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ON AN EXTENSION OF THE CONCEPT OF DEFICIENCY IN THE THEORY OF MEROMORPHIC FUNCTIONS

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1. Let $f(z)$ be a non-constant meromorphic function in the domain $|z| < R \leqq \infty$. We use the standard notations (see [4]):

$$m(r, a) = (2\pi)^{-1} \int_0^{2\pi} \overset{+}{\log} |f(re^{i\varphi}) - a|^{-1} d\varphi \qquad (a \neq \infty),$$

$$m(r, \infty) = (2\pi)^{-1} \int_0^{2\pi} \overset{+}{\log} |f(re^{i\varphi})| \, d\varphi,$$

$$N(r, a) = \int_0^r n(r, a) \, d\log r,$$

$n(r, a)$ being the number of the roots of the equation $f(z) = a$ in the disk $|z| \leqq r$, each root being counted according to multiplicity. Then

$$T(r) = m(r, \infty) + N(r, \infty)$$

is Nevanlinna's characteristic function for the function $f(z)$.

The foundation of the modern value distribution theory for meromorphic functions is provided by Nevanlinna's first fundamental theorem:

$$m(r, a) + N(r, a) = T(r) + O(1) \,. \tag{1}$$

In addition to its intrinsic interest, this relation gives rise to the considerations which lead to Nevanlinna's second fundamental theorem and in particular to the definitions of deficiency and of normal and exceptional values.

However, the relation (1) is interesting only in case $T(r)$ is unbounded, and the above-mentioned consequences lose all meaning if $T(r)$ is bounded. In order to avoid this disadvantage we shall derive in this paper a relation which is closely related to (1) but is so formulated as to be of interest also for functions of bounded characteristic. Thus we are able to define the deficiency, and hence the normality or anomaly of a value, in a uni-

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fied manner for functions of both bounded and unbounded characteristic. Since we shall deal with the theory of functions of bounded characteristic in another paper, we here confine ourselves to some short remarks concerning the consequences of the relation obtained.

2. In order to establish the desired relation we start from the argument principle. By this principle,

$$\int_0^{2\pi} d \arg \left(\frac{f(re^{i\varphi})-a}{f(re^{i\varphi})-\zeta}\right) = 2\pi\big(n(r, a)-n(r, \zeta)\big),$$

where a and ζ are arbitrary complex numbers. By the Cauchy-Riemann equations, this relation can also be written:

$$\text{(2)}\quad r\frac{d}{dr}\left\{\int_0^{2\pi}\big(\log|f(re^{i\varphi})-a| - \log|f(re^{i\varphi})-\zeta|\big)\, d\varphi\right\} = 2\pi\big(n(r, a)-n(r, \zeta)\big).$$

Let μ be a completely additive set function defined for all Borel subsets of a closed set S. Multiplying (2) by $d\mu$ and integrating over S with respect to ζ, we conclude that

$$\text{(3)}\qquad \frac{r}{2\pi}\frac{d}{dr}\left\{\int_0^{2\pi} u(f(re^{i\varphi}))\, d\varphi - \mu(S)\int_0^{2\pi} \log|f(re^{i\varphi})-a|\, d\varphi\right\} = n(r, a)\mu(S) - \int_S n(r, \zeta)\, d\mu(\zeta),$$

where

$$u(w) = -\int_S \log|w-\zeta|\, d\mu(\zeta)$$

denotes the logarithmic potential corresponding to the set function μ. Let us suppose $u(w)$ is continuous at $w = f(0)$. Dividing (3) by r and integrating with respect to r we then finally get the relation (cf. Frostman [1])

$$u(f(0)) - (2\pi)^{-1}\int_0^{2\pi} u\big(f(re^{i\varphi})\big)\, d\varphi + \left\{\log|f(0)-a| - (2\pi)^{-1}\int_0^{2\pi}\log|f(re^{i\varphi})-a|\, d\varphi\right\}\mu(S)$$

$$\text{(4)}\qquad = \int_S N(r, \zeta)\, d\mu(\zeta) - N(r, a)\mu(S) \qquad \big(a \neq f(0)\big),$$

which offers a convenient starting point for several studies in value distribution theory, in addition to the questions dealt with in this paper.

3. Let G be an arbitrary domain in the w-plane whose boundary C is of positive capacity. We apply the relation (4) to $f(z)$, choosing the point a

in the region G and taking as set S the boundary of G. For the set function μ we put

$$\mu(e) = \omega(a, e, G) ,$$

where $\omega(a, e, G)$ is the value of the harmonic measure of the set e at the point a, with respect to the domain G. For this choice of μ,

$$\mu(S) = \omega(a, C, G) = 1 \tag{5}$$

and

$$u(w) = \begin{cases} - \log |w-a| - g(w, a, G) \\ - \log |w-a| , \end{cases}$$

where $g(w, a, G)$ is the Green's function of G with pole at $w = a$, and where the upper equation holds if w lies in G and the lower holds if w is a regular point of C (with respect to Green's function) or an inner point of the complement of G (if such points exist).

For simplicity, we first suppose that the boundary of G consists of regular points only. Letting $g^+(w, a, G)$ equal $g(w, a, G)$ or zero, according as w belongs to G or to the complement of G, we may write, for every w,

$$u(w) = - \log |w-a| - g^+(w, a, G) . \tag{6}$$

By (5) and (6), it now follows from (4) that

$$\Phi(r, a)+N(r, a) = \int_C N(r, \zeta)\, d\omega(a, \zeta, G)+g^+\big(a, f(0), G\big) , \tag{7}$$

where

$$\Phi(r, a) = (2\pi)^{-1} \int_0^{2\pi} g^+\big(f(re^{i\varphi}), a, G\big)\, d\varphi .$$

The integral

$$p(r, a) = \int_C N(r, \zeta)\, d\omega(a, \zeta, G)$$

represents a non-negative function which is harmonic in G and possesses the boundary values $N(r, a)$ on C. Since $\Phi \geqq 0$, we conclude from (7) that the function

$$P(r, a) = p(r, a) + g^+(a, f(0), G)$$

is a harmonic majorant for the subharmonic function $N(r, a)$. It follows at once from the boundary behaviour of $P(r, a)$ that $P(r, a)$ is the *least* harmonic majorant of $N(r, a)$ in G.

4. In the preceding section we have assumed that all boundary points of G are regular with respect to the Green's function. If G possesses irregular boundary points, no essential difficulties are encountered. This is due to the fact that by a theorem of Frostman [1], the set of irregular boundary points always is of capacity zero. Provided that $w = f(0)$ is not an irregular boundary point of G, it is therefore easily seen that all the above conclusions remain valid.

The case in which $w = f(0)$ is an irregular boundary point of G requires a little more consideration. We introduce a sequence of complex numbers $t_\nu, \nu = 1, 2, \ldots$, where each t_ν is of modulus less than one, $t_\nu \to 0$ for $\nu \to \infty$, and every point $w = f(t_\nu)$ lies in G. Approximating $f(z)$ by the functions

$$f_\nu(z) = f\left(\frac{z+t_\nu}{1+\bar{t}_\nu z}\right)$$

we again find that $\Phi+N$ equals the least harmonic majorant of N.

Summarizing the above results, we obtain the following

THEOREM. *Let $f(z)$ be a non-constant meromorphic function in $|z| < R \leqq \infty$, and let G be an arbitrary domain whose boundary is of positive capacity. Let $g^+(w, a, G)$ equal $g(w, a, G)$ or zero, according as w belongs to G or to the complement of G, and form the function*

$$\Phi(r, a) = (2\pi)^{-1} \int_0^{2\pi} g^+\big(f(re^{i\varphi}), a, G\big)\, d\varphi ,$$

which measures the convergence in the mean of $f(z)$ towards the value a on $|z| = r$. Then, for every a in G,

(8) $$\Phi(r, a)+N(r, a) = P(r, a) ,$$

where $P(r, a)$ is the least harmonic majorant of $N(r, a)$ in G.

5. Let us consider this theorem first for the case in which $f(z)$ is of unbounded characteristic. It follows immediately that, irrespective of the choice of the domain G,

$$\Phi(r, a) = m(r, a)+O(1) .$$

Hence, by (1) and (8),

(9) $$T(r) = P(r, a)+O(1) ,$$

no matter how the domain G and the point $a\,(\neq f(0))$ are chosen.

The equation (9) is of double interest. On the one hand, it provides a new characterization of the function $T(r)$, and on the other hand, it yields information about the asymptotic behaviour of $P(r, a)$ for $r \to R$.

6. Suppose now that $w = f(z)$ is of bounded characteristic. Since $f(z)$

is non-constant, we must assume $R < \infty$, and it is well known that $f(z)$ possesses boundary values almost everywhere on $|z| = R$.

Let Γ be a closed point set in the w-plane containing the boundary values $w = f(Re^{i\varphi})$ for almost all φ. For instance, Γ may be the closure of all boundary values of $f(z)$. By a well-known theorem of Nevanlinna and Frostman, Γ always is of positive capacity. We assume that the complement of Γ is not void; it then consists of open domains.

We have previously proved [3] that, regarding the distribution of values of $f(z)$, only the following two alternatives can occur: If D is any one of the above-mentioned domains outside Γ, then $f(z)$ either takes no value which belongs to D or it takes every value in D, except perhaps a set of capacity zero. Supposing that the latter alternative is true, we choose, in the above theorem, $G = D$. Because $T(r)$ is bounded, the passage to the limit, $r \to R$, can be performed in (8), and we obtain

$$\Phi(R, a)+N(R, a) = P(R, a)\,. \tag{10}$$

As above, $P(R, a)$ is the least harmonic majorant of $N(R, a)$ in D.

Now we can prove (this will be done in a forthcoming paper) that $\Phi(R, a) = 0$, except perhaps for a set of values a of capacity zero. Hence, up to such an exceptional set, $N(R, a)$ is harmonic in D. This implies that if $D' \subset D$ and if we construct the least harmonic majorant for $N(R, a)$ in D', we have for every a in D',

$$P(R, a, D) = P(R, a, D')\,. \tag{11}$$

In view of the result that $\Phi = 0$ up to a set of capacity zero and considering (11), it is natural to define the deficiency $\delta(a)$ of the value a for a function of bounded characteristic in the following manner:

$$\delta(a) = 1-\frac{N(R, a)}{P(R, a)}.$$

By (9), this definition corresponds to the classical definition

$$\delta(a) = 1-\limsup_{r\to R}\frac{N(r, a)}{T(r)}$$

for functions of unbounded characteristic.

In analogy with the case that $f(z)$ is of unbounded characteristic we call a value a normal if $\delta(a) = 0$, exceptional if $\delta(a) > 0$ or if a belongs to a domain outside Γ which does not contain any value of $f(z)$. This division of the values of functions of bounded characteristic into two categories is further justified by the fact that the normality of a value does not depend on the conformal mappings of $|z| < R$ onto itself.

It appears from (10) that if for a value a the deficiency $\delta(a) > 0$, then $f(z)$ strongly approximates this value in the neighbourhood of $|z| = R$. In fact, we can prove that for a function of bounded characteristic such an exceptional value is an asymptotic value.

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