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ON THE ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES AND EIGENFUNCTIONS OF ELLIPTIC DIFFERENTIAL OPERATORS

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Introduction. Let $a = a(x, D)$ be a differential operator of the form

$$a(x, D) = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha \qquad (m \geqq 1)\,,$$

where $x = (x_1, \ldots, x_n)$ is a point in real n-space, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a differentiation index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$D^\alpha = D_x^{\ \alpha} = i^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\ \alpha_1} \ldots \partial x_n^{\ \alpha_n}}. \tag{0.1}$$

The coefficients $a_\alpha(x)$ are supposed to be infinitely differentiable in an open region T, and the operator a is supposed to be elliptic so that

$$a_0(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \qquad (\xi^\alpha = \xi_1^{\ \alpha_1} \ldots \xi_n^{\ \alpha_n}) \tag{0.2}$$

is a positive definite polynomial in ξ for all x in T.

Let $H = H(S)$ be the set of all infinitely differentiable functions vanishing outside compact subsets of an open bounded set S whose closure is contained[1] in T. Put

$$((f, g)) = \int_S \sum_{|\alpha| \leqq m} f_\alpha(x) \overline{g_\alpha(x)}\, dx, \qquad \|f\|^2 = ((f, f)), \qquad f_\alpha = D^\alpha f\,,$$

and

$$(f, g) = \int_S f(x)\, \overline{g(x)}\, dx, \qquad |f|^2 = (f, f)\,.$$

Closing $H(S)$ in the norm $\|f\|$ we get a Hilbert space $\mathfrak{H} = \mathfrak{H}(S)$ which may be described roughly as the set of all functions in S having square integrable derivatives of any order $\leqq m$, those of order $< m$ vanishing at the boundary of S.

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[1] The assumptions on the differential operator a may be weakened by considering it only in S and by letting the coefficients be only sufficiently differentiable, but we do not go into the details.

Consider (af, g) where f and g are in $H(S)$. By integrations by parts it may be written in the form

$$a(f, g) = \int_S \Sigma\, a_{\alpha\beta}'(x) f_\alpha(x) \overline{g_\beta(x)}\, dx \qquad (|\alpha| \leqq m, |\beta| \leqq m)\,,$$

and hence it is obviously a bounded function of f and g in $\mathfrak{H}$. As shown by the author [3], the form $a(f, g)$ is also bounded from below in the following sense. Let t be a large positive number and put

$$a_t(f, g) = a(f, g) + t(f, g)$$

and

$$((f, g))_t = ((f, g)) + t(f, g)\,.$$

Then there exist a number t_0 and a number $c > 0$ so that[2]

$$c^{-1}\,((f, f))_t \leqq |a_t(f, f)| \leqq c\,((f, f))_t \qquad (t > t_0) \tag{0.3}$$

for all f in $\mathfrak{H}$. Hence the bounded linear operator N_t from $\mathfrak{H}$ to $\mathfrak{H}$ defined by

$$a_t(f, f') = ((N_t f, f'))_t, \qquad f, f', N_t f \in \mathfrak{H}\,,$$

has a bounded inverse N_t^{-1}. Let $\mathfrak{H}_0 = \mathfrak{H}_0(S)$ be the set of all square integrable functions defined in S. As is easily seen, the equation

$$(f, f') = ((M_t f, f'))_t, \qquad f \in \mathfrak{H}_0, \quad M_t f, f' \in \mathfrak{H}\,,$$

defines a completely continuous linear operator from $\mathfrak{H}_0$ to $\mathfrak{H}$. Hence

$$(f, f') = a_t(G_t f, f'), \qquad f \in \mathfrak{H}_0, \quad f' \in \mathfrak{H}\,,$$

where $G_t = N_t^{-1} M_t$, defines a completely continuous linear operator G_t from $\mathfrak{H}_0$ to $\mathfrak{H}$ which will be called Green's transformation corresponding to the differential operator $a_t = a + t$ and the linear subset $\mathfrak{H}$ of $\mathfrak{H}_0$. The reason is that G_t transforms $\mathfrak{H}_0$ into $\mathfrak{H}$ and that G_t^{-1} is an extension of the differential operator a_t whose graph is the set of all pairs $\{f, a_t f\}$ where $f \in \mathfrak{H}$ is $2m$ times continuously differentiable and $a_t f$ is in $\mathfrak{H}_0$. In fact, if h is in H we have $a_t(G_t a_t f, h) = (a_t f, h) = a_t(f, h)$ so that $G_t a_t f = f$.

Let k be an integer > 0. We shall prove that G_t^k has a kernel $g_t^{(k)}(x, y)$ so that

$$(G_t^k f, f') = \int_{S\times S} g_t^{(k)}(x, y) f(x) \overline{f'(y)}\, dx\, dy$$

[2] In [3] the operator a is denoted by q. Introducing the operator R_t defined in the proof of Lemma 4.1 (*l. c.* p. 69), we may write $q_t(f, f)$ as $p_t(f + R_t f, f)$. Hence we get

$$p_t(f, f)(1 - |R_t|_t) \leqq |q_t(f, f)| \leqq p_t(f, f)(1 + |R_t|_t)\,,$$

the norm $|R_t|_t$ being defined in *l. c.* p. 69. Now $|R_t|_t$ tends to zero with $1/t$, and hence Theorem 2.2 of *l. c.* proves that the formula (0.3) above is true.

when $f, f' \in H(S)$. The kernel is infinitely differentiable when $x \neq y$ and has the singularity to be expected when $|x-y|$ is small. If $2km > n$ it is continuous and satisfies

$$\lim_{t \to \infty} t^{k-\nu} g_t^{(k)}(x, y) = \delta_{xy} (2\pi)^{-n} \int \left(a_0(x, \xi) + 1\right)^{-k} d\xi \tag{0.4}$$

where $\delta_{xy} = 0$ when $x \neq y$ and $\delta_{xx} = 1$ and $\nu = n/(2m)$. If a is formally self-adjoint, that is, if (af, f) is real for all f in H, then G_t is also self-adjoint,

$$\operatorname{tr} G_t^k = \int_S g_t^{(k)}(x, x)\, dx < \infty$$

if $2mk > n$, and

$$\lim t^{k-\nu} \operatorname{tr} G_t^k = (2\pi)^{-n} \int_S dx \int \left(a_0(x, \xi) + 1\right)^{-k} d\xi \,. \tag{0.5}$$

Moreover, if a is self-adjoint, there exists a set $\varphi_1, \varphi_2, \ldots$ of eigenfunctions of every G_t with eigenvalues

$$(\lambda_1 + t)^{-1}, (\lambda_2 + t)^{-1}, \ldots \qquad (\lambda_1 \leqq \lambda_2 \leqq \ldots).$$

The eigenfunctions form a complete orthonormal system in $\mathfrak{H}_0$. If $2mk > n$ we have

$$g_t^{(k)}(x, y) = \Sigma\, \overline{\varphi_j(x)}\, \varphi_j(y)\, (\lambda_j + t)^{-k}$$

and

$$\operatorname{tr} G_t^k = \Sigma\, (\lambda_j + t)^{-k} .$$

We can now apply the method of Carleman [2] to deduce some asymptotic formulas for the eigenfunctions and eigenvalues. Our last two formulas combined with (0.4) and (0.5) and a Tauberian theorem of Hardy and Littlewood [4] in the formulation of Pleijel [9] show in fact that

$$N(t) = \sum_{\lambda_j \leqq t} 1 = (2\pi)^{-n} w_a(S)\, t^{n/(2m)} \left(1 + o(1)\right) \tag{0.6}$$

and

$$\lim_{N \to \infty} N^{-1} \sum_1^N \overline{\varphi_j(x)}\, \varphi_j(y) = \delta_{xy}\, w_a(x)/w_a(S) \,, \tag{0.7}$$

where

$$w_a(x) = \int_{a_0(x, \xi) < 1} d\xi \qquad \text{and} \qquad w_a(S) = \int_S w_a(x)\, dx \,.$$

These formulas are well known in various special cases. They were stated by the author [3] with indications of a proof[3] which works when $2m > n$.

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[3] Essentially this proof has been published in detail by Browder [1].

In this paper we shall follow another line of attack to obtain the key formulas (0.4) and (0.5).

— Our results combined with those announced by Keldych [7] prove that the asymptotic formula (0.6) is true also when a is not self-adjoint, provided that we replace λ_j by $\Re\lambda_j$.

1. Two lemmas. We shall use the theory of generalized Fourier transforms (Schwartz [10]). Let $F = F(x) = F(x_1, \ldots, x_n)$ be an infinitely differentiable function on real n-space vanishing outside a compact set. It has a Fourier transform f given by

$$f(\xi) = \int e^{ix\xi} F(x)\, dx\ ,$$

where $x\xi = x_1\xi_1 + \ldots$. It is well known that $f(\xi) = O\big((1+|\xi|)^{-N}\big)$ for every N $\big(|\xi| = (\xi_1{}^2 + \ldots)^{\frac{1}{2}}\big)$. The inverse formula reads

$$F(x) = (2\pi)^{-n}\int e^{-ix\xi} f(\xi)\, d\xi\ .$$

Let $a(\xi)$ be a locally integrable function which is $O(1+|\xi|^N)$ for some $N > 0$. Then

$$A(F) = (2\pi)^{-n}\int a(\xi)\,\overline{f(\xi)}\, d\xi$$

defines an antilinear functional A of F called the generalized inverse Fourier transform of a. If $(2\pi)^{-n}\int |a(\xi)|\, d\xi = c < \infty$, the function

$$A(x) = (2\pi)^{-n}\int e^{-ix\xi}\, a(\xi)\, d\xi$$

is continuous and defined for all x, and $|A(x)| \leqq c$. In this case

$$A(F) = \int A(x)\,\overline{F(x)}\, dx\ .$$

More generally, we say that A is a function $A(x)$ in a region R if there exists a locally integrable function $A(x)$ in R for which this equation holds when F vanishes outside a compact set in R.

Returning to the general case, let D^α be defined by (0.1). The derivative $D^\alpha A$ of A is defined by

$$D^\alpha A(F) = (2\pi)^{-n}\int \xi^\alpha\, a(\xi)\,\overline{f(\xi)}\, d\xi\ ,$$

where ξ^α is defined in (0.2). The product λA of A and a polynomial $\lambda(x)$ is defined by

$$\lambda A(F) = (2\pi)^{-n}\int a(\xi)\ \lambda(D_\xi)\overline{f(\xi)}\, d\xi\ ,$$

where $\lambda(D_\xi) = \lambda(i\partial/\partial\xi_1, \ldots, i\partial/\partial\xi_n)$.

Among the results of Schwartz we quote the following ones. If the product of A and a polynomial λ which is not zero in a region R is a function $B(x)$ in R, then A is the function $B(x)/\lambda(x)$ in R. If all the derivatives of A are functions in a region R, then A is infinitely differentiable there and its ordinary derivatives $D^\alpha A(x)$ are related to $D^\alpha A(F)$ by the formula

$$D^\alpha A(F) = \int D^\alpha A(x)\, \overline{F(x)}\, dx \qquad (F \in H(R)).$$

We are now in a position to prove the following lemma.

Lemma 1. *Let $p(\xi)$ be a polynomial of degree μ whose coefficients are majorized by a number c_1, and suppose that $|p(\xi)| \geqq c_2(1+|\xi|^\mu)$. Then the generalized inverse Fourier transform of $1/p(\xi)$ is an infinitely differentiable*[4] *function $P(x)$ in the region $x \neq 0$ satisfying*

$$|D^\alpha P(x)| \leqq C\, e_{|\alpha|}(x) \left(1+|x|^N\right)^{-1}, \begin{cases} e_{|\alpha|}(x) = 1 & \text{when } \mu-|\alpha|-n > 0, \\ e_{|\alpha|}(x) = |x|^{\mu-|\alpha|-n-\varepsilon} & \text{when } \mu-|\alpha|-n \leqq 0. \end{cases}$$

Here $N \geqq 0$ and $1 > \varepsilon > 0$ are arbitrary, and the number C depends on c_1, c_2, $|\alpha|$, N, and ε, but is otherwise independent of the polynomial p.

Proof. Let λ be a polynomial. Then

$$\lambda D^\alpha P(F) = (2\pi)^{-n} \int \xi^\alpha p^{-1}(\xi)\, \lambda(D_\xi) \overline{f(\xi)}\, d\xi\,.$$

By virtue of the properties of p, we may integrate by parts and get

$$\lambda D^\alpha P(F) = (2\pi)^{-n} \int \overline{f(\xi)}\, \lambda(-D_\xi)\big(\xi^\alpha p^{-1}(\xi)\big)\, d\xi\,.$$

Let λ be homogeneous of degree $k \leqq N$ and let its coefficients be $\leqq 1$ in absolute value. Let C denote a suitable number, not always the same, but depending only on $|\alpha|$, c_1, c_2, N, and later also on ε. It is clear that

$$\left|\lambda(-D_\xi)\big(\xi^\alpha p^{-1}(\xi)\big)\right| \leqq C(1+|\xi|)^{|\alpha|-\mu-k}.$$

Hence, if $|\alpha|-\mu < -n$ and λ has the properties stated, it follows that $\lambda D^\alpha P$ is a function majorized by C. This proves the first half of the lemma.

Next consider the case $|\alpha|-\mu \geqq -n$. Supposing that the coefficients of λ are $\leqq 1$ in absolute value, we let λ be homogeneous of degree $k+n+|\alpha|-\mu$ where $1 \leqq k \leqq N+1$. Then

$$\left|\lambda(-D_\xi)\big(\xi^\alpha p^{-1}(\xi)\big)\right| \leqq C(1+|\xi|)^{-n-k},$$

and if $\lambda D^\alpha P$ is the function

$$(2\pi)^{-n} \int e^{-ix\xi}\, \lambda(-D_\xi)\big(\xi^\alpha p^{-1}(\xi)\big)\, d\xi\,.$$

[4] Actually, it is analytic (Schwartz [10], F. John [6]).

This function vanishes when $x = 0$, the integrand being a sum of exact differentials, and hence we may replace $e^{-ix\xi}$ by $e^{-ix\xi}-1$ whose absolute value is less than $C|x|^{1-\varepsilon}|\xi|^{1-\varepsilon}$, so that $\lambda\, D^\alpha P$ is majorized by $C|x|^{1-\varepsilon}$.

Hence, if λ is a polynomial with the properties stated above, then $\lambda D^\alpha P$ is a function bounded by $C|x|^{1-\varepsilon}$. The second half of the lemma follows.

The function $P(x)$ is a fundamental solution of the differential operator $p(-D_x)$ in the sense that

$$\int P(x-y)\, p(D_y)F(y)\, dy = F(x)\,.$$

We shall express the connection between P and p by the symbolic notation

$$P(x) \sim (2\pi)^{-n} \int e^{-ix\xi}\, p^{-1}(\xi)\, d\xi\,.$$

It is clear that linear coordinate transformations preserve the sense of this formula.

Obviously, $e_{|\alpha|}(x)$ is majorized by $e_{|\alpha'|}(x)$ if $|\alpha| \leqq |\alpha'|$ and x is bounded. For large $|x|$, $|D^\alpha P(x)| \leqq C|x|^{-N}$ for arbitrary N, and hence we obtain from Lemma 1 the supplementary

LEMMA 2. *Under the hypotheses of Lemma* 1, *the following estimates hold*:

$$|D^\alpha P(x)| \leqq C|x|^{1-\varepsilon-n}(1+|x|^N)^{-1} \qquad (|\alpha| < \mu)$$

and

$$|D^\alpha P(x)| \leqq C|x|^{-\varepsilon-n}(1+|x|^N)^{-1} \qquad (|\alpha| \leqq \mu)\,.$$

Here $N \geqq 0$ and $1 > \varepsilon > 0$ are arbitrary, and the number C depends on c_1, c_2, N, and ε, but is otherwise independent of the polynomial p.

2. Estimates of certain fundamental solutions. Let τ be a large positive real parameter and consider a differential operator of order μ

$$b = b(\tau, x, D_x) = \sum_{|\alpha| \leqq \mu} b_\alpha(\tau, x)\, \tau^{-|\alpha|}\, D_x{}^\alpha,$$

where $b_\alpha(\tau, x)$ is a polynomial in τ^{-1} whose coefficients are infinitely differentiable functions in an open region T. It is assumed that the following polynomial in ξ

$$b_0(\tau, x, \xi) = b(\tau, x, \tau\xi) = \sum b_\alpha(\tau, x)\xi^\alpha$$

has the property that

(2.1) $$b_0^{-1}(\infty, x, \xi) = O(1)\,(1+|\xi|^{\mu})^{-1}\,,$$

uniformly on compact subsets of T.

The algebraic adjoint b^* of b is defined by the identity

$$\int b^* f(x) f'(x)\,dx = \int f(x)\, bf'(x)\,dx, \qquad (f, f' \in H(T))\,.$$

It is readily seen to have the same form as b itself,

$$b^*(\tau, x, D_x) = \sum_{|\alpha| \leqq \mu} b_\alpha^*(\tau, x)\, \tau^{-|\alpha|}\, D_x^{\alpha}\,,$$

where $b_\alpha^*(\tau, x)$ is a polynomial in τ^{-1} and $b_0^*(\infty, x, \xi)$ satisfies (2.1) since, in fact, we have

(2.2) $$b_0^*(\infty, x, \xi) = b_0(\infty, x, -\xi)\,.$$

Let U be an arbitrary open subset of T whose closure $\bar{U}$ is contained in T. When τ is large enough we are going to construct a fundamental solution Γ of the differential operator b, that is, a function $\Gamma(\tau, x, z)$ defined on $U \times U$ and having the property that

$$\int \Gamma(\tau, z, x)\, b^*(\tau, x, D) f(x)\,dx = f(z) \qquad (f \in H(U))\,.$$

The point z is called the pole of Γ. We shall establish the estimates

(2.3) $$\Gamma(\tau, z, x) = O(1)\tau^n\, e_0(\tau(x-z))\,(1+|\tau(x-z)|^N)^{-1}$$

(where $e_0(y) = |y|^{\mu-n-\varepsilon}$ when $\mu - n \leqq 0$, and $e_0(y) = 1$ otherwise) and

(2.4) $$D_x^{\alpha}\Gamma(\tau, z, x) = O(1)\tau^{|\alpha|+1-\varepsilon}|x-z|^{1-\varepsilon-n}(1+|\tau(x-z)|^N)^{-1} \qquad (|\alpha| < \mu)\,.$$

In these formulas $N \geqq 0$ and $1 > \varepsilon > 0$ are arbitrary, and the estimate $O(1)$ for large τ is uniform in $U \times U$. We shall also prove that, if $\mu > n$, then

(2.5) $$\lim_{\tau \to \infty} \tau^{-n}\Gamma(\tau, x, x) = (2\pi)^{-n} \int d\xi / b_0(\infty, x, \xi)\,,$$

uniformly on the diagonal $\Delta(U \times U)$ of $U \times U$. Precisely the same estimates hold for a similarly constructed fundamental solution $\Gamma^*(\tau, x, z)$ with pole in z defined in $U \times U$ for large τ and satisfying[5]

$$\int \Gamma^*(\tau, x, z)\, b(\tau, x, D) f(x)\,dx = f(z)$$

when f is in $H(U)$. For future reference we write them down:

(2.6) $$\Gamma^*(\tau, x, z) = O(1)\,\tau^n\, e_0(\tau(x-z))\,(1+|\tau(x-z)|^N)^{-1}\,,$$

[5] It is convenient to change the parts played by the last two variables in Γ and Γ^*.

(2.7) $D_x^\alpha \Gamma^\star(\tau, x, z) = O(1)\tau^{|\alpha|+1-\varepsilon}\,|x-z|^{1-\varepsilon-n}\left(1+|\tau(x-z)|^N\right)^{-1} \qquad (|\alpha| < \mu)$,

(2.8) $$\lim_{\tau\to\infty} \tau^{-n}\,\Gamma^\star(\tau, x, x) = (2\pi)^{-n}\int d\xi/b_0{}^\star(\infty, x, \xi)\,.$$

By virtue of (2.2), the right sides of (2.5) and (2.8) are the same.

We shall use the current parametrix method of Hilbert [5] and E. E. Levi [8]. The parametrix is a fundamental solution $B(\tau, z, x)$ of the differential operator $b(\tau, z, -D_x)$ with its pole at z. Put

(2.9) $$B(\tau, z, x) \sim (2\pi)^{-n}\int e^{-i(x-z)\xi}\, b^{-1}(\tau, z, \xi)\, d\xi \;\sim\; \tau^n\,(2\pi)^{-n}\int e^{-ix'\xi}\, b^{-1}(\tau, z, \tau\xi)\, d\xi$$

where $x' = \tau(x-z)$. By virtue of (2.1) the polynomial $b(\tau, z, \tau\xi)$ never vanishes if τ is large enough, so that

$$\int B(\tau, z, x)\, b(\tau, z, -D_x) f(x)\, dx = f(z) \qquad (f \in H(T))\,.$$

Also, by virtue of (2.1), the polynomial $b(\tau, z, \tau\xi)$ satisfies the requirements of Lemma 1 on compact subsets of T. Hence by Lemma 2,

$$D_{x'}{}^\alpha B(\tau, z, x) = O(1)\,\tau^n\,|x'|^{1-\varepsilon-n}\left(1+|x'|^N\right)^{-1} \qquad (|\alpha| < \mu)\,,$$

so that

(2.10) $D_x^\alpha B(\tau, z, x) = O(1)\,\tau^{|\alpha|+1-\varepsilon}|x-z|^{1-\varepsilon-n}\left(1+|\tau(x-z)|^N\right)^{-1} \qquad (|\alpha| < \mu)$.

Similarly

(2.11) $D_x^\alpha B(\tau, z, x) = O(1)\,\tau^{|\alpha|-\varepsilon}\,|x-z|^{-\varepsilon-n}\left(1+|\tau(x-z)|^N\right)^{-1} \qquad (|\alpha| \leqq \mu)$.

Both of these estimates are valid for arbitrary $N \geqq 0$ and $1 > \varepsilon > 0$, uniformly for sufficiently large τ, all x and all z on any compact part of T. Let us put

$$\beta(\tau, z, x) = \left(b(\tau, z, D_x) - b(\tau, x, D_x)\right) B(\tau, z, x)\,,$$

and let $u(\tau, z, x)$ be a solution of the integral equation

(2.12) $$u(\tau, z, x) - \int_U u(\tau, z, y)\,\beta(\tau, y, x)\, dy = \beta(\tau, z, x)\,.$$

Then

(2.13) $$\Gamma(\tau, z, x) = B(\tau, z, x) + \int_U u(\tau, z, y)\, B(\tau, y, x)\, dy$$

is a fundamental solution in $U\times U$ (see F. John [6]). We shall investigate the possibility of solving (2.12). Writing $\beta(\tau, z, x)$ more explicitly as

$$\Sigma\left(b_\alpha(\tau, z) - b_\alpha(\tau, x)\right)\tau^{-|\alpha|}\, D_x^\alpha B(\tau, z, x)$$

and using (2.10) and (2.11), we get

$$\beta(\tau, z, x) = O(1)\,\tau^{-\varepsilon}\,|x-z|^{1-\varepsilon-n}\,(1+|\tau(x-z)|^N)^{-1}$$

for arbitrary N and ε, uniformly when τ is large and x, z belongs to a compact part of $T\times T$. Hence the Neumann series of (2.12), namely

$$\beta(\tau, z, x) + \int_U \beta(\tau, z, y)\beta(\tau, y, x)dy + \dots ,$$

is majorized by

$$(1+|\tau(x-z)|^N)^{-1}\Big\{C\tau^{-\varepsilon}\,|x-z|^{1-\varepsilon-n} + (C\tau^{-\varepsilon})^2 \int_U |z-y|^{1-\varepsilon-n}\,|y-x|^{1-\varepsilon-n}dy + \dots\Big\},$$

where C is a constant. We have here used the simple inequality

$$(1+|\tau(z-y)|^N)^{-1}(1+|\tau(y-x)|^N)^{-1} \leqq (1+|\tau(z-x)|^N)^{-1} .$$

Hence, if τ is large enough, the integral equation may be solved by its Neumann series, and we get

$$u(\tau, z, x) = O(1)\,\tau^{-\varepsilon}|x-z|^{1-\varepsilon-n}(1+|\tau(x-z)|^N)^{-1} \tag{2.14}$$

for large τ, uniformly on $U\times U$. This estimate together with (2.13) proves the desired estimate (2.7), because, as it stands, we may clearly differentiate (2.13) with respect to x less than μ times. From (2.9) and Lemma 1 follows

$$B(\tau, z, x) = O(1)\tau^n e_0(\tau(x-z))\,(1+|\tau(x-z)|^N)^{-1}, \tag{2.15}$$

which together with (2.12) and (2.13) gives (2.6).

It remains to prove (2.8). From (2.9) and the properties of b it follows that

$$\lim_{\tau\to\infty} \tau^{-n}B(\tau, x, x) = (2\pi)^{-n}\int d\xi/b_0(\infty, x, \xi),$$

uniformly on any compact subset of $\Delta(T\times T)$. Now if $\mu > n$, (2.15) reads

$$B(\tau, z, x) = O(1)\tau^n\,(1+|\tau(x-z)|^N)^{-1}.$$

Combining this with the estimate (2.14) of u, we see that τ^{-n} times the integral in (2.13) is uniformly small on $U\times U$ if τ is large enough. Hence (2.8) follows. For Γ^* the construction and the proofs are the same.

At last we remark that according to the results of F. John, $\Gamma(\tau, z, x)$ is infinitely differentiable in $U\times U$ if $x \neq z$, and if f is in $H(U)$ then

$$\Gamma f(x) = \int \Gamma(\tau, z, x)f(z)dz \tag{2.16}$$

is infinitely differentiable in U. Moreover,

$$b(\tau, x, D_x)\Gamma(\tau, z, x) = 0 \qquad (x \neq z), \tag{2.17}$$

and

$$b(\tau, x, D_x)\Gamma f(x) = f(x) . \tag{2.18}$$

Analogous results hold for $\Gamma^\star(\tau, x, z)$ and

$$\Gamma^\star f(x) = \int \Gamma^\star(\tau, x, z) f(z) dz .$$

3. Estimates of a certain kernel. Let us use the notations of the preceding section, and let $\mathfrak{H}_0(T)$ be the set of all square integrable functions on T. Assume that a bilinear form

$$C(f, g) = C(\tau, f, g)$$

is given on $\mathfrak{H}_0 \times \mathfrak{H}_0$ and that it is uniformly bounded for large τ and satisfies the identity

$$C(bf, g) = C(f, b^\star g) = \int f(x) g(x) dx \tag{3.1}$$

on $H(T) \times H(T)$. Then we shall prove that C has a kernel $c(\tau, x, y)$,

$$C(f, g) = \int_{T\times T} c(\tau, x, y) f(x) g(y)\, dx dy \qquad (f, g \in H(T)),$$

which is infinitely differentiable when $x \neq y$ and satisfies

$$c(\tau, x, y) = O(1)\, \tau^n e_0\big(\tau(x-y)\big)\big(1 + |\tau(x-y)|^N\big)^{-1}, \tag{3.2}$$

where as usual

$$e_0(z) = |z|^{\mu-n-\varepsilon} \quad \text{when } \mu - n \leqq 0,$$
$$e_0(z) = 1 \quad \text{when } \mu - n > 0 .$$

The estimate $O(1)$ is uniform for large τ and compact subsets of $T \times T$, but it depends on the numbers N and ε which may be chosen except for the conditions $N \geqq 0$ and $1 > \varepsilon > 0$. When $\mu - n > 0$,

$$\lim_{\tau\to\infty} \tau^{-n} c(\tau, x, y) = (2\pi)^{-n} \delta_{xy} \int d\xi / b_0(\infty, x, \xi) , \tag{3.3}$$

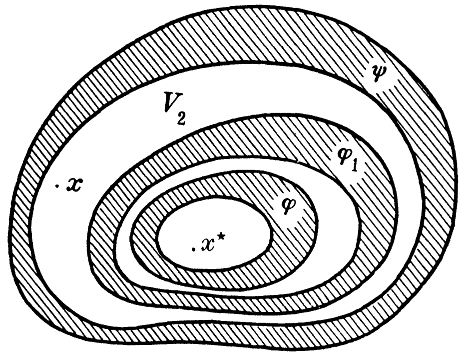
uniformly on compact subsets of $T \times T$, the symbol δ_{xy} denoting 0 when $x \neq y$ and 1 otherwise.

Let V be an arbitrary open subset of T whose closure $\overline{V}$ is contained in T, and choose three larger telescoping open subsets V_1, V_2 and V_3 such that $V \subset \overline{V} \subset V_1 \subset \overline{V}_1 \subset V_2 \subset \overline{V}_2 \subset V_3 \subset \overline{V}_3 \subset T$. Construct fundamental solutions Γ and $\Gamma^\star$ in $U = V_3$ satisfying (2.3) to (2.8). Let ψ be in $H(V_3)$ and let it be 1 on V_2. We want to prove that

$$\begin{aligned} \varrho(\tau, x, x^\star) &= \Gamma^\star(\tau, x, x^\star) - \Gamma(\tau, x, x^\star) \\ &= \int_{V_3 - V_2} \Gamma(\tau, x, z)\, b^\star(\tau, z, D_z)\big(\psi(z)\Gamma^\star(\tau, z, x)\big) dz . \end{aligned} \tag{3.4}$$

when x and $x^\star$ are in V_2. This formula in another form is due to F. John [6], but for the convenience of the reader we give a proof.

First let $x \neq x^\star$ and choose three open telescoping neighborhoods W, W_1 and W_2 of $x^\star$ such that $\overline{W} \subset W_1 \subset \overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset V_2$ and $\overline{W}_2$ does not contain x (see the figure).



The shaded rings are those where the corresponding functions are $\neq 0$ and $\neq 1$. Each function equals 1 inside its ring and 0 outside it.

Let $\varphi \in H(W_1)$ be 1 on W and $\varphi_1 \in H(W_2)$ be 1 on W_1. Put $f(z) = \Gamma(\tau, x, z)$ and $f^\star(z) = \Gamma^\star(\tau, z, x^\star)$. Then, by the properties of Γ and $\Gamma^\star$,

$$[f, b^\star((\psi-\varphi)f)] = f^\star(x) \qquad \left([f, g] = \int f(x)g(x)\,dx\right).$$

Since $b^\star f^\star = 0$ except at the point $z = x$, the left side can be written as

$$[f, b^\star(\psi f^\star)]+[f, b^\star(1-\varphi)f^\star]\,.$$

But

$$[f, b^\star(1-\varphi)f^\star] = [\varphi_1 f, b^\star(1-\varphi)f^\star] = [b\varphi_1 f, (1-\varphi)f^\star] = [b\varphi_1 f, f^\star] = f(x^\star),$$

and hence

$$f^\star(x)-f(x^\star) = [f, b^\star(\psi f^\star)]\,,$$

which is the formula (3.4). By continuity, the formula just proved is valid also when $x = x^\star$. This shows that ϱ is infinitely differentiable on $V_2 \times V_2$. Put

$$\Gamma^\star(\tau, f, g) = \int \Gamma^\star(\tau, x, x^\star)f(x)g(x^\star)\,dx\,dx^\star$$

and

$$C'(f, g) = C(f, g) - \Gamma^\star(\tau, f, g)\,.$$

By virtue of (2.4) with $|\alpha| = 0$, the norm

$$|C'| = \sup |C'(f, g)|\, |f|^{-1}|g|^{-1} \qquad \Big(|f|^2 = \int_{V_3} |f(x)|^2 dx; \quad f, g \in \mathfrak{H}_0(V_3)\Big)$$

of the bilinear form C' is $O(1)\tau^{1-\varepsilon}$. Moreover, by the properties of $\Gamma^\star$,

$$C'(bf, g) = [f, g] - [f, g] = 0 ,$$

and if $\Gamma(\tau, f, g)$ and $\varrho(\tau, f, g)$ are defined in analogy to the above definition of $\Gamma^\star(\tau, f, g)$,

$$C'(f, b^\star g) = C(f, b^\star g) - \Gamma(\tau, f, b^\star g) - \varrho(\tau, f, b^\star g) = -\varrho(\tau, f, b^\star g)$$

on $H(V_2)\times H(V_2)$. Put

$$p(x, z) = b(\tau, z, D_z)\big(\Gamma(\tau, x, z)(1-\varphi(z))\big) \tag{3.5}$$

and

$$p^\star(z, x^\star) = b(\tau, z, D_z)\big(\Gamma^\star(\tau, z, x^\star)(1-\varphi(z))\big) , \tag{3.6}$$

where $\varphi \in H(V_2)$ and equals 1 on V_1. For given x and $x^\star$ in V_1, the functions $p(x, z)$ and $p^\star(z, x^\star)$ vanish except on the ring $V_2 - V_1$. Put further

$$c'(\tau, x, x^\star) = C'\big(p(x,.), p^\star(., x^\star)\big) . \tag{3.7}$$

By the continuity of p, $p^\star$ and C',

$$c'(\tau, f, g) = \int c'(\tau, x, x^\star) f(x) g(x^\star)\, dx\, dx^\star = C'(f - b\varphi\Gamma f, g - b^\star\varphi\Gamma^\star g) ,$$

where f and g are in $H(V)$,

$$\Gamma f(z) = \int \Gamma(\tau, x, z) f(x)\, dx ,$$

and

$$\Gamma^\star g(z) = \int \Gamma^\star(\tau, z, x^\star) g(x^\star)\, dx^\star .$$

By the properties of C',

$$c'(\tau, f, g) = C'(f, g) + \varrho(\tau, f, b^\star\varphi\Gamma^\star g) .$$

Now

$$\begin{aligned} b^\star\varphi\Gamma^\star g(z) &= b^\star(\tau, z, D_z)\Big(\varphi(z)\int \Gamma^\star(\tau, z, x^\star) g(x^\star)\, dx^\star\Big) \\ &= g(z) + \int b^\star(\tau, z, D_z)\big(\varphi(z)\Gamma^\star(\tau, z, x^\star)\big)\, g(x^\star)\, dx^\star, \end{aligned}$$

where the kernel

$$\sigma(\tau, z, x^\star) = b^\star(\tau, z, D_z)\big(\varphi(z)\Gamma^\star(\tau, z, x^\star)\big) \tag{3.8}$$

is different from zero only when z is in $V_2 - V_1$. Hence

$$C(f, g) = \Gamma^\star(\tau, f, g) + c'(\tau, f, g) + \varrho(\tau, f, g) + \lambda(\tau, f, g) ,$$

where

$$\lambda(\tau, x, x^\star) = \int_{V_2 - V_1} \varrho(\tau, x, z)\sigma(\tau, z, x^\star)\, dz . \tag{3.9}$$

Consider now the functions $p(x, z)$ and $p^\star(z, x^\star)$. Since

$$b(\tau, z, D_z)\Gamma(\tau, x, z) = 0$$

when $x \neq z$, and

$$b^\star(\tau, z, D_z)\Gamma^\star(\tau, z, x^\star) = 0$$

when $x^\star \neq z$, only derivatives of orders $< \mu$ of Γ and $\Gamma^\star$ really enter into p and $p^\star$, respectively. Hence by (2.4)

$$p(x, z) = O(1)\,\tau^{1-\varepsilon}r^{1-\varepsilon-n}\big(1+|\tau r|^N\big)^{-1},$$

where r is the distance from x to V_1, and analogously,

$$p(z, x) = O(1)\,\tau^{1-\varepsilon}r^{\star\,1-\varepsilon-n}\big(1+|\tau r^\star|^N\big)^{-1},$$

where $r^\star$ is the distance from $x^\star$ to V_1. Hence

(3.10) $$p(x, z) = O(1)\,\tau^{1-\varepsilon-N}$$

and

(3.11) $$p^\star(z, x^\star) = O(1)\,\tau^{1-\varepsilon-N}$$

uniformly on $V\times(V_2-V_1)$ and $(V_2-V_1)\times V$, respectively. In a similar fashion we infer from (3.4) that

(3.12) $$\varrho(\tau, x, z) = O(1)\tau^{2(1-\varepsilon)-N}$$

uniformly on $V_1\times V_2$, and

(3.13) $$\varrho(\tau, x, x^\star) = O(1)\tau^{2(1-\varepsilon-N)}$$

uniformly on $V_1\times V_1$. Further,

$$\sigma(\tau, z, x) = O(1)\tau^{1-\varepsilon-N}$$

uniformly on $V_2\times V_1$. Hence by (3.9),

(3.14) $$\lambda(\tau, x, x^\star) = O(1)\tau^{3(1-\varepsilon)-2N}$$

uniformly on $V_1\times V_1$. Combining (3.10) and (3.11) with the estimate $O(1)\,\tau^{1-\varepsilon}$ of $|C'|$, we get

(3.15) $$c'(\tau, x, x^\star) = O(1)\tau^{3(1-\varepsilon)-2N}$$

uniformly on $V\times V$. Combining (3.13), (3.14), and (3.15) it follows that $C(f, g)$ has a kernel

$$c(\tau, x, x^\star) = \Gamma^\star(\tau, x, x^\star) + O(1)\tau^{3(1-\varepsilon)-2N},$$

$O(1)$ being uniform on $V\times V$. Hence the desired formulas (3.2) and (3.3) follow from (2.6) and (2.8), respectively.

4. Estimates of Green's function. In the introduction, Green's transformation G_t was defined for sufficiently large t by

$$(f, f') = a_t(G_t f, f'), \tag{4.1}$$

where $f \in \mathfrak{H}_0(S)$ and $G_t f, f' \in \mathfrak{H}(S)$. It is clear that $\|f\|_t^2 \geqq t|f|^2$ and hence by (0.3),

$$tc^{-1}|G_t f|^2 \leqq c^{-1}\|G_t f\|_t^2 \leqq |a_t(G_t f, G_t f)| \leqq |f||G_t f|,$$

so that

$$|G_t f| \leqq ct^{-1}|f|.$$

Considering G_t as an operator from $\mathfrak{H}_0$ to $\mathfrak{H}_0$ we therefore have

$$|G_t| \leqq ct^{-1}.$$

We have already shown that if f is in $H(S)$ then

$$G_t a_t f = f. \tag{4.2}$$

Let a_t^* be the complex conjugate adjoint of a_t defined by

$$(a_t f, f') = (f, a_t^* f') \qquad (f, f' \in H(S)).$$

If $f \in \mathfrak{H}(S)$ and $f' \in H(S)$, we then have

$$(G_t f, a_t^* f') = a_t(G_t f, f') = (f, f'). \tag{4.3}$$

Let k be a positive integer and put

$$C(f, f') = (G_t^k f, \bar{f}')t^k$$

and $b = t^{-k}a_t^k$. Then $C(f, f')$ is bilinear and bounded,

$$|C(f, f')| \leqq c|f||f'|,$$

and by virtue of (4.1),

$$C(bf, f') = (G_t^k a_t^k f, \bar{f}') = (f, \bar{f}') \qquad (f, f' \in H(S)).$$

If $b^\star$ is the algebraic adjoint of b, it follows from (4.3) that

$$C(f, b^\star f') = (G_t^k f, a_t^{*k}\bar{f}') = (f, \bar{f}') \qquad (f, f' \in H(S)).$$

Moreover, putting $\tau^{2m} = t$, it is clear that

$$b = b(\tau, x, D_x) = \Sigma\, b_\alpha(\tau, x)\tau^{-|\alpha|}D_x^\alpha$$

has the property (2.1). Hence, applying the results of the preceding section we see that G_t^k has a continuous kernel $g_t^{(k)}(x, y)$ with the property

$$\lim_{t\to\infty} t^{k-\nu}g_t^{(k)}(x, y) = \delta_{xy}(2\pi)^{-n}\int \big(a_0(x, \xi)+1\big)^{-k}d\xi, \tag{4.4}$$

provided that $2mk > n \qquad (\nu = n/(2m))$.

If a_t is self-adjoint, that is, if $a_t^* = a_t$, then $a_t(f,f) = (a_t f, f) = (f, a_t^* f)$ is real when $f \in H(S)$. Hence $a_t(f,f)$ is real when $f \in \mathfrak{H}(S)$ and consequently $(G_t f, f) = a_t(G_t f, G_t f)$ is real when $f \in \mathfrak{H}_0$. By virtue of (0.3) also $(G_t f, f) \geqq 0$ when t is large enough, and hence G_t is a self-adjoint positive transformation. But then $g_t^{(k)}(x, x) \geqq 0$, and Fatou's theorem gives

$$\underline{\lim}\, t^{k-\nu} \int_S g_t^k(x, x)\, dx \geqq (2\pi)^{-n} \int_S dx \int \big(a_0(x, \xi) + 1\big)^{-k} d\xi\,. \tag{4.5}$$

It remains to prove the converse inequality.

Let $\dot{S}$ be an open set in T containing the closure of S, and let $\dot{G}_t$ be Green's transformation corresponding to $\dot{S}$. By virtue of (4.1) we have

$$(G_t f, f) = a_t(G_t f, G_t f) = \sup |(f, g)|^2 / a_t(g, g) \qquad (g \in \mathfrak{H}(S)\,.$$

Since $\mathfrak{H}(\dot{S})$ contains $\mathfrak{H}(S)$ this means that

$$(G_t f, f) \leqq (\dot{G}_t f, f) = (E\dot{G}_t f, f) \qquad (f \in \mathfrak{H}_0(S))\,,$$

where E is the projection of $\mathfrak{H}_0(\dot{S})$ upon $\mathfrak{H}_0(S)$. Hence all the eigenvalues of G_t taken in descending order are less than or equal to the corresponding eigenvalues of the restriction $\Gamma_t = E\dot{G}_t E$ of $\dot{G}_t$ to $\mathfrak{H}_0(S)$. Hence

$$\operatorname{tr} G_t^k \leqq \operatorname{tr} \Gamma_t^k$$

for all k. We want to obtain an estimate for the right side.

To begin with, it is clear that the bilinear forms $(\dot{G}_t^k f, \bar{f}')t^k$ and $(\Gamma_t^k f, \bar{f}')t^k$ both satisfy the requirements of the preceding section with respect to the differential operator $b = t^{-k} a_t^k$ and the regions $\dot{S}$ and S, respectively. In particular, Γ_t^k has a kernel $\gamma_t^{(k)}(x, y)$ which is continuous when $2mk > n$ and satisfies

$$\lim t^{k-\nu} \gamma_t^{(k)}(x, x) = (2\pi)^{-n} \int \big(a_0(x, \xi) + 1\big)^{-k} d\xi\,. \tag{4.6}$$

The kernel $\gamma_t(x, y)$ of Γ_t is the restriction of the kernel $\dot{g}_t(x, y)$ of $\dot{G}_t$ to S, and hence from (3.2) we get the estimate

$$\gamma_t(x, y) = t^{\nu-1} O(1)\, e\big(\tau(x-y)\big)\big(1 + |\tau(x-y)|^N\big)^{-1}$$

with arbitrary $N \geqq 0$ and $1 > \varepsilon > 0$, uniformly on $S \times S$ for large t ($\tau^{2m} = t$). Here $e(x) = 1$ if $2m - n > 0$, and $e(x) = |x|^{2m-n-\varepsilon}$ otherwise. By virtue of this estimate and Fubini's theorem, the kernel of Γ_t^k is

$$\gamma_t^{(k)}(x, y) = \int_{S^{k-1}} \gamma_t(x, y_1)\, \gamma_t(y_1, y_2) \cdots \gamma_t(y_{k-1}, y)\, dy_1 \cdots dy_{k-1}\,.$$

Putting $y = x$ and $y_j = x + \tau^{-1} z_j$ we get

$$0 \leqq t^{k-\nu}\gamma_t^{(k)}(x, x)$$
$$\leqq O(1)\int e(z_1)e(z_1-z_2)\ldots e(z_{k-1})\left(1+|z_1|^N\right)^{-1}\left(1+|z_1-z_2|^N\right)^{-1}\ldots$$
$$\left(1+|z_{k-1}|^N\right)^{-1} dz_1 \ldots dz_{k-1}.$$

If $2mk > n$, we can make the right side finite by choosing N large enough and ε so small that $2mk-\varepsilon k > n$. Combining this formula with (4.7) and applying Lebesgue's theorem it follows that

$$\lim t^{k-\nu}\int_S \gamma_t^{(k)}(x, x)\,dx = (2\pi)^{-n}\int_S dx\int \left(a_0(x, \xi)+1\right)^{-k} d\xi\,.$$

Since $\gamma_t^{(k)}(x, y)$ is a continuous and positive kernel, the integral on the left side is the trace of Γ_t^k, and hence

$$\overline{\lim}\, t^{k-\nu} \operatorname{tr} G_t^k \leqq (2\pi)^{-n}\int_S dx\int \left(a_0(x, \xi)+1\right)^{-k} d\xi\,.$$

It follows form this formula and (4.5) that

$$\lim_{t\to\infty} t^{k-\nu} \operatorname{tr} G_t^k = (2\pi)^{-n}\int_S dx\int \left(a_0(x, \xi)+1\right)^{-k} d\xi \tag{4.7}$$

when $2mk > n$.

Since G_t is self-adjoint and positive, there exists a complete system $\varphi_1, \varphi_2, \ldots$ of eigenfunctions of G_t (t fixed) with positive eigenvalues $(\lambda_1+t)^{-1} \geqq (\lambda_2+t)^{-1} \geqq \ldots$. In view of the properties of G_t, a necessary and sufficient condition that $\varphi \in \mathfrak{H}_0$ and $G_t\varphi = (\lambda+t)^{-1}\varphi$ is that $\varphi \in \mathfrak{H}$ and that $(\lambda+t)(\varphi, f) = a_t(\varphi, f)$ for all f in $\mathfrak{H}$. Hence $G_t\varphi = (\lambda+t)^{-1}\varphi$ implies $G_s\varphi = (\lambda+s)^{-1}\varphi$ and conversely if t and s are large enough. Hence

$$G_t\varphi_j = (\lambda_j+t)^{-1}\varphi_j$$

for all t. Moreover, $(\lambda_j+t)(\varphi_j, f) = a_t(\varphi_j, f)$ means that $\left(\varphi_j, (\lambda_j-a)f\right) = 0$ when f is in $H(S)$, and consequently by Schwartz's theorem on weak solutions of elliptic differential equations [10] and John's construction of a fundamental solution, φ_j is infinitely differentiable and

$$a\varphi_j = \lambda_j\varphi_j\,.$$

Suppose now for a moment that we have shown that

$$g_t^{(k)}(x, y) = \Sigma\,(\lambda_j+t)^{-k}\overline{\varphi_j(x)}\,\varphi_j(y) \tag{4.8}$$

when $2mk > n$. Then

$$\int_S g_t^{(k)}(x, x)\,dx = \Sigma\,(\lambda_j+t)^{-k} = \operatorname{tr} G_t^k\,. \tag{4.9}$$

Let us consider the integral

$$w_a^{(k)}(x) = (2\pi)^{-n}\int\bigl(a_0(x,\xi)+1\bigr)^{-k}d\xi\,.$$

Introducing polar coordinates in the integral by the formula $d\xi = d\varrho^n d\omega$, where $\varrho^{2m} = a_0(x, \xi)$, we get, since $d\omega(\xi)$ is homogeneous of order zero,

$$w_a(x) = \int\limits_{a_0(x,\xi)<1} d\xi = \int\limits_0^1 d\varrho^n \int d\omega = \int d\omega\,,$$

and consequently

$$w_a^{(k)}(x) = w_a(x)\,(2\pi)^{-n}\int\limits_0^\infty (\varrho^{2m}+1)^{-k}d\varrho^n =$$

$$= w_a(x)\,(2\pi)^{-n}\Gamma(\nu+1)\,\Gamma(k-\nu)\bigl(\Gamma(k)\bigr)^{-1}.$$

Hence, by (4.4) and (4.8)

$$\Sigma\,(\lambda_j+t)^{-k}\overline{\varphi_j(x)}\varphi_j(y) = (2\pi)^{-n}\delta_{xy}w_a(x)\,\Gamma(\nu+1)\Gamma(k-\nu)\bigl(\Gamma(k)\bigr)^{-1}t^{\nu-k}\bigl(1+o(1)\bigr)\,. \tag{4.10}$$

Putting $x = y$ and integrating, we obtain by virtue of (4.9) and (4.7),

$$\Sigma\,(\lambda_j+t)^{-k} = (2\pi)^{-n}w_a(S)\,\Gamma(\nu+1)\Gamma(k-\nu)\bigl(\Gamma(k)\bigr)^{-1}t^{\nu-k}\bigl(1+o(1)\bigr),$$

where $w_a(S) = \int_S w_a(x)\,dx$.

By application of a Tauberian theorem of Hardy and Littlewood [4] in the formulation of Pleijel [9], we arrive at the desired formula (0.6) of the introduction. Applying the same theorem to (4.4) with $x = y$ we get

$$\sum_{\lambda_j\leq t}|\varphi_j(x)|^2 = (2\pi)^{-n}w_a(x)\,t^\nu\bigl(1+o(1)\bigr)\,, \tag{4.11}$$

and applying it to

$$\Sigma\,(\lambda_j+t)^{-k}|\varphi_j(x)+\theta\varphi_j(y)|^2 = (2\pi)^{-n}\bigl(w_a(x)+w_a(y)\bigr)\,\Gamma(\nu+1)\Gamma(k-\nu)\bigl(\Gamma(k)\bigr)^{-1}t^{\nu-k}\bigl(1+o(1)\bigr)$$

($|\theta| = 1$), which when $x \neq y$ follows from (4.4), we obtain

$$\sum_{\lambda_j\leq t}|\varphi_j(x)+\theta\varphi_j(y)|^2 = (2\pi)^{-n}\bigl(w_a(x)+w_a(y)\bigr)\,t^\nu\bigl(1+o(1)\bigr)\qquad(|\theta| = 1)\,. \tag{4.12}$$

The validity of the formulas (4.11), (4.12) and (0.5) prove the desired formula (0.7).

It remains to prove that (4.8) holds. Consider the kernel $g_t^{(k)}(x, y)$. Locally it is $O(1)$ in $S\times S$ if $2mk > n$ and $O(1)\,|x-y|^{2mk-n-\varepsilon}$ if $2mk \leq n$. Hence, if $2mk-n-\varepsilon > -\frac{1}{2}n$, that is, if $2mk > \frac{1}{2}n+\varepsilon$, the integral

$$\int\limits_V |g_t^{(k)}(x, y)|^2\,dy$$

is finite provided that $\bar{V}$ is contained in S. We want to prove that also

$$\int_S |g_t^{(k)}(x, y)|^2 dy < \infty . \tag{4.13}$$

Let us put $C(f, g) = t^k (G_t^k f, \bar{g})$ and $b = t^{-k} a_t^k$, and apply the methods of the preceding section. Let U be an open subset of S whose closure $\bar{U}$ is contained in S and choose another open set U_1 such that $\bar{U} \subset U_1 \subset \bar{U}_1 \subset S$. Let $\eta \in H(U_1)$ be 1 on U and put with large $\tau = t^{1/(2m)}$

$$q(x, z) = b(\tau, z, D_z)\big(\Gamma(\tau, x, z)(1-\eta(z))\big), \qquad x \in U ,$$

and

$$r(x) = C\big(q(x, \cdot), g\big) , \qquad x \in U ,$$

where $g \in H(S-\bar{U}_1)$. If $f \in H(U)$, it follows from the properties of C and Γ that

$$\begin{aligned}\int r(x) f(x) dx &= C\big(q(f,.), g\big) = C\big(b(\Gamma(\tau, f,.)(1-\eta(.)), g\big) \\ &= C(f, g) - \big(\Gamma(\tau, f,.)\eta(.), g\big) = C(f, g) .\end{aligned}$$

Now the bilinear form C has a kernel $c(\tau, x, z)$ so that the last result can be written in the form

$$\int r(x) f(x) dx = \int c(\tau, x, z) f(x) g(z) dx dz .$$

Since f is arbitrary in $H(U)$, r is continuous and c continuous when $x \neq z$, we get

$$C\big(q(x,.), g\big) = \int c(\tau, x, z) g(z) dz$$

when $x \in \bar{U}$ and $g \in H(S-\bar{U}_1)$. This proves that

$$\int_{S-U_1} |c(\tau, x, z)|^2 dz \leqq |C|^2 |q(x,.)|^2$$

when x is in U. Because $g_t^{(k)}(x, y) = t^{-k} c(\tau, x, y)$, the formula (4.13) follows and we also see that $g_t^{(k)}(x,.)$ considered as an element of $\mathfrak{H}_0$ is uniformly continuous in x on compact subsets of S. Now by Fubini's theorem and the properties of G_t,

$$\begin{aligned}(\lambda_j+t)^{-k}(f, \varphi_j) = (G_t^k f, \varphi_j) &= \int\left\{\int g_t^{(k)}(x, z) f(x) dx\right\} \overline{\varphi_j(z)} dz \\ &= \int\left\{\int g_t^{(k)}(x, z) \overline{\varphi_j(z)} dz\right\} f(x) dx\end{aligned}$$

when $f \in H(S)$. This means that

$$\int g_t^{(k)}(x, z) \overline{\varphi_j(z)}\, dz = (\lambda_j+t)^{-k} \overline{\varphi_j(x)} ,$$

both sides being continuous in x. Hence, by Parseval's formula,

$$(4.14) \qquad \left(g_t^{(k)}(x,.), g_t^{(k)}(y,.)\right) = \Sigma\, (\lambda_j+t)^{-2k}\, \overline{\varphi_j(x)}\, \varphi_j(y) \, .$$

By Fubini's theorem

$$\int f(x)\overline{f(y)}\, dx\, dy \int g_t^{(k)}(x, z)\, \overline{g_t^{(k)}(y, z)}\, dz$$
$$= \int dz \left\{\int g_t^{(k)}(x, z) f(x)\, dx\right\} \overline{\left\{\int g_t^{(k)}(y, z) f(y)\, dy\right\}} = (G_t^k f, G_t^k f)$$
$$= (G_t^{2k} f, f) = \int g_t^{(2k)}(x, y) f(x) \overline{f(y)}\, dx\, dy$$

when f is in $H(S)$. Hence the left side of (4.14) equals $g_t^{(2k)}(x, y)$, and since $\varepsilon > 0$ is arbitrary, this proves the desired formula (4.8) when $2mk > n$.

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