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CONGRUOUS AND INCONGRUOUS ONE-TO-ONE CORRESPONDENCES

FREDERICK BAGEMIHL

In this note we first prove a general theorem concerning one-to-one correspondences between a set and itself, relative to decompositions of the set into subsets, and then obtain some related results dealing with more special correspondences involving real or rational numbers.

Let Γ be a one-to-one correspondence defined between the elements of the (not necessarily distinct) sets M and N, or a one-to-one transformation defined on M and mapping M onto N, and let $m \in M$ and $n \in N$ be (not necessarily distinct) corresponding elements under Γ. If $m \neq n$, we call m a free element of M, under Γ. There are obviously the same number of free elements, under Γ, in M as in N. We say that Γ has $\mathfrak{k}$ free elements, if the cardinal number of the set of free elements in M, under Γ, is $\mathfrak{k}$. Let $\mathfrak{K}$ be an arbitrary class of mutually exclusive sets, with $|\mathfrak{K}| \geqq 2$, and let $\mathfrak{p} \geqq 1$. If m and n belong to the same element of $\mathfrak{K}$, we call them a congruous pair of elements relative to $\mathfrak{K}$; if they belong to distinct element of $\mathfrak{K}$, we call them an incongruous pair of elements relative to $\mathfrak{K}$. We say that Γ is

(a) congruous,
(b) $\mathfrak{p}$-congruous,
(c) at least $\mathfrak{p}$-incongruous,
(d) $\mathfrak{p}$-incongruous,

relative to $\mathfrak{K}$, according as the following conditions are satisfied:

(a) all pairs of corresponding elements are congruous relative to $\mathfrak{K}$;
(b) Γ is congruous relative to $\mathfrak{K}$, and every element of $\mathfrak{K}$ contains exactly $\mathfrak{p}$ pairs of corresponding elements;
(c) if $X \in \mathfrak{K}$, $Y \in \mathfrak{K}$, $X \neq Y$, then there are at least $\mathfrak{p}$ incongruous pairs of corresponding elements relative to the class $\{X, Y\}$;
(d) if $X \in \mathfrak{K}$, $Y \in \mathfrak{K}$, $X \neq Y$, then there are exactly $\mathfrak{p}$ incongruous pairs of corresponding elements relative to the class $\{X, Y\}$.

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A $\mathfrak{p}$-decomposition ($\mathfrak{p} \geqq 1$) of a set E is a class, $\mathfrak{P}$, of $\mathfrak{p}$ nonempty mutually exclusive sets whose union is E.

C denotes an ordered set whose order type is that of the real numbers in their natural order. R denotes an ordered set whose order type is that of the rational numbers in their natural order, except in Theorem 5, where R stands for the set of rational points on a straight line.

By an interval of an ordered set S we mean either S itself, or the subset of S preceding some element of S, or the subset of S succeeding some element of S, or the subset of S between two distinct elements of S. A subset, S', of S is dense in S provided that every interval of S contains an element of S'.

THEOREM 1. *Let $\mathfrak{G}$ be a class of $\aleph_\alpha$ one-to-one correspondences each of which has at least $\aleph_\alpha$ free elements, E be the union of the sets between which the elements of $\mathfrak{G}$ are defined, and $\mathfrak{p}$ be a cardinal number with $2 \leqq \mathfrak{p} \leqq \aleph_\alpha$. Then there exists a $\mathfrak{p}$-decomposition, $\mathfrak{P}$, of E such that every element of $\mathfrak{G}$ is at least $\aleph_\alpha$-incongruous relative to $\mathfrak{P}$.*

PROOF: Consider $\aleph_\alpha$ replicas of every element of $\mathfrak{G}$, well-order the resulting complex of $\aleph_\alpha{}^2 = \aleph_\alpha$ one-to-one correspondences to form a sequence

(1) $$\Gamma_0, \Gamma_1, \ldots, \Gamma_\xi, \ldots \qquad (\xi < \omega_\alpha)\,,$$

and denote by M_ξ, N_ξ the (not necessarily distinct) sets between which Γ_ξ $(\xi < \omega_\alpha)$ is defined. Let ϱ be the smallest ordinal number such that $|\varrho| = \mathfrak{p}$. The power of the set of all ordered pairs, (γ, δ), of ordinal numbers with $\gamma < \delta < \varrho$, is not greater than $\aleph_\alpha$. Consider $\aleph_\alpha$ replicas of each of these ordered pairs, and well-order the resulting complex of $\aleph_\alpha$ ordered pairs to form a sequence

(2) $$p_0, p_1, \ldots, p_\xi, \ldots \qquad (\xi < \omega_\alpha)\,.$$

Let $m_0 \in M_0$, $n_0 \in N_0$ be free corresponding elements under Γ_0. Suppose that $0 < \xi < \omega_\alpha$, and that elements $m_\sigma \in M_\sigma$, $n_\sigma \in N_\sigma$ have been defined for every $\sigma < \xi$. If we put $V_\xi = \{m_\sigma\}_{\sigma<\xi} \cup \{n_\sigma\}_{\sigma<\xi}$, then $|V_\xi| < \aleph_\alpha$. Let $m_\xi \in M_\xi - V_\xi$, $n_\xi \in N_\xi - V_\xi$ be free corresponding elements under Γ_ξ; they exist because Γ_ξ, by hypothesis, has at least $\aleph_\alpha$ free elements. The set $V_{\omega_\alpha} = \{m_\xi\}_{\xi<\omega_\alpha} \cup \{n_\xi\}_{\xi<\omega_\alpha}$ is thus defined by transfinite induction.

We express V_{ω_α} as the union of $\mathfrak{p}$ mutually exclusive sets B_π $(\pi < \varrho)$, as follows: Suppose that $\xi < \omega_\alpha$. The terms of (1) which are identical with Γ_ξ but whose subscripts are less than ξ, form a subsequence, of some order type $\tau_\xi < \omega_\alpha$, of (1). If $p_{\tau_\xi} = (\gamma, \delta)$, assign m_ξ to B_γ and n_ξ to B_δ.

Express $E-(\{m_\xi\}_{\xi<\omega_\alpha} \cup \{n_\xi\}_{\xi<\omega_\alpha})$ in an arbitrary manner as the union of $\mathfrak{p}$ mutually exclusive sets B_π' $(\pi < \varrho)$, and put $A_\pi = B_\pi \cup B_\pi'$ $(\pi < \varrho)$. The class $\mathfrak{P} = \{A_\pi\}_{\pi<\varrho}$ is a $\mathfrak{p}$-decomposition of E.

Now let $\Gamma \in \mathfrak{G}$, and suppose that A_γ and A_δ $(\gamma < \delta < \varrho)$ are any two distinct elements of $\mathfrak{P}$. This Γ appears $\aleph_\alpha$ times in (1); let

$$\Gamma_{\mu_0}, \Gamma_{\mu_1}, \ldots, \Gamma_{\mu_\xi}, \ldots \qquad (\xi < \omega_\alpha)$$

be the subsequence of (1), whose terms are identical with Γ. The ordered pair (γ, δ) appears $\aleph_\alpha$ times in (2); let

$$p_{\nu_0}, p_{\nu_1}, \ldots, p_{\tau_\xi}, \ldots \qquad (\xi < \omega_\alpha)$$

be the subsequence of (2), whose terms are identical with (γ, δ). According to the definition of the sets B_π $(\pi < \varrho)$, we have $m_{\mu_{\nu_\xi}} \in B_\gamma$, $n_{\mu_{\nu_\xi}} \in B_\delta$ $(\xi < \omega_\alpha)$, which means that Γ is at least $\aleph_\alpha$-incongruous relative to the class $\{A_\gamma, A_\delta\}$. This completes the proof of Theorem 1.

THEOREM 2. *Let* $2 \leqq \mathfrak{p} \leqq 2^{\aleph_0}$. *Then there exists a* $\mathfrak{p}$*-decomposition,* $\mathfrak{P}$, *of* C, *such that every antisimilarity or nonidentical similarity between two (not necessarily distinct) intervals of* C *is* $2^{\aleph_0}$*-incongruous relative to* $\mathfrak{P}$.

PROOF: Let $\mathfrak{G}$ be the class of all antisimilarities and nonidentical similarities between all pairs of (not necessarily distinct) intervals of C. There are $2^{\aleph_0}$ intervals of C, and $2^{\aleph_0}$ antisimilarities and nonidentical similarities defined between every pair of these intervals. It follows that $|\mathfrak{G}| = 2^{\aleph_0}$. Every element of $\mathfrak{G}$, since it is not the identity, has a whole interval of free elements. If we now put $\aleph_\alpha = 2^{\aleph_0}$ and $E = C$, then the hypotheses of Theorem 1 are satisfied, and Theorem 2 is an immediate consequence of the conclusion of Theorem 1.

Let R' be a set which is similar to R, R be the union of the mutually exclusive sets A_ν $(\nu < \varrho, 1 \leqq \varrho \leqq \omega)$ each of which is dense in R, and R' be the union of the mutually exclusive sets A_ν' $(\nu < \varrho)$ each of which is dense in R'. According to Skolem [1, pp. 30–36], there exists a similarity mapping, Γ, of R onto R', such that $\Gamma(A_\nu) = A_\nu'$ for every $\nu < \varrho$. Let B_0 be a subset of A_0 which is cofinal with A_0 and has order type ω. One can obtain $2^{\aleph_0}$ subsets of A_0' each of which is cofinal with A_0' and has order type ω. Let B_0' denote one of these subsets of A_0'. Then it is not difficult to see that the Γ in Skolem's theorem can be chosen so as to have the additional property that $\Gamma(B_0) = B_0'$. Furthermore, each choice of B_0' leads to a different Γ. Thus, as an immediate consequence of Skolem's theorem, we have

THEOREM 3. *Let* $\mathfrak{P}$ *be a* $\mathfrak{p}$*-decomposition of* R*, where* $1 \leqq \mathfrak{p} \leqq \aleph_0$*, such that every element of* $\mathfrak{P}$ *is dense in* R*. Then, between any two (not necessarily distinct) intervals of* R*, there exist* $2^{\aleph_0}$ *similarities which are* $\aleph_0$*-congruous relative to* $\mathfrak{P}$.

Theorem 3 may no longer hold if we drop the condition that every element of $\mathfrak{P}$ be dense in R. To see this, it suffices to well-order the elements of R to form a sequence, $r_0, r_1, \ldots, r_\nu, \ldots$ ($\nu < \omega$), put $A_\nu = \{r_\nu\}$ for every $\nu < \omega$, and take $\mathfrak{P} = \{A_\nu\}_{\nu<\omega}$. Then every similarity (except the identity) defined on any interval of R is 1-incongruous relative to some infinite subclass (depending on the similarity) of $\mathfrak{P}$. If, however, $\mathfrak{p}$ is finite, then it is possible to obtain the following result:

THEOREM 4. *Let* $\mathfrak{P}$ *be a* $\mathfrak{p}$*-decomposition of* R*, where* $1 \leqq \mathfrak{p} < \aleph_0$*. Then, between every interval of* R *and itself, there exist* $2^{\aleph_0}$ *similarities which are congruous relative to* $\mathfrak{P}$.

PROOF: Let I be an arbitrary interval of R. Since R is the union of the finitely many elements of $\mathfrak{P}$, there exists an interval, I_1, of I such that, for some element, say A_1, of $\mathfrak{P}$, $A_1 \cap I_1$ is dense in I_1. Let $A_2, A_3, \ldots, A_\varrho$, where $|\varrho| = \mathfrak{p}$, be the remaining elements of $\mathfrak{P}$, in case $\mathfrak{p} > 1$. We define an interval I_ϱ, by induction, as follows: Suppose that $1 \leqq \nu < \varrho$ and that the intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_\nu$ have been defined so that, if $1 \leqq \tau \leqq \nu$, $A_\tau \cap I_\nu$ is either dense in I_ν or empty. If there exists an interval, J, of I_ν such that $A_{\nu+1} \cap J = 0$, we define $I_{\nu+1}$ to be J; otherwise, we put $I_{\nu+1} = I_\nu$, in which case $A_{\nu+1} \cap I_{\nu+1}$ is dense in $I_{\nu+1}$. Let $B_\nu = A_\nu \cap I_\varrho$ ($1 \leqq \nu \leqq \varrho$); evidently B_ν is either dense in I_ϱ or empty, and B_1, certainly, is dense in I_ϱ. According to Theorem 3, there exist $2^{\aleph_0}$ similarities between I_ϱ and itself, which are congruous relative to that subclass of $\mathfrak{P}$ consisting of those elements A_ν for which the corresponding set B_ν is not empty. Each of these similarities when extended to I so as to be the identity on $I - I_\varrho$, is congruous relative to $\mathfrak{P}$, and hence the proof of the theorem is complete.

Let R be the set of rational points on a straight line. There are $\aleph_0$ displacements between R and itself, and at most 2 displacements between two (not necessarily distinct) congruent intervals of R if these intervals are different from R. If such a displacement is not the identity, it has $\aleph_0$ free elements. A simple application of Theorem 1 yields

THEOREM 5. *If* $2 \leqq \mathfrak{p} \leqq \aleph_0$*, then there exists a* $\mathfrak{p}$*-decomposition,* $\mathfrak{P}$*, of* R*, such that every nonidentical displacement between two (not necessarily distinct) congruent intervals of* R *is* $\aleph_0$*-incongruous relative to* $\mathfrak{P}$.

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INSTITUTE FOR ADVANCED STUDY PRINCETON, N. J., U.S.A.