ then the following seven statements hold:

(1) $\langle u, v\rangle \in G(x)$ if and only if $\langle u, v\rangle \in F(x)$.
(2) $\langle u, r\rangle \in G(x)$ if and only if $\langle u, p\rangle \in F(x)$ and $a \leq x$.
(3) $\langle u, s\rangle \in G(x)$ if and only if $\langle u, p\rangle \in F(x)$, $a \leq x$ and $b \leq x$.
(4) $\langle u, t\rangle \in G(x)$ if and only if $\langle u, q\rangle \in F(x)$ and $b \leq x$.
(5) $\langle r, s\rangle \in G(x)$ if and only if $b \leq x$.
(6) $\langle r, t\rangle \in G(x)$ if and only if $a \leq x$ and $b \leq x$.
(7) $\langle s, t\rangle \in G(x)$ if and only if $a \leq x$.

As in the proof of Lemma 2.4, we infer that $\langle G, V\rangle$ is a weak representation of A and an extension of $\langle F, U\rangle$. Finally

$$\langle p, r\rangle \in G(a), \quad \langle r, s\rangle \in G(b), \quad \langle s, t\rangle \in G(a), \quad \langle t, q\rangle \in G(b) ,$$

so that

$$\langle p, q\rangle \in G(a);G(b);G(a);G(b) .$$

LEMMA 4.2. *Suppose A is a lattice and $\langle F, U\rangle$ is a weak representation of A. Then there exists an extension $\langle G, V\rangle$ of $\langle F, U\rangle$ such that*

$$F(a+b) \subseteq G(a);G(b);G(a);G(b) \quad \text{for } a, b \in A .$$

PROOF. We proceed as in the proof of Lemma 3.6, using Lemma 4.1 in place of Lemma 3.4.

THEOREM 4.3. *Every lattice has a representation of type* 3.

PROOF. We proceed as in the proof of Theorem 3.7, using Lemma 4.2 in place of Lemma 3.6.

5. Unsolved problems. In connection with the above results it is natural to ask whether the class of all lattices which are isomorphic to lattices of commuting equivalence relations can be characterized by means of identities. It can be shown that a lattice of dimension 4 or less has a representation of type 1 if and only if it satisfies the modular law and the condition (α) of Lemma 2.1, but is is not known whether these conditions are sufficient in the case of a lattice of higher dimension. Similar questions can be raised concerning lattices which are isomorphic to lattices of normal subgroups of arbitrary groups or to lattices of subgroups of Abelian groups. In particular, it would be interesting to know whether these three classes of lattices are actually distinct.

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[4] *Added in proofs:* Dr. Schützenberger has pointed out to me that the inequality (2.1) is actually equivalent to $\mathfrak{U}_l$. He has also shown that every lattice of normal subgroups of a group satisfies $\mathfrak{U}_L$, thus obtaining an independent solution of Problem 27 of Birkhoff [1]. By a slight modification of his reasoning it can be shown that $\mathfrak{U}_L$ holds in every lattice of commuting equivalence relations.