TWO SUMMATION FORMULAE FOR PRODUCT SUMS OF BINOMIAL COEFFICIENTS

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It is well known that the formula

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n} \tag{1}$$

holds for arbitrary a and b and integral n. A simple way of proving it, is to consider the relation

$$(1+x)^a(1+x)^b = (1+x)^{a+b}. \tag{2}$$

If we introduce the binomial series for $(1+x)^a$, $(1+x)^b$, and $(1+x)^{a+b}$ and compare the coefficients of x^n on both sides of (2), then (1) follows immediately.

Since I have found no formula for the sum

$$\sum_{i=0}^{k} \binom{a}{i}\binom{b}{n-i}, \qquad k = 1, \ldots, n-1\,, \tag{4}$$

in the literature, I think that the following two formulae may be of some interest. They cover the special cases $b = -a$ and $b = 1-a$ and were developed for use in a problem in probability. (See the following, paper, pp. 276 and 278.)

The formulae, will be proved by induction, are:

$$\sum_{i=0}^{k} \binom{a}{i}\binom{-a}{n-i} = \frac{n-k}{n}\binom{a-1}{k}\binom{-a}{n-k}, \qquad \begin{array}{l} n = 1, 2, \ldots, \\ k = 0, 1, \ldots, n\,, \end{array} \tag{5}$$

and

$$\sum_{i=0}^{k} \binom{a}{i}\binom{1-a}{n-i} = \frac{(n-1)(1-a)-k}{n(n-1)}\binom{a-1}{k}\binom{-a}{n-k-1} \tag{6}$$

$$n = 2, 3, \ldots, \qquad k = 0, 1, \ldots, n-1\,.$$

Proof of (5): For $n = 1, 2, \ldots$ and $k = 0$ the formula evidently holds, since both sides equal $\binom{-a}{n}$. We assume that (5) holds for n ($\geqq 1$) and k ($< n$) and get

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$$(7)\quad \sum_{i=0}^{k+1}\binom{a}{i}\binom{-a}{n-i}=\frac{n-k}{n}\binom{a-1}{k}\binom{-a}{n-k}+\binom{a}{k+1}\binom{-a}{n-k-1}$$

$$=\binom{a-1}{k}\binom{-a}{n-k-1}\left(\frac{(n-k)(-a-n+k+1)}{n(n-k)}+\frac{a}{k+1}\right)$$

$$=\binom{a-1}{k}\binom{-a}{n-k-1}\frac{(a-k-1)(n-k-1)}{n(k+1)}=\frac{n-k-1}{n}\binom{a-1}{k+1}\binom{-a}{n-k-1},$$

and (5) follows by induction with respect to k.

Proof of (6): For $n = 2, 3, \ldots$ and $k = 0$ formula (6) holds since both sides equal $\binom{1-a}{n}$. We assume that (6) holds for n ($\geqq 2$) and k ($< n-1$) and get

$$(8)\quad \sum_{i=0}^{k+1}\binom{a}{i}\binom{1-a}{n-i}$$

$$=\frac{(n-1)(1-a)-k}{n(n-1)}\binom{a-1}{k}\binom{-a}{n-k-1}+\binom{a}{k+1}\binom{1-a}{n-k-1}$$

$$=\binom{a-1}{k}\binom{-a}{n-k-2}\left\{\frac{[(n-1)(1-a)-k](-a-n+k+2)}{n(n-1)(n-k-1)}+\frac{a(1-a)}{(k+1)(n-k-1)}\right\}$$

$$=\binom{a-1}{k}\binom{-a}{n-k-2}\frac{(n-k-1)[(n-1)(1-a)-k-1](a-k-1)}{n(n-1)(k+1)(n-k-1)}$$

$$=\frac{(n-1)(1-a)-k-1}{n(n-1)}\binom{a-1}{k+1}\binom{-a}{n-k-2}.$$

By induction with respect to k the formula (6) follows for $k = 1, 2, \ldots, n-1$. For $k = n-1$ the operations in (8) are invalid.

It will be noted in both proofs that for each n the induction only proceedes through a finite number of steps.

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