MATH. SCAND. 1 (1953)

SOME CONSIDERATIONS CONCERNING RECURSIVE FUNCTIONS

TH. SKOLEM

This paper is concerned with some scattered remarks regarding different facts in recursive arithmetic. The contents of the six sections are as follows: In § 1 a very simple way is shown how to set up a class of numbers for which no general decision procedure is possible. In § 2 I give a parametric representation of variables connected by a recursive relation. In § 3 an elimination of general recursive functions is explained. Further research will be desirable here. § 4 contains a scruple one can nourish with regard to the first incompleteness theorem of Gödel. In § 5 it is shown that a sketch of a proof set forth in an earlier paper by the author can be given a more general form. Finally, in § 6 a possibly new normal form is given for general recursive functions of one variable. However, the chief purpose of the whole paper is to provide very simple proofs.

1. In a recent article J. Myhill (see [4], pp. 50–53) uses E. L. Post's theory of formal systems to prove an undecidability theorem. We suppose known the concepts primitive recursive, recursively enumerable, and general recursive both with regard to functions and relations. Further we suppose known that a relation $R(x_1, \ldots, x_n)$ is general recursive, if and only if both $R(x_1, \ldots, x_n)$ and its negation $\bar{R}(x_1, \ldots, x_n)$ are recursively enumerable. Then it appears to me that the simplest way to prove such a theorem is the following.

Let $g(x, y, z)$ be a general recursive function which enumerates the primitive recursive functions of x and y, i.e., putting $z = 0, 1, 2, \ldots$ in $g(x, y, z)$ we get all primitive recursive functions of x and y. Such a function $g(x, y, z)$ can easily be defined. Using a theorem of R. Péter ([5], p. 48, see also the similar result in [6], p. 59) on primitive recursive functions I have given a very simple definition in an earlier paper (see Skolem [9], p. 6). However, it is not necessary to use Péter's theorem. A similar, but somewhat more complicated, recursive definition is possible which is adapted to the schemes of substitution and primitive recursion in the

Received October 2, 1953.

manner explained in my cited paper. Then the equation $g(x, y, z) = 0$ will yield all primitive recursive relations between x and y when z runs through all non-negative integers, because every primitive recursive relation has the form $f(x, y) = 0$ for some primitive recursive function f. Further, every recursively enumerable class has the form $(Ey)(f(x, y)=0)$, f primitive recursive. Therefore we get all recursively enumerable classes by putting $z = 0, 1, \ldots$ in $H(x, z)$, where $H(x, z)$ stands for the expression

$$(Ey)\big(g(x, y, z) = 0\big) .$$

Now the relation $\overline{H}(x, x)$ cannot be recursively enumerable. Let us assume that it were. Then $\overline{H}(x, x)$ would for some value of z, $z = a$ say, be identical with the class

$$(Ey)\big(g(x, y, a) = 0\big) ,$$

that is, $H(x, a)$. But the equivalence

$$\overline{H}(x, x) \rightleftarrows H(x, a)$$

becomes a contradiction, if we put $x = a$.

Now $H(x, x)$, that is, $(Ey)\big(g(x, y, x) = 0\big)$, is recursively enumerable. In fact, it is a known theorem that $(Ey)R(x, y)$ is recursively enumerable, even if R is not primitive recursive. But $H(x, x)$ is not recursive, because if it were recursive, both it and its negation should be recursively enumerable. This means that $H(x, x)$ is an unsolvable class, that is, a uniform procedure to decide whether a number belongs to it or not is impossible. There is also a number b such that $H(x, x)$ is the same class as $H(x, b)$. Thus $g(x, y, b) = 0$ is a primitive recursive relation such that no general method is possible to decide for any given x, whether a y exists or not such that $g(x, y, b) = 0$. Further, a more general formulation of the proof is easily obtained by the same kind of reasoning. Indeed, let $f(x)$ be a function which attains any value for some x, then $H\big(x, f(x)\big)$ is an unsolvable class. In fact, if $H\big(x, f(x)\big)$, which certainly is a recursively enumerable relation, should be recursive, the negation $\overline{H}\big(x, f(x)\big)$ must be recursively enumerable, that is, a number c must exist such that the equivalence

$$\overline{H}\big(x, f(x)\big) \rightleftarrows H(x, c)$$

be valid. But according to hypothesis a number d exists such that $f(d) = c$. Putting $x = d$ in the equivalence we get a contradiction.

There are, of course, infinitely many a such that $H(x, a)$ is a solvable class, and the same is true for the b for which $H(x, b)$ is unsolvable. It is reasonable to believe that the latter case is the most general and the former the more exceptional, but a thorough investigation of this may perhaps be difficult.

This proof of an undecidability theorem seems to me especially simple, since it only presupposes the ordinary and most well-known theorems on recursive functions. The most difficult theorem used is that $(Ey)R(x, y)$ is recursively enumerable, even if R is not primitive recursive. For in order to prove this statement, Kleene's normal form of general recursive functions has, as far as I know, hitherto always been used, and the proof of this normal form is based on the Gödel numbering method. (However, see the announcement at the end of this paper.)

There are stronger theorems of undecidability but here I will only mention that the strongest result hitherto, as far as I know, is that obtained by J. Myhill on p. 55 in his paper cited above. His result is that there is no general decision procedure for the questions of the form

$$\min_{x_1, \ldots, x_n \leqq y} f(x_1, x_2, \ldots, x_n, y) = 0 ,$$

where f is a polynomial with integral coefficients.

2. It is well known, that if a non-void set is general recursively enumerable, then it is also primitive recursively enumerable. This theorem can be used to set up a simple proof of the fact that the values of the variables satisfying a general recursive relation can be given as primitive recursive functions of a single parameter. In the corresponding problem of algebra we know that we need n parameters if the dimension of the algebraic variety is n.

Let $f(x_1, \ldots, x_n)$ be a recursive function. Putting

$$\binom{x_1+\ldots+x_n+n-1}{n}+\binom{x_1+\ldots+x_{n-1}+n-2}{n-1}+\ldots+\binom{x_1}{1}=y ,$$

which gives an enumeration without repetitions of all n-tuples $x_1, \ldots, x_n$, we have

(1) $$x_1 = \tau_1(y), \ldots, x_n = \tau_n(y) ,$$

where $\tau_1, \ldots, \tau_n$ are certain primitive recursive functions (see Skolem [9], p. 5). Putting

$$g(y) = f(\tau_1(y), \ldots, \tau_n(y)) ,$$

the equation

(2) $$f(x_1, \ldots, x_n) = 0$$

can be written

(3) $$g(y) = 0 .$$

If we now suppose known that the relation (2) is not void, then there is at least one value of y, k say, such that $g(k) = 0$. Then the numbers satisfying (3) can be enumerated, for example, by the formula

$$y = k \operatorname{sg} g(t) + t(1 - \operatorname{sg} g(t)) = h(t) . \tag{4}$$

If first we suppose that f is primitive recursive, so are g and h. Then all n-tuples $x_1, \ldots, x_n$ satisfying (2) are given by the equations

$$x_1 = h_1(t), \ldots, x_n = h_n(t) , \tag{5}$$

where, for $i = 1, \ldots, n$,

$$h_i(t) = \tau_i(h(t)) . \tag{6}$$

Obviously, all h_i are primitive recursive so that my assertion is proved very simply in this case. If, however, f is general recursive, then g is only known to be general recursive, and the same is the case with regard to h. However, if we use the theorem that a non-void set which is general recursively enumerable also can be enumerated by a primitive recursive function, we may nevertheless replace (4) by

$$y = \eta(t) ,$$

where η is a primitive recursive function. Thus all $x_1, \ldots, x_n$ satisfying (2) are given by (5) when (6) is replaced by

$$h_i(t) = \tau_i(\eta(t)) .$$

Therefore, also in this general case, there is a primitive recursive parametric representation of the values of the variables for which the given relation is valid.

3. In connection with this I should like to mention the following. In my opinion, it is a rather remarkable fact, that if we only want to set up an arithmetic without quantifiers, the explicit introduction of general recursive functions is in a certain sense superfluous. We need only the primitive recursive functions and relations. It is even possible to restrict the theory to elementary functions in Kalmar's sense, but I will not enter into that.

I will first consider an example. Let $\varphi(x, y)$ be general, but not primitive recursive, with, however, $z = \varphi(x, y)$ a primitive recursive relation. This may occur on occasion. Let it be proved that

$$(x_1 < x_2) \rightarrow (\varphi(x_1, y) < \varphi(x_2, y)) . \tag{7}$$

Here the function φ which is not primitive occurs. But since $z = \varphi(x, y)$ is a primitive recursive relation, it can be written in the form $\varrho(x, y, z) = 0$, where ϱ is primitive recursive. Then (7) can be written

(8) $\big(\varrho(x_1, y, z_1) = 0\big) \;\&\; \big(\varrho(x_2, y, z_2) = 0\big) \;\&\; (x_1 < x_2) \to (z_1 < z_2)$,

and here we have only the primitive recursive function ϱ. If it is agreed upon that (7) and (8) are equivalent, we have succeeded in eliminating the function φ. However, there may still be a difference between (7) and (8). Whether there is such a difference or not depends on the interpretation of the expressions written as values of functions. Writing $f(a, b)$, we may mean the value of f for the arguments a and b, tacitly assuming that such a value exists. But we may also write $f(a, b)$ meaning the value of f for the arguments a and b, provided such a value exists. Thus we may distinguish an existential use and a hypothetical use of the function symbols. If these symbols are interpreted hypothetically, then, as I shall show below, the recursive functions which are not primitive recursive can be eliminated. Indeed, they can be eliminated in a fashion demonstrated in the example above. On the other hand, if the function symbols are conceived existentially, the elimination is not possible if we use only free variables. For example, the proposition $R(x, \varphi(x))$, where R may be a primitive recursive relation and φ not primitive recursive, expresses according to the existential conception that for arbitrary x, a y exists such that $R(x, y)$, namely $y = \varphi(x)$. But to express this in free variables with only primitive recursive functions will certainly be impossible, if, for example, for every x only $y = \varphi(x)$ is such that $R(x, y)$ is valid.

The elimination can be carried out thus. Let a formula U be given in which the function $\varphi(x, y)$ occurs that is not primitive. Suppose first that $z = \varphi(x, y)$ is a primitive recursive relation so that it can be written $\varrho(x, y, z) = 0$, ϱ primitive recursive. Then let z be a variable not occurring in U. We replace every occurrence of $\varphi(x, y)$ in U by the letter z, U thereby being changed into U'. Then U is replaced by the implication

$$\big(\varrho(x, y, z) = 0\big) \to U' .$$

Otherwise we have (see Skolem [7], p. 103) $\varphi(x, y) = \psi\big(\iota z(\varrho(x, y, z) = 0)\big)$ where ψ and ϱ are primitive. Then let u be a variable not occurring in U. We replace every occurrence of $\varphi(x, y)$ by $\psi(u)$ with the effect that U is transformed into U'. Then instead of U we may write

$$\big(\varrho(x, y, u) = 0\big) \to U' .$$

By repeated application of this procedure we get at last a propositional formula containing only primitive functions.

It is natural to ask whether or not all proofs in recursive arithmetic—with restriction to free variables only and with the hypothetical conception of the functions of course—can be performed already in primitive recursive arithmetic when we eliminate all non-primitive functions in the indicated manner. I hope to return to this question later.

4. Regarding arithmetical proofs, it is natural to mention the so-called Gödel's first theorem of non-deducibility (see Hilbert-Bernays [1], p. 272–273). It is proved by Gödel-numbering that in a formal system S containing recursive arithmetic there is a primitive recursive function $\mathfrak{F}$ such that $\mathfrak{F}(\mathfrak{a}) = 0$ for every numeral $\mathfrak{a}$, whereas the general formula $\mathfrak{F}(a) = 0$, where a is a variable, is not deducible in S. I would like to remark that this property of non-deducibility is not necessarily invariant with respect to replacement of $\mathfrak{F}$ by another function f which attains just the same values. Indeed, f denoting a function which may be defined quite otherwise than $\mathfrak{F}$, we might have the equation

$$f(\mathfrak{a}) = \mathfrak{F}(\mathfrak{a})$$

for every numeral $\mathfrak{a}$, whereas the formula

$$f(a) = \mathfrak{F}(a)$$

is not deducible in S. But then there is nothing which excludes $f(a) = 0$ from being deducible in S. Therefore one might wonder how much Gödel's first non-deducibility theorem really hampers the possibilities of proof.

5. In an earlier paper (see Skolem [8], p. 9) I gave a simple version of the proof of Kleene's theorem that every recursive function has the form $\psi(\mu y R(x_1, \ldots, x_n, y))$, where $(x_1)\ldots(x_n)(Ey)R(x_1, \ldots, x_n, y)$ (see Kleene [2], p. 732, or [3], p. 292), although only the case $n = 1$ was considered by me. This, however, is irrelevant. I proved that ψ could be chosen as τ_2, defined in (1) for $n = 2$. Markov has proved that every function of large oscillation can replace ψ in this connection, a function being of large oscillation when it attains every value infinitely often. I should like to show that my reasoning in [8] can just as well be carried out so that ψ is chosen at once as an arbitrary function of large oscillation. First I will prove the following lemma:

If the function $f(t)$ is of large oscillation, another function $g(t)$ of large oscillation, primitive recursive with respect to f, can be found such that, putting

(9) $$x = f(t), y = g(t) ,$$

(x, y) runs through all pairs of non-negative integers without repetitions when t runs through all integers $\geqq 0$.

Indeed, it suffices to put $g(0) = 0$ and for $t > 0$,

$$g(t) = \sum_{t'=0}^{t-1} \overline{\mathrm{sg}}\left((f(t') \dot{-} f(t)) + (f(t) \dot{-} f(t'))\right).$$

One observes that $g(t) = y$ means that there exist y numbers smaller than t such that f attains the same value for all these as for t. That is, we have

$$t_1 < t_2 < \ldots < t_y < t$$

and

$$f(t_1) = f(t_2) = \ldots = f(t) .$$

Since for every x an infinite sequence $t_1 < t_2 < \ldots$ exists such that $f(t_r) = x$, the function $g(t)$ attains all integers y as values for the successive t_r with $f(t_r) = x$. Thus we get all pairs x, y in the form (9). For given x there is a one-to-one correspondence between the values of y and those of t. A similar situation prevails between x and t for given y. Thus the lemma is proved. Further, it is evident that g is primitive recursive if f is.

By Gödel-numbering one obtains a primitive recursive relation $U(x_1, \ldots, x_n, y, z, u)$ with the meaning that u is the number of a sequence of formulas leading to the value y of the principal function $f(x_1, \ldots, x_n)$ with the number z. Then if z is the number of a general recursive function, we must have

$$(x_1)\ldots(x_n)(Ey)(Eu)U(x_1, \ldots, x_n, y, z, u) . \tag{10}$$

Now let $g(t)$ be any primitive recursive function of large oscillation and $h(t)$ another primitive recursive function chosen as in the proof of the lemma such that

$$y = g(t), u = h(t)$$

yield all pairs y, u when t runs through the non-negative integers. Then (10) can be written

$$(x_1)\ldots(x_n)(Et)U\left(x_1, \ldots, x_n, g(t), z, h(t)\right) ,$$

and the function with number z is given by the expression

$$y = g\left(\mu t\, U\left(x_1, \ldots, x_n, g(t), z, h(t)\right)\right) .$$

Thus we have got the normal form with the arbitrary function g of large oscillation.

6. I have proved earlier (see Skolem [7], p. 103) that every general recursive function y of $x_1, \ldots, x_n$ can also be expressed in the form

$$y = \psi\left(\iota t R(x_1, \ldots, x_n, t)\right) ,$$

where ψ and R are primitive recursive and R is such that for all $x_1, \ldots, x_n$

there is one and only one t with $R(x_1, \ldots, x_n, t)$. It is clear after the result just obtained, that here also ψ may be chosen as an arbitrary function of large oscillation.

Let us consider the case $n = 1$. I will show that every general recursive function y of x can be written

$$y = \psi\big(\mu t(x = f(t))\big) \tag{11}$$

for some primitive recursive functions ψ and f, where f attains every value.

Indeed, as we have just noticed we can write first

$$y = \psi_1\big(\iota z R(x, z)\big)$$

where $(x)(E!z)R(x, z)$. On the other hand, there is for arbitrary x and z just one t such that

$$x = \tau_1(t), z = \tau_2(t) ,$$

τ_1 and τ_2 being determined as in (1) for $n = 2$. Then we have

$$y = \psi_1\Big(\tau_2\Big(\iota t\big(R\big(\tau_1(t), \tau_2(t)\big) \ \& \ \big(x = \tau_1(t)\big)\big)\Big)\Big) .$$

But the numbers t such that $R\big(\tau_1(t), \tau_2(t)\big)$ can be represented as the values of a primitive recursive function $h(u)$. Hence

$$y = \psi_1\Big(\tau_2\Big(h\Big(\mu u\big(x = \tau_1(h(u))\big)\Big)\Big)\Big) .$$

Writing

$$\psi(v) = \psi_1\big(\tau_2\big(h(v)\big)\big), \quad f(u) = \tau_1\big(h(u)\big) ,$$

we obtain

$$y = \psi\big(\mu u(x = f(u))\big)$$

which is (11).

Probably the necessary and sufficient condition for ψ to be able to represent any general recursive function of x in the form (11) for some primitive recursive f is again that it is of large oscillation, but I have not investigated it.

Without use of the normal form and thus independently of the Gödel-numbering method, I intend to prove in another paper the theorem that every general recursive relation is recursively enumerable.

REFERENCES

1. D. Hilbert and P. Bernays, *Grundlagen der Mathematik* II, Berlin, 1939.
2. S. C. Kleene, *General recursive functions of natural numbers,* Math. Ann. 112 (1936), 727–742.
3. S. C. Kleene, *Introduction to metamathematics,* Amsterdam, 1952.

4. J. Myhill, *Three contributions to recursive function theory*, Proceedings of the 11th International Congress of Philosophy 14, Amsterdam 1953, 50–59.
5. R. Péter, *Konstruktion nichtrekursiver Funktionen*, Math. Ann. 111 (1935), 42–60.
6. R. Péter, *Rekursive Funktionen*, Budapest, 1951.
7. Th. Skolem, *Some remarks on recursive arithmetic*, Det Kgl. Norske Vid. Selskab Forh. 17 (1944), Nr. 26.
8. Th. Skolem, *Den rekursive aritmetikk*, Norsk Mat. Tidsskr. 1946, 1–12.
9. Th. Skolem, *The development of recursive arithmetic*, Den 10. skandinaviske Matematikerkongres i København 1946, 1–16.

UNIVERSITY OF OSLO, NORWAY