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SOME EXTREMAL PROPERTIES OF LAPLACE TRANSFORMS

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To the Memory of HARALD BOHR

In the following we shall take up for discussion some of the problems posed by G. Doetsch in his handbook on the Laplace transformation [3]. Several of these problems will be solved essentially by solving the corresponding problems for Laplace-Stieltjes transforms. We take as our point of departure a Dirichlet series studied by H. Bohr in his dissertation [2, pp. 32–34, Sætning XVII].

1. Bohr's series. This Dirichlet series is constructed with the aid of four sequences of positive integers, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{y_n\}$, such that

$$\alpha_1 < y_1 < \beta_1 < \gamma_1 < \alpha_2 < \ldots < \alpha_n < y_n < \beta_n < \gamma_n < \alpha_{n+1} < \ldots .$$

Here

$$\alpha_n \leqq {y_n}^{1/2}, \qquad \beta_n = {y_n}^{1+\delta_n}, \qquad \lim_{n\to\infty} \delta_n = 0, \qquad \lim_{n\to\infty} {y_n}^{-\delta_n} = 0, \qquad {y_n}^2 \leqq \gamma_n .$$

We set

$$\varphi(z) = \sum_{m=1}^{\infty} a_m m^{-z} , \tag{1.1}$$

where the partial sums of the series $\Sigma_1^\infty a_n$ are determined by the conditions

$$\begin{aligned} S_m &= 0, & \alpha_n &\leqq m < \beta_n ,\\ S_m &= m^{iy_n}, & \beta_n &\leqq m \leqq \gamma_n ,\\ S_m &= 1, & \gamma_n &< m < \alpha_{n+1} . \end{aligned}$$

Since $|S_m| \leqq 1$ and S_m does not converge, the series (1.1) has the abscissa of convergence $\beta_0 = 0$. Bohr showed that the Lindelöf mu-function of $\varphi(z)$, $z = x+iy$, equals

$$\mu(x;\varphi) = \begin{cases} 1-x, & 0 < x < 1, \\ 0 \;, & 1 \leqq x . \end{cases} \tag{1.2}$$

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Bohr derived this result from a detailed study of the asymptotic behavior of $\varphi(x+iy_n)$ as $n \to \infty$. We shall need more details about this behavior as well as some inequalities obtainable from the same considerations.

Suppose that $y_n \leqq y \leqq ny_n$ and $0 < x \leqq 1-\delta < 1$. We have

$$\varphi(z) = \sum_{m=1}^{\infty} S_m [m^{-z} - (m+1)^{-z}]$$

and for $|z| < m$, hence certainly for $m \geqq \beta_n$ if n is large,

$$m^{-z} - (m+1)^{-z} = zm^{-1-z} + z^2 m^{-2-z} F(z, m)$$

with $|F(z, m)| \leqq M$, a fixed finite quantity. Thus

$$\begin{aligned}\varphi(z) &= \sum_{m=1}^{\alpha n-1} S_m [m^{-z} - (m+1)^{-z}] + z \sum_{m=\beta n}^{\gamma n} S_m m^{-1-z} \\ &\quad + z \sum_{m=\gamma n+1}^{\infty} S_m m^{-1-z} + z^2 \sum_{m=\beta n}^{\infty} F(z, m)\, m^{-2-z} \\ &\equiv \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4 .\end{aligned}$$

Here

$$|\textstyle\sum_1| \leqq 2 \displaystyle\sum_{m=1}^{\alpha n-1} m^{-x} < 2(1-x)^{-1} \alpha_n^{1-x} \leqq 2(1-x)^{-1} y_n^{\frac{1}{2}(1-x)}, \tag{1.3}$$

$$|\textstyle\sum_3| \leqq (1+ny_n) \displaystyle\sum_{m=\gamma n+1}^{\infty} m^{-1-x} < 2x^{-1} ny_n^{1-2x}, \tag{1.4}$$

$$\begin{aligned}|\textstyle\sum_4| &\leqq (1+ny_n)^2 M \displaystyle\sum_{m=\beta n}^{\infty} m^{-2-x} \\ &< M(1+ny_n)^2 (1+x)^{-1} \beta_n^{-1-x} < 4Mn^2 y_n^{1-(1+\delta_n)x-\delta_n}.\end{aligned} \tag{1.5}$$

Setting

$$\sigma_n(x) = \sum_{m=\beta n}^{\gamma n} m^{-1-x}, \tag{1.6}$$

we get

$$\begin{aligned}\sigma_n(x) &= x^{-1}[\beta_n^{-x} - \gamma_n^{-x}] + O(\beta_n^{-1-x}) \\ &= x^{-1}\beta_n^{-x}[1 - (\beta_n \gamma_n^{-1})^x + xO(\beta_n^{-1})] .\end{aligned}$$

Here the second term in the last bracket does not exceed $y_n^{-x(1-\delta_n)}$. Replacing x by x/k and noting that $y_n^{-1/n} \to 0$ we conclude that for fixed x

$$\sigma_n(x/k) = kx^{-1} y_n^{-(1+\delta_n)x/k} [1+o(1)], \quad k = 1, 2, \ldots, n, \tag{1.7}$$

as $n \to \infty$, uniformly with respect to k.

We have now

$$\textstyle\sum_2 = (x+iy) \displaystyle\sum_{m=\beta n}^{\gamma n} m^{-1-x+i(y_n-y)}.$$

If $y = y_n$ this gives

$$\Sigma_2 = (x+iy_n)\,\sigma_n(x) = ix^{-1}y_n^{\;1-(1+\delta_n)x}\,[1+o(1)]\,.$$

It is clear that for this value of y the sum Σ_2 dominates the other sums Σ_k so that

$$\varphi(x+iy_n) = ix^{-1}y_n^{\;1-(1+\delta_n)x}\,[1+o(1)] \tag{1.8}$$

for fixed x. This is Bohr's result save for the omitted factor x^{-1}.

It is essential for the following that we may replace x by x/n in this formula, obtaining

$$\varphi(x/n+iy_n) = inx^{-1}y_n^{\;1-(1+\delta_n)x/n}\,[1+o(1)] \tag{1.9}$$

as $n \to \infty$, x being fixed, $0 < x \leqq a < \infty$. This follows from the fact that (1.7) holds and that the resulting value of Σ_2 still dominates the other Σ_k's for n large.

Suppose now that $y_n < y \leqq ny_n$. We have then

$$|\Sigma_2| \leqq |x+iy|\,\sigma_n(x) \leqq 2nx^{-1}y_n^{\;1-(1+\delta_n)x} \tag{1.10}$$

for $n > n_0$, independently of x, $x > 0$. Combining (1.3)–(1.5) with (1.10) one gets for $0 < x < k$, $1 \leqq k \leqq n$,

$$\begin{aligned} |\varphi\big((x+iny_n)/k\big)| \leqq\; & 2k(k-x)^{-1}y_n^{\;\frac{1}{2}(1-x/k)} + 2kx^{-1}ny_n^{\;1-2x/k} \\ & +4Mn^2y_n^{\;1-(1+\delta_n)x/k-\delta_n} + 2nkx^{-1}y_n^{\;1-(1+\delta_n)x/k}; \end{aligned}$$

whence for large n

$$|\varphi\big((x+iny_n)/k\big)| \leqq Cn^2x^{-1}y_n^{\;1-(1+\delta_n)x/(n-1)}, \qquad k = 1, 2, \ldots, n-1. \tag{1.11}$$

The assumption that $0 < x < 1$ is clearly not necessary since the series (1.1) is absolutely convergent for $x > 1$ so that the estimate (1.11) is trivially true for $x > 1$. Cf. (1.12) below[1].

Finally we note the trivial but useful estimate

$$|\varphi(z)| \leqq |z|\,x^{-1}, \quad x > 0\,. \tag{1.12}$$

This follows immediately from the representation of $\varphi(z)/z$ as a Laplace integral.

2. Maximal order of Laplace integrals on vertical lines. It is known that a Laplace transform is at most $o(|y|)$ on vertical lines. The following theorem shows that this is the best possible estimate thus answering a question posed by G. Doetsch [3, p. 175].

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[1] The author is indebted to Dr. Erling Følner for calling his attention to errors in the original argument used in proving (1.9) and (1.11).

THEOREM 1. *There exists a Laplace transform* $q(z)$, *converging for* $\operatorname{Re}(z) > 0$, *such that*

$$\mu(x;q) = \begin{cases} 1-x, & 0 < x < 1, \\ 0\ , & 1 \leqq x\,. \end{cases} \tag{2.1}$$

We use Bohr's function $\varphi(z)$ and define

$$q(z) = \varphi(z)\big(\log\,(z+2)\big)^{-1}. \tag{2.2}$$

The function $\big(\log\,(z+2)\big)^{-1}$ is holomorphic for $\operatorname{Re}(z) > -1$ and is a Laplace transform in this half-plane. In fact, from

$$\big(\Gamma(\alpha)\big)^{-1}\int_0^\infty e^{-(z+2)t}t^{\alpha-1}dt = (z+2)^{-\alpha},$$

one obtains by integration with respect to α

$$\big(\log\,(z+2)\big)^{-1} = \int_0^\infty e^{-zt}\left\{e^{-2t}\int_0^\infty t^{\alpha-1}\big(\Gamma(\alpha)\big)^{-1}\,d\alpha\right\}dt \equiv \int_0^\infty e^{-zt}L(t)\,dt \tag{2.3}$$

which is the desired representation. Here $L(t)$ is real positive. We have then

$$q(z) = \int_0^\infty e^{-zt}Q(t)\,dt \tag{2.4}$$

with

$$Q(t) = \int_0^t L(t-s)\,dA(s) \tag{2.5}$$

if

$$\varphi(z) = \int_0^\infty e^{-zt}\,dA(t)\,. \tag{2.6}$$

Here

$$A(t) = \sum_{\log n \leqq t} a_n = S_{[e^t]}. \tag{2.7}$$

It is clear that (2.4) converges for $\operatorname{Re}(z) > 0$ as the product of two convergent Laplace-Stieltjes integrals of which one is absolutely convergent and the value of $\mu(x;q)$ equals that of $\mu(x;\varphi)$ which is given by (1.2).

3. Laplace transforms of maximal order in a half-plane. Actually we can do much more with Bohr's function than is indicated in Theorem 1. On p. 181 of [3] Doetsch raised the question of the existence of a Laplace transform bounded in no right half-plane. In the mean time P.H. Bloch [1] has constructed such a function, but we shall find one with still more extreme properties.

THEOREM 2. *There exists a Laplace transform $f(z)$, converging for* $\operatorname{Re}(z) > 0$, *such that*

$$\mu(x; f) \equiv 1, \quad x > 0\,. \tag{3.1}$$

For the construction we use Bohr's series again, but it is convenient to specify the basic sequences of integers. We shall take, for instance,

$$\alpha_n^{\ 2} = y_n, \quad y_n = 2^{2^{n^2}}, \quad \gamma_n = y_n^{\ 2}, \quad \delta_n = 2^{-n}\,. \tag{3.2}$$

This choice is evidently consistent with the conditions imposed in section 2. We then set

$$g(z) = \sum_{n=1}^{\infty} y_{n-1}^{\ -1}\varphi(z/n)\,. \tag{3.3}$$

In view of (1.12) the series converges absolutely and uniformly in every sector $|z| \geqq \varepsilon$, $|\arg z| \leqq \frac{1}{2}\pi - \varepsilon$, $\varepsilon > 0$.

Since

$$\varphi(z/n) = \int_0^\infty e^{-zt}\,dA(nt)\,,$$

we have

$$g(z) = \sum_{n=1}^{\infty} y_{n-1}^{\ -1}\int_0^\infty e^{-zt}\,dA(nt)$$

and, formally,

$$g(z) = \int_0^\infty e^{-zt}\,dG(t) \tag{3.4}$$

with

$$G(t) = \sum_{n=1}^{\infty} y_{n-1}^{\ -1}A(nt)\,. \tag{3.5}$$

The series (3.5) converges for $0 \leqq t < \infty$, uniformly in every finite interval, since $|A(nt)| \leqq 1$. Further

$$V_0^{\,t}[A(ns)] = \sum_{\log m \leqq nt} |a_m| \leqq 2 \sum_{\log m \leqq nt} 1 < 2e^{nt},$$

so that

$$V_0^{\,t}[G(s)] \leqq 2\sum_{n=1}^{\infty} y_{n-1}^{\ -1}e^{nt}$$

and $G(t)$ is of bounded variation in every finite interval. On the other hand, the variation grows faster than any function $e^{\omega t}$ as $t \to +\infty$. It follows that the integral in (3.4) converges for $\operatorname{Re}(z) > 0$ but has no half-plane of absolute convergence. It is a simple matter to verify that the integral actually represents $g(z)$, for instance, by computing the saltus corresponding to $t = n^{-1}\log m$ and comparing it with the coefficient of $m^{-z/n}$ in the double series

$$\sum_{n=1}^{\infty} y_{n-1}^{-1} \sum_{m=1}^{\infty} a_m m^{-z/n}.$$

This being accomplished we shall verify that

$$\mu(x; g) \equiv 1, \quad x > 0. \tag{3.6}$$

For this purpose we write

$$g(z) = \sum_{k=1}^{n-1} y_{k-1}^{-1}\varphi(z/k) + y_{n-1}^{-1}\varphi(z/n) + \sum_{k=n+1}^{\infty} y_{k-1}^{-1}\varphi(z/k)$$

and choose $z = x+iny_n$. For n large we can then use (1.9) and obtain

$$\begin{aligned} y_{n-1}^{-1}\varphi(x/n+iy_n) &= ix^{-1}y_{n-1}^{-1}ny_n^{1-(1+\delta_n)x/n}[1+o(1)] \\ &= ix^{-1}y_n^{1-(1+\delta_n)x/n-2\delta_n^2}[1+o(1)]. \end{aligned} \tag{3.7}$$

The finite sum is estimated with the aid of (1.11) which shows that it is small in comparison with the contribution from the nth term. The infinite remainder is estimated using (1.12) and is found to be dominated by a constant times n/x. Combining the three estimates one gets

$$g(x+iny_n) = ix^{-1}ny_n^{1-(1+\delta_n)x/n-2\delta_n^2}[1+o(1)] \tag{3.8}$$

uniformly in x, $0 < x \leqq \omega < \infty$. It follows that for every fixed ϱ, $0 \leqq \varrho < 1$, and fixed x, $0 < x < \infty$, one has

$$\limsup_{y\to+\infty} y^{-\varrho}\,|g(x+iy)| = +\infty.$$

Hence $\mu(x; g) \geqq 1$. But the converse inequality must hold in the half-plane of convergence of a Laplace-Stieltjes transform. This proves (3.6).

We now form

$$f(z) = g(z)\big(\log(z+2)\big)^{-1} = \int_0^{\infty} e^{-zt}F(t)\,dt \tag{3.9}$$

with

$$F(t) = \int_0^t L(t-s)\,dG(s). \tag{3.10}$$

By the analogue of Mertens' theorem the integral in (3.9) converges for $\mathrm{Re}(z) > 0$. It follows that $f(z)$ is a convergent Laplace transform satisfying (3.1) and the theorem is proved.

4. Summable Laplace transforms of maximal order in a half-plane. On p. 333 of [3] Doetsch raised the question if the estimate $f(x+iy) = o(|y|^{k+1})$ is the best possible for $x \geqq \beta_k+\varepsilon$ if $f(z)$ is representable as a Laplace transform, summable (C, k) for $x > \beta_k$. We shall show that this is indeed the case for integral values of k.

THEOREM 3. *For each positive integer k there exists a function $f_k(z)$ which is representable by a Laplace integral, summable (C, k) for $x > 0$, such that*

$$(4.1)\qquad \mu(x; f_k) \equiv k+1, \qquad x > 0 .$$

We set

$$(4.2)\qquad f_k(z) = [f(z)]^{k+1}, \qquad k = 1, 2, 3, \ldots,$$

where $f(z)$ is the function defined by (3.9). Obviously it satisfies (4.1). For $k = 1, f_1(z) = f(z)f(z)$ is the product of two convergent Laplace integrals. Consequently we have formally

$$(4.3)\qquad f_1(z) = \int_0^\infty e^{-zt} F_1(t)\, dt$$

with

$$(4.4)\qquad F_1(t) = \int_0^t F(t-u) F(u) du .$$

Here the integral (4.3) cannot converge for any z since $f_1(x+iy)$ is not $o(|y|)$. But by a well-known theorem (see Doetsch [3], p. 351) the product of two convergent Laplace integrals is certainly summable $(C, 1)$ so that

$$(4.5)\qquad f_1(z) = z \int_0^\infty e^{-zt} \int_0^t F_1(s) ds\, dt ,$$

the integral being convergent for $\text{Re}(z) > 0$. Since $|f_1(x+iy)|$ is not $o(|y|^{1+\alpha})$ for any $\alpha < 1$, it follows that (4.3) cannot be summable (C, α) with an $\alpha < 1$. This settles the case $k = 1$.

Since

$$f_{k+1}(z) = f(z) f_k(z) ,$$

we can apply an obvious induction argument based on the fact that if two Laplace integrals are summable (C, α) and (C, β) respectively, then their product is summable $(C, \alpha+\beta+1)$ at least. The details may be left to the reader.

5. On the theorem of Landau. It was proved by E. Landau [4, p. 546], cf. Doetsch [3, p. 153], that if the Laplace transform $f(z) = \mathfrak{L}\{F\}$ of a positive function $F(t)$ has a half-plane of convergence, $\text{Re}(z) > \beta_0$, then the point $z = \beta_0$ is a singular point of $f(z)$. On p. 331 of [3], Doetsch raised the question of the character of the point $z = \beta_k$, if it is known that $\mathfrak{L}\{F\}$ does not have a half-plane of convergence, $\beta_0 = +\infty$, but there exists a $k > 0$ such that the integral is summable (C, k) for $\text{Re}(z) > \beta_k$, $\beta_k < +\infty$, $F(t)$ being ultimately positive.

We shall show that this case cannot arise. It is no restriction to assume that $F(t) \geqq 0$ for all t. By assumption

$$\limsup_{\omega\to\infty} \omega^{-1} \log \int_0^\omega F(t)\,dt = +\infty\,. \tag{5.1}$$

If $\mathfrak{L}^{(k)}\{F\}$ converges for some $k > 0$ and some real $x > 0$, we would have for this x (see [3, p. 315, formula (3)])

$$f(x) = x^{k+1}\big(\Gamma(k+1)\big)^{-1} \int_0^\infty e^{-xt} \int_0^t (t-s)^k F(s)\,ds\,dt \tag{5.2}$$

and

$$\limsup_{\omega\to\infty} \omega^{-1} \log \int_0^\omega \int_0^t (t-s)^k F(s)\,ds\,dt \leqq x < \infty\,. \tag{5.3}$$

Suppose that A is given arbitrarily large but at least $> 2x$. By assumption we can find arbitrarily large values ω such that

$$\int_0^\omega F(t)\,dt > e^{2A\omega}.$$

But this says that there are intervals $(\omega, 2\omega)$ with arbitrarily large values of ω such that

$$\int_0^u F(t)\,dt > e^{Au} \quad \text{for} \quad \omega \leqq u \leqq 2\omega\,.$$

Next we observe that

$$\int_0^t (t-s)^k F(s)\,ds > \int_0^{t-1} F(s)\,ds > e^{A(t-1)}$$

for $\omega+1 \leqq t \leqq 2\omega+1$. Hence

$$\int_0^{2\omega+1} \int_0^t (t-s)^k F(s)\,ds\,dt > \int_{\omega+1}^{2\omega+1} e^{A(t-1)}\,dt > \omega e^{A\omega},$$

so that

$$(2\omega+1)^{-1} \log \int_0^{2\omega+1} \int_0^t (t-s)^k F(s)\,ds\,dt > (2\omega+1)^{-1} A\omega + (2\omega+1)^{-1} \log \omega\,.$$

It follows that the superior limit of the left side for $\omega \to \infty$ is at least $\frac{1}{2}A > x$. This is a contradiction and shows that $\beta_k = +\infty$ for every k.

6. The abscissas of finite order and of holomorphism. Let $f(z) = \mathfrak{L}\{F\}$ be a Laplace transform with $x = \beta_0 < +\infty$ as abscissa of convergence.

If there is no singular point on the line $x = \beta_0$, it may happen that for some $k > 0$ the abscissa of (C, k)-summability β_k is less than β_0. Since β_k is a decreasing function of k

$$\lim_{k\to\infty} \beta_k \equiv \beta_\infty \tag{6.1}$$

exists and $\eta \leqq \beta_\infty$, where η is the abscissa of holomorphism of $f(z)$, that is, every strip $\eta-\varepsilon \leqq x \leqq \eta$ contains at least one singular point of $f(z)$. Here β_∞ is also characterized by function theoretical properties of $f(z)$; indeed, β_∞ is the abscissa of finite order, that is, $|f(x+iy)| = O\{|y|^{k(\varepsilon)}\}$ for $x \geqq \beta_\infty+\varepsilon$, $\varepsilon > 0$, but ceases to be of finite order in any half-plane $x \geqq \beta_\infty-\varepsilon$. On p. 331 of [3] Doetsch asked if η could be less than β_∞ and, if so, that an example should be found. This will be done here.

THEOREM 4. *There exists an entire function $f(z)$ such that β_k is identically zero, $0 \leqq k \leqq +\infty$.*

For the proof we use the function of Mittag-Leffler of order α

$$E_\alpha(z) = \sum_{n=0}^{\infty} z^n \big(\Gamma(1+\alpha n)\big)^{-1}. \tag{6.2}$$

It is known that

$$|E_\alpha(z)| \leqq M_1(\alpha), \quad \tfrac{1}{2}\pi\alpha \leqq \arg z \leqq 2\pi-\tfrac{1}{2}\pi\alpha\,, \tag{6.3}$$

$$|E_\alpha(z)-\alpha^{-1}e^{z^{1/\alpha}}| \leqq |z|^{-1}M_2(\alpha), \quad |\arg z| \leqq \tfrac{1}{2}\pi\alpha\,. \tag{6.4}$$

Here we take $\alpha = \frac{1}{2}$ and form the function

$$f(z) = z^{-1}[E_{\frac{1}{2}}(\omega z)-1], \qquad \omega = e^{\frac{3}{4}\pi i}. \tag{6.5}$$

This is an entire function so $\eta = -\infty$. Further

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 dy \leqq C, \qquad x \geqq 0\,, \tag{6.6}$$

by virtue of (6.3). By a well-known theorem (see Doetsch [3, p. 422]), $f(z)$ is then the Laplace transform of a function $F(t)$ in $L_2(0, \infty)$ so that $\beta_0 = 0$. Now (6.4) shows that for $x < 0$

$$f(x+iy) = 2(x+iy)^{-1}\{e^{2xy-i(x^2-y^2)}+O(1)\} \tag{6.7}$$

so that $\beta_\infty = 0$. This completes the proof.

Addendum (October 24, 1953). Dr. Følner has kindly called my attention to a paper by Tim Jansson, *Über die Grössenordnung Dirichletscher Reihen*, Arkiv för Mat., Astr. o. Fysik, 15, no. 6 (1920), 11 pp. In this paper Dr. Jansson used Bohr's method to construct a Dirichlet series with $\lambda_n = \log \log n$, convergent for $x > 0$ and having $\mu(x) \equiv 1$. Moreover, he used the same device to construct a continuous function whose Laplace transform has $\mu(x) \equiv 1$ and he observed that the square of the transform is summable (C, α) for $\alpha \geqq 1$ but not for $\alpha < 1$. Thus the problems that I set out to solve in sections 3 and 4 were solved years ago. The observation that multiplication by $[\log (z+2)]^{-1}$ or a similar slowly decreasing logarithmico-exponential function carries a Laplace-Stieltjes integral into a Laplace integral with the same mu-function, is possibly new, however. Perhaps the publishing of this paper will bring back to light some more forgotten results.

BIBLIOGRAPHY

1. P. H. Bloch, *Über eine Laplace-Transformierte, welche in keiner Halbebene beschränkt ist,* Compositio Mathematica 9 (1951), 289–292.
2. Harald Bohr, *Bidrag til de Dirichlet'ske Rækkers Theori,* København, 1910.
3. G. Doetsch, *Handbuch der Laplace-Transformation* I, Basel, 1950.
4. E. Landau, *Über einen Satz von Tschebyschef,* Math. Ann. 61 (1905), 527–550.

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