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ON THE FLUCTUATIONS OF SUMS OF RANDOM VARIABLES

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1. The problems treated here are connected with certain random variables defined in terms of the sums $S_i = X_1 + \ldots + X_i$ of a sequence of random variables $X_1, X_2, \ldots$. These random variables are:

$1°$ the index L_n at which S_i for the first time attains the maximum value $\max(S_0 = 0, S_1, \ldots, S_n)$,

$2°$ the index M_n at which S_i for the last time attains the minimum value $\min(S_0, S_1, \ldots, S_n)$,

$3°$ the number N_n of sums $S_1, \ldots, S_n$, which are > 0.

In papers by K. L. Chung and W. Feller [3] and by M. Lipschutz [7] the distribution of N_n has been treated under the assumption $S_n = 0$. Chung and Feller consider the special case where $\Pr\{X_i = 1\} = \Pr\{X_i = -1\} = \frac{1}{2}$, $i = 1, 2, \ldots$, and show that if we let N_n' denote the number of sums S_i, $i = 1, 2, \ldots, n$, for which either $S_i > 0$, or $S_i = 0$ and $S_{i-1} > 0$, then

$$\Pr\{N_{2n}' \leqq 2r \mid S_{2n} = 0\} = (r+1)(n+1)^{-1} .$$

$\Pr\{A \mid B\}$ denotes the conditional probability of A under the hypothesis B. In the paper of Lipschutz this result is generalized to independent identically distributed random variables, which assume only integer values (lattice distributions). M. Lipschutz shows that if the distribution of the random variables has mean zero, variance 1, and finite fourth moment, and the minimum distance between the jumps is one unit, then

$$(1.1)\quad \Pr\{N_n' \leqq \alpha n \mid S_n = 0\} = ([\alpha n]+1)(n+1)^{-1} + g(n), \qquad 0 \leqq \alpha \leqq 1 ,$$

where

$$g(n) = O(n^{-1/30}\log n)$$

if the random variables have third moment zero and

$$g(n) = O(n^{-1/72}\log n)$$

if the random variables have a third moment differing form zero.

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2. In this paper the conditional distributions of L_n, M_n and N_n are treated under the hypothesis $S_{n+1} = 0$. First some results are proved for finite n. Thereafter it is shown that the conditional distributions of L_n, M_n and N_n are asymptotically uniform under the hypothesis $S_{n+1} = 0$, if only certain rather weak conditions are satisfied by $X_1, X_2, \ldots$.

The method used in the proofs is completely different from the method used by Lipschutz and is based on the idea of symmetrically dependent random variables. This concept is a generalization of the concept of independent and identically distributed random variables. From the paper [1] we shall quote the definitions and results which we need in the present paper.

We consider a finite or infinite sequence $X_1, X_2, \ldots$ of real-valued random variables. As sample space E, we denote the product set $(R, R, \ldots)$, where R is the set of real numbers. By $[\ldots]$, where $\ldots$ indicates a number of relations involving $X_1, X_2, \ldots$, we denote the event in E, at which the relations are satisfied.

The joint distribution function of $X_1, \ldots, X_n$ is denoted by

$$F_n(x_1, \ldots, x_n) = \Pr \left\{ \bigcap_{i=1}^{n} [X_i \leqq x_i] \right\}.$$

If $F_n(x_1, \ldots, x_n)$ is a symmetric function of $x_1, \ldots, x_n$, we shall say that the variables $X_1, \ldots, X_n$ are symmetrically dependent. If the sequence $X_1, X_2, \ldots$ is infinite and the variables $X_1, \ldots, X_n$ are symmetrically dependent for $n = 2, 3, \ldots$, we shall say that the variables $X_1, X_2, \ldots$ are symmetrically dependent.

If an event C is invariant under permutations of the variables $x_1, \ldots, x_n$ or $x_1, x_2, \ldots$ we shall say that the event is symmetric with respect to $X_1, \ldots, X_n$ or $X_1, X_2, \ldots$ respectively.

For symmetrically dependent random variables we have the following lemma and theorem (see the paper [1]).

LEMMA 1. *Let $X_1, X_2, \ldots$ be symmetrically dependent random variables. Let A and B be events, both defined by relations in $X_1, \ldots, X_n$ such that the relations defining A are transformed into the relations defining B by a permutation of the variables $X_1, \ldots, X_n$. Let C be an event which is symmetric with respect to $X_1, \ldots, X_n$. Then* $\Pr\{AC\} = \Pr\{BC\}$.

THEOREM 1. *Let $X_1, X_2, \ldots$ be symmetrically dependent random variables. Let C_n be an event, which is symmetric with respect to $X_1, \ldots, X_n$. Then*

$$(3.3)\quad \Pr\{[N_n = m]C_n\} = \Pr\{[L_n = m]C_n\} = \Pr\{[M_n = n-m]C_n\}, \quad m = 0, 1, \ldots, n.$$

Beside the variables $X_1, X_2, \ldots$ and $S_0 = 0, S_1 = X_1, S_2 = X_1+X_2, \ldots,$ we shall also study the variables

$$X_i^{(n)} = X_i - (n+1)^{-1}S_{n+1}, \qquad i = 1, \ldots, n+1,$$

and

$$S_i^{(n)} = X_1^{(n)} + \ldots + X_i^{(n)} = S_i - i(n+1)^{-1}S_{n+1}, \qquad i = 1, \ldots, n+1.$$

The variables $X_i^{(n)}$ evidently satisfy the relation $S_{n+1}^{(n)} = 0$.

3. We shall first prove the following simple theorem:

THEOREM 2. *Let $X_1, X_2, \ldots$ be symmetrically dependent random variables and let C be an event which is symmetric with respect to $X_1, \ldots, X_{n+1}$. Let K_n be one of the variables L_n, M_n or N_n. Then, under the assumption* $\Pr\{[S_{n+1} = 0]C\} > 0$,

$$\text{(3.1)} \quad \Pr\{K_n = m \mid [S_{n+1} = 0]C\} = (n+1)^{-1}, \qquad m = 0, 1, \ldots, n,$$

if and only if

$$\text{(3.2)} \qquad \Pr\{[S_i = 0][S_{n+1} = 0]C\} = 0, \qquad i = 1, 2, \ldots, n.$$

REMARK: Since we have, when $S_{n+1} = 0$, the relations $L_n = L_{n+1}$, and $N_n = N_{n+1}$ and furthermore $M_n = M_{n+1}$, except when $M_n = 0$, in which case $M_{n+1} = n+1$, it actually does not matter whether we consider the distribution of K_n under the assumption $S_n = 0$ or under the assumption $S_{n+1} = 0$. When we consider K_n under the assumption $S_{n+1} = 0$, we may, however, state the result in the following way: under the assumptions of the theorem there is uniform distribution among the possible values $0, 1, \ldots, n$ of the three variables:

1° the index L_n of the maximum sum,
2° the index M_n of the minimum sum,
3° the number N_n of positive sums.

PROOF OF THEOREM 2: It follows from Theorem 1 that it is sufficient to prove (3.1) for $K_n = L_n$. Let m denote one of the numbers $0, 1, \ldots, n$. We transform the event $A_m = [L_n = m][S_{n+1} = 0]$ by the permutation

$$X_i \to X_{i+1}, \qquad i = 1, 2, \ldots, n,$$
$$X_{n+1} \to X_1.$$

This permutation carries S_i, $i = 1, 2, \ldots, n$, into $S_{i+1} - X_1$ and leaves S_{n+1} unchanged. From the definition of L_n it follows that[1]

[1] If C_i, $i = 1, 2, \ldots,$ are events in the sample space E, and $k < j$, then $\bigcap_{i=j}^{k} C_i = E$.

$$[L_n = m][S_{n+1} = 0] = \bigcap_{i=0}^{m-1} [S_i < S_m] \bigcap_{i=m+1}^{n} [S_i \leqq S_m][S_{n+1} = 0] .$$

The event $A_m = [L_n = m][S_{n+1} = 0]$ is therefore carried into the event

$$\begin{aligned} B_m &= \bigcap_{i=0}^{m-1} [S_{i+1} - X_1 < S_{m+1} - X_1] \bigcap_{i=m+1}^{n} [S_{i+1} - X_1 \leqq S_{m+1} - X_1][S_{n+1} = 0] \\ &= \bigcap_{i=0}^{m-1} [S_{i+1} < S_{m+1}] \bigcap_{i=m+1}^{n} [S_{i+1} \leqq S_{m+1}][S_{n+1} = 0] \\ &= \bigcap_{i=1}^{m} [S_i < S_{m+1}] \bigcap_{i=m+2}^{n+1} [S_i \leqq S_{m+1}][S_{n+1} = 0] . \end{aligned}$$

Since $S_0 = S_{n+1}$ in $[S_{n+1} = 0]$ it follows that

$$B_m = [S_0 \leqq S_{m+1}] \bigcap_{i=1}^{m} [S_i < S_{m+1}] \bigcap_{i=m+2}^{n} [S_i \leqq S_{m+1}][S_{n+1} = 0] .$$

We therefore obtain

$$\begin{aligned} B_m &= \left([S_0 < S_{m+1}] \bigcap_{i=1}^{m} [S_i < S_{m+1}] \bigcap_{i=m+2}^{n} [S_i \leqq S_{m+1}][S_{n+1} = 0]\right) \\ &\qquad \cup \left([S_0 = S_{m+1}] \bigcap_{i=1}^{m} [S_i < S_{m+1}] \bigcap_{i=m+2}^{n} [S_i \leqq S_{m+1}][S_{n+1} = 0]\right) \\ &= \left([L_n = m+1][S_{n+1} = 0]\right) \cup \left([S_{m+1} = 0] \bigcap_{i=1}^{m} [S_i < 0] \bigcap_{i=m+2}^{n} [S_i \leqq 0][S_{n+1} = 0]\right) \\ &= \left([L_n = m+1][S_{n+1} = 0]\right) \cup \left([L_n = 0][S_{m+1} = 0] \bigcap_{i=1}^{m} [S_i < 0][S_{n+1} = 0]\right) , \end{aligned}$$

where the two components are non-overlapping. If $m = n$, the first component is empty. From Lemma 1 applied to A_m, B_m, and C it now follows that for $m = 0, 1, \ldots, n$

$$\begin{aligned} (3.3)\quad & \Pr\{[L_n = m][S_{n+1} = 0]C\} \\ &= \Pr\{[L_n = m+1][S_{n+1} = 0]C\} \\ &\qquad + \Pr\left\{[L_n = 0][S_{m+1} = 0] \bigcap_{i=1}^{m} [S_i < 0][S_{n+1} = 0]C\right\}. \end{aligned}$$

In the proof of (3.3) we have only used that the variables $X_1, \ldots, X_{n+1}$ are symmetrically dependent; this we shall use later in the proof of Theorem 4 below.

We shall now use the fact that (3.2) implies, for $m = 0, 1, \ldots, n-1$,

$$(3.4)\qquad \Pr\left\{[L_n = 0][S_{m+1} = 0] \bigcap_{i=1}^{m} [S_i < 0][S_{n+1} = 0]C\right\} = 0 ,$$

since

$$(3.5)\quad [L_n = 0][S_{m+1} = 0] \bigcap_{i=1}^{m} [S_i < 0][S_{n+1} = 0]C \subseteqq [S_{m+1} = 0][S_{n+1} = 0]C \,.$$

From (3.3) and (3.4) follows

$$\begin{aligned}\Pr \{[L_n = 0][S_{n+1} = 0]C\} &= \Pr \{[L_n = 1][S_{n+1} = 0]C\} \\ &= \ldots = \Pr \{[L_n = n][S_{n+1} = 0]C\} \,.\end{aligned}$$

We therefore obtain, since

$$\sum_{m=0}^{n} \Pr \{[L_n = m][S_{n+1} = 0]C\} = \Pr \{[S_{n+1} = 0]C\} \,,$$

that

$$(3.6)\qquad \Pr \{[L_n = m][S_{n+1} = 0]C\} = (n+1)^{-1} \Pr \{[S_{n+1} = 0]C\}, \qquad m = 0, 1 \ldots, n \,.$$

From (3.6) follows (3.1) for $K_n = L_n$ immediately when

$$\Pr \{[S_{n+1} = 0]C\} > 0 \,.$$

It remains to be shown that (3.1) can hold only if (3.2) holds. We assume that (3.1) holds and shall show that the events $[S_i = 0][S_{n+1} = 0]C$ have probability zero for $i = 1, \ldots, n$. Since we have

$$\begin{aligned}&[S_i = 0][S_{n+1} = 0]C \\ &= \bigcap_{j=0}^{i-1} \bigcap_{k=i}^{n} [S_j = \max (S_0, \ldots, S_{i-1})][S_k = \max (S_i, \ldots, S_n)][S_i = 0][S_{n+1} = 0]C \,,\end{aligned}$$

we need only show that the event AC, where

$$A = [S_j = \max (S_0, \ldots, S_{i-1})][S_k = \max (S_i, \ldots, S_n)][S_i = 0][S_{n+1} = 0] \,,$$

has probability zero for $0 \leqq j < i \leqq k \leqq n$. If we transform A by the permutation

$$\begin{aligned}X_h &\to X_{h+i-j} &&\text{for}\quad h = 1, \ldots, j \,, \\ X_h &\to X_{h-j} &&\quad\text{,,}\quad\ h = j+1, \ldots, i \,, \\ X_h &\to X_{h+n+1-k} &&\quad\text{,,}\quad\ h = i+1, \ldots, k \,, \\ X_h &\to X_{h+i-k} &&\quad\text{,,}\quad\ h = k+1, \ldots, n+1 \,,\end{aligned}$$

we obtain the new event

$$B = \bigcap_{h=1}^{n} [S_h \leqq 0][S_i = 0][S_{n+1} = 0]$$
$$= [L_n = 0][S_{m+1} = 0] \bigcap_{h=1}^{m} [S_h < 0][S_{n+1} = 0] ,$$

for some non-negative integer m less than i. From Lemma 1 follows that $\Pr\{AC\} = \Pr\{BC\}$, so that we only have to show that $\Pr\{BC\} = 0$. We now use (3.3) and obtain

$$\Pr\{BC\} = \Pr\{[L_n = m][S_{n+1} = 0]C\} - \Pr\{[L_n = m+1][S_{n+1} = 0]C\}$$
$$= (n+1)^{-1}\Pr\{[S_{n+1} = 0]C\} - (n+1)^{-1}\Pr\{[S_{n+1} = 0]C\} = 0 ,$$

since we have assumed that (3.1) holds. This completes the proof of Theorem 2.

In most cases Theorem 2 is not directly applicable since (3.2) is seldom satisfied. We may, however, apply Theorem 2 to the variables $X_1^{(n)}, \ldots, X_{n+1}^{(n)}$, when $X_1, \ldots, X_{n+1}$ are symmetrically dependent, since we have:

LEMMA 2. *If $X_1, \ldots, X_{n+1}$ are symmetrically dependent random variables, then the variables $X_1^{(n)}, \ldots, X_m^{(n)}$ are also symmetrically dependent for $m = 2, \ldots, n+1$.*

PROOF: We have to show that if $j_1, \ldots, j_m$ is a permutation of the numbers $1, \ldots, m$ then

$$\Pr\left\{\bigcap_{i=1}^{m} [X_{j_i}^{(n)} \leqq x_i]\right\} = \Pr\left\{\bigcap_{i=1}^{m} [X_i^{(n)} \leqq x_i]\right\}. \tag{3.7}$$

From the definition of $x_i^{(n)}$, it follows that

$$\bigcap_{i=1}^{m} [X_i^{(n)} \leqq x_i] = \bigcap_{i=1}^{m} [X_i - (n+1)^{-1}S_{n+1} \leqq x_i] .$$

We transform this event by the permutation

$$X_i \rightarrow X_{j_i}, \qquad i = 1, \ldots, m ,$$
$$X_i \rightarrow X_i , \qquad i = m+1, \ldots, n+1 .$$

This permutation carries the event into the event

$$\bigcap_{i=1}^{m} [X_{j_i} - (n+1)^{-1}S_{n+1} \leqq x_i] = \bigcap_{i=1}^{m} [X_{j_i}^{(n)} \leqq x_i] .$$

Relation (3.7) now follows from Lemma 1, with $C = E$.

If $m > n+1$ and $X_1, \ldots, X_m$ are symmetrically dependent random variables, then $X_1^{(n)}, \ldots, X_m^{(n)}$ are not symmetrically dependent, as can easily be seen.

From Theorem 2 and Lemma 2 we obtain the following:

THEOREM 3. *Let $X_1, \ldots, X_{n+1}$ be symmetrically dependent random variables and let C be an event, which is symmetric with respect to $X_1, \ldots, X_{n+1}$. Let $N_n{}^*$ be the number of points (j, S_j), $j = 1, \ldots, n$, which lie above the straight line from* $(0, 0)$ *to* $(n+1, S_{n+1})$. *Then, for* $\Pr\{C\} > 0$,

$$\Pr\{N_n{}^* = m \mid C\} = (n+1)^{-1}, \qquad m = 0, 1, \ldots, n, \tag{3.8}$$

if and only if

$$\Pr\{[i^{-1}S_i = (n+1)^{-1}S_{n+1}]C\} = 0, \qquad i = 1, 2, \ldots, n. \tag{3.9}$$

REMARK: $N_n{}^*$ may be replaced by $L_n{}^*$, the first index for which

$$S_i - i(n+1)^{-1}S_{n+1} = \max_{i=0,\ldots,n} (S_i - i(n+1)^{-1}S_{n+1}),$$

or $M_n{}^*$, the last index for which

$$S_i - i(n+1)^{-1}S_{n+1} = \min_{i=0,\ldots,n} (S_i - i(n+1)^{-1}S_{n+1}).$$

PROOF OF THEOREM 3: Apply Theorem 1 to $X_1^{(n)}, \ldots, X_{n+1}^{(n)}$, with $K_n = N_n$; then $N_n{}^* = K_n$. We have $\Pr\{[S_{n+1}^{(n)} = 0]C\} > 0$, since $[S_{n+1}^{(n)} = 0] = E$; and

$$\begin{aligned}[S_i^{(n)} = 0][S_{n+1}^{(n)} = 0]C &= [S_i - i(n+1)^{-1}S_{n+1} = 0]C \\ &= [i^{-1}S_i = (n+1)^{-1}S_{n+1}]C\end{aligned}$$

so that (3.2) is satisfied for $X_1^{(n)}, \ldots, X_{n+1}^{(n)}$ if and only if (3.9) is satisfied for $X_1, \ldots, X_{n+1}$.

COROLLARY 1. *If the random variables $X_1, \ldots, X_{n+1}$ are independent and each has a continuous distribution, or if the random variables are symmetrically dependent and the joint distribution function is absolutely continuous, then for any C which is symmetric with respect to $X_1, \ldots, X_{n+1}$ and has* $\Pr\{C\} > 0$ *(as, for example, $C = [S_{n+1} > 0]$ if* $\Pr\{X_1 > 0\} > 0$*), we have*

$$\Pr\{N_n{}^* = m \mid C\} = (n+1)^{-1}, \qquad m = 0, 1, \ldots, n.$$

COROLLARY 2. *If the random variables $X_1, \ldots, X_{n+1}$ are symmetrically dependent and assume only integer values and $C \subset [S_{n+1} = 1]$, then, if* $\Pr\{C\} > 0$, *we have*

$$\text{(3.10)} \qquad \Pr\{K_n = m \mid C\} = (n+1)^{-1}, \qquad m = 0, 1, \ldots, n\,,$$

where K_n stands for one of the variables L_n, M_n or N_n.

Corollary 1 is an almost immediate consequence of Theorem 2.

Corollary 2, however, may need a proof. We first observe that when S_i is an integer and $S_{n+1} = 1$, we cannot have $S_i = i(n+1)^{-1}S_{n+1}$. We therefore have (3.10) with K_n replaced by $N_n{}^*$. But from $S_{n+1} = 1$ it follows for $i = 1, \ldots, n$, that

$$[S_i > i(n+1)^{-1}S_{n+1}] = [S_i > i(n+1)^{-1}] = [S_i > 0]\,,$$

since S_i is an integer. We therefore obtain (3.10) with $K_n = N_n$. From Theorem 1 then follows (3.10) also for $K_n = L_n$ or M_n.

4. If condition (3.9) is not satisfied, the situation is much more complicated. We shall therefore consider only independent and identically distributed random variables. Furthermore we shall assume $C = E$. We shall first prove:

THEOREM 4. *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables. Let K_n be one of the variables L_n, M_n, or N_n. Then, for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots, n$,*

$$\text{(4.1)} \qquad \Pr\{[K_n = m][S_{n+1} = 0]\}$$
$$= \sum_{k=m}^{n} \Pr\left\{\bigcap_{i=1}^{k} [S_i < 0][S_{k+1} = 0]\right\} \Pr\{[L_{n-k} = 0][S_{n-k} = 0]\}\,.$$

PROOF: By repeated use of equation (3.3), which was proved without use of condition (3.2), we obtain, with $C = E$,

$$\text{(4.2)} \qquad \Pr\{[L_n = m][S_{n+1} = 0]\}$$
$$= \sum_{k=m}^{n} \Pr\left\{[L_n = 0][S_{k+1} = 0] \bigcap_{i=1}^{k} [S_i < 0][S_{n+1} = 0]\right\},$$

for $m = 0, 1, \ldots, n$. For each k the event

$$A_{n,k} = [L_n = 0][S_{k+1} = 0] \bigcap_{i=1}^{k} [S_i < 0][S_{n+1} = 0]$$
$$= \bigcap_{i=1}^{k} [S_i < 0][S_{k+1} = 0] \bigcap_{i=k+2}^{n} [S_i \leqq S_{k+1}][S_{k+1} = S_{n+1}]$$

is the intersection of the events

$$B_k = \bigcap_{i=1}^{k} [S_i < 0][S_{k+1} = 0] \quad \text{and} \quad C_{n,k} = \bigcap_{i=k+2}^{n} [S_i \leqq S_{k+1}][S_{k+1} = S_{n+1}] .$$

The event B_k depends only on $X_1, \ldots, X_{k+1}$ and the event $C_{n,k}$ depends only on $X_{k+2}, \ldots, X_{n+1}$. Since the variables $X_1, \ldots, X_{n+1}$ are independent we obtain

$$\text{(4.3)} \qquad \Pr\{A_{n,k}\} = \Pr\{B_k\} \Pr\{C_{n,k}\} .$$

We now transform the event $C_{n,k}$ by the permutation

$$X_i \to X_{i+n-k}, \qquad i = 1, \ldots, k+1,$$
$$X_i \to X_{i-k-1}, \qquad i = k+2, \ldots, n+1 .$$

This permutation carries

$$C_{n,k} = \bigcap_{i=k+2}^{n} [X_{k+2}+\ldots+X_i \leqq 0][X_{k+2}+\ldots+X_{n+1} = 0]$$

into

$$D_{n-k} = \bigcap_{i=k+2}^{n} [X_1+\ldots+X_{i-k-1} \leqq 0][X_i+\ldots+X_{n-k} = 0]$$
$$= \bigcap_{i=1}^{n-k-1} [X_1+\ldots+X_i \leqq 0][X_1+\ldots+X_{n-k} = 0]$$
$$= [L_{n-k} = 0][S_{n-k} = 0] .$$

Lemma 1 then gives

$$\text{(4.4)} \qquad \Pr\{C_{n,k}\} = \Pr\{D_{n-k}\} .$$

From (4.2), (4.3), (4.4), and Theorem 1 we immediately obtain (4.1).

We now introduce the symbol Σ^*. This symbol shall indicate that the summation is restricted to those values of the summation variables $\alpha_1, \ldots, \alpha_n$, which are non-negative and satisfy the relation $\alpha_1+2\alpha_2+\ldots+n\alpha_n = n$. We are then able to give explicit formulae for $\Pr\{B_n\}$ and $\Pr\{D_n\}$ in terms of $\Pr\{S_n = 0\}$, $n = 1, 2, \ldots$. We state the results in:

THEOREM 5. *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables. Let $c_n = \Pr\{S_n = 0\}$, $n = 1, 2, \ldots$. Let*

$$f_n = \Pr\left\{\bigcap_{i=1}^{n-1} [S_i < 0][S_n = 0]\right\}, \qquad n = 1, 2, \ldots,$$

and

$$u_n = \Pr\{[L_n = 0][S_n = 0]\}, \qquad n = 0, 1, 2, \ldots \qquad (u_0 = 1)\,.$$

Then

$$u_n = \sum_{k=1}^{n} f_k u_{n-k}, \qquad n = 1, 2, \ldots, \tag{4.5}$$

$$c_n = \sum_{k=1}^{n} k f_k u_{n-k}, \qquad n = 1, 2, \ldots, \tag{4.6}$$

$$f_n = -\sum_{\alpha_1, \ldots, \alpha_n}^{*} \prod_{\nu=1}^{n} (\alpha_\nu!)^{-1}(-c_\nu)^{\alpha_\nu}\nu^{-\alpha_\nu}, \qquad n = 1, 2, \ldots, \tag{4.7}$$

and

$$u_n = \sum_{\alpha_1, \ldots, \alpha_n}^{*} \prod_{\nu=1}^{n} (\alpha_\nu!)^{-1} c_\nu^{\alpha_\nu}\nu^{-\alpha_\nu}, \qquad n = 1, 2, \ldots. \tag{4.8}$$

Furthermore the generating functions

$$C(s) = \sum_{n=1}^{\infty} c_n s^n, \qquad F(s) = \sum_{n=1}^{\infty} f_n s^n, \qquad U(s) = \sum_{n=0}^{\infty} u_n s^n$$

satisfy, for $|s| < 1$, *the relations*

$$U(s) - 1 = F(s)\,U(s)\,, \tag{4.9}$$

$$C(s) = sF'(s)\,U(s)\,, \tag{4.10}$$

$$F(s) = 1 - \exp\left(-\int_0^s t^{-1}C(t)\,dt\right), \tag{4.11}$$

and

$$U(s) = \exp\left(\int_0^s t^{-1}C(t)\,dt\right). \tag{4.12}$$

Throughout the following we assume, as in Theorem 5, that $|s| < 1$.

Proof of Theorem 5: Equations (4.5) and (4.9) follow from the fact that the events $[L_n = 0][S_n = 0]$ are recurrent events, see for example Feller [4, Chapter 12]. We may, however, deduce (4.5) directly from (4.1) with $m = 0$ and n replaced by $n-1$, since we have, for $n = 1, 2, \ldots,$

$$\begin{aligned} u_n &= \Pr\{[L_n = 0][S_n = 0]\} = \Pr\{[L_{n-1} = 0][S_n = 0]\} \\ &= \sum_{k=0}^{n-1} \Pr\left\{\bigcap_{i=0}^{k} [S_i < 0][S_{k+1} = 0]\right\} \Pr\{[L_{n-k-1} = 0][S_{n-k-1} = 0]\} \\ &= \sum_{k=0}^{n-1} f_{k+1} u_{n-k-1} = \sum_{k=1}^{n} f_k u_{n-k}\,. \end{aligned}$$

Equation (4.9) follows from (4.5), since

$$\begin{aligned} U(s)-1 &= \sum_{n=1}^{\infty} u_n s^n = \sum_{n=1}^{\infty} s^n \sum_{k=1}^{n} f_k u_{n-k} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} (f_k s^k)(u_{n-k} s^{n-k}) = \left(\sum_{n=1}^{\infty} f_n s^n\right)\left(\sum_{n=0}^{\infty} u_n s^n\right) \\ &= F(s)\,U(s)\,. \end{aligned}$$

The series for $C(s)$ and $F(s)$ converge for $|s| < 1$, since the coefficients, being probabilities, are bounded.

Next we shall prove (4.6) and (4.10). We sum (4.1) for $m = 0, 1, \ldots, n$ and obtain, for $n = 0, 1, 2, \ldots,$

$$\begin{aligned} \text{(4.13)}\quad & \sum_{m=0}^{n} \Pr\{[K_n = m][S_{n+1} = 0]\} \\ &= \sum_{m=0}^{n} \sum_{k=m}^{n} \Pr\left\{\bigcap_{i=1}^{k} [S_i < 0][S_{k+1} = 0]\right\} \Pr\{[L_{n-k} = 0][S_{n-k} = 0]\}\,. \end{aligned}$$

In (4.13) we replace n by $n-1$ and introduce f_k and u_{n-k}. We then obtain, for $n = 1, 2, \ldots,$

$$\begin{aligned} \text{(4.14)}\quad \sum_{m=0}^{n-1} \Pr\{[K_{n-1} = m][S_n = 0]\} &= \sum_{m=0}^{n-1} \sum_{k=m}^{n-1} f_{k+1} u_{n-k-1} \\ &= \sum_{k=0}^{n-1} (k+1) f_{k+1} u_{n-k-1} = \sum_{k=1}^{n} k f_k u_{n-k}\,. \end{aligned}$$

Since $\bigcup_{m=0}^{n-1} [K_{n-1} = m] = E$ and the events $[K_{n-1} = m]$ are non-overlapping, we obtain

$$\text{(4.15)}\qquad \sum_{m=0}^{n-1} \Pr\{[K_{n-1} = m][S_n = 0]\} = \Pr\{S_n = 0\} = c_n\,.$$

From (4.14) and (4.15) follows (4.6). We now obtain (4.10) from (4.6) in almost the same way, in which (4.9) was obtained from (4.5). We have

$$\begin{aligned} C(s) &= \sum_{n=1}^{\infty} c_n s^n = \sum_{n=1}^{\infty} s^n \sum_{k=1}^{n} k f_k u_{n-k} = s \sum_{n=1}^{\infty} \sum_{k=1}^{n} (k f_k s^{k-1})(u_{n-k} s^{n-k}) \\ &= s\left(\sum_{n=1}^{\infty} n f_n s^{n-1}\right)\left(\sum_{n=0}^{\infty} u_n s^n\right) = s F'(s)\, U(s)\,. \end{aligned}$$

If we eliminate $U(s)$ from (4.9) and (4.10), we obtain

$$\text{(4.16)}\qquad C(s)\big(1-F(s)\big) = s F'(s)\,.$$

The events $\bigcap_{i=1}^{n-1} [S_i < 0][S_n = 0]$, $n = 1, 2, \ldots,$ are non-overlapping. We therefore obtain

$$|F(s)| = \left|\sum_{n=1}^{\infty} f_n s^n\right| \leqq \sum_{n=1}^{\infty} f_n |s|^n < \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \Pr\left\{\bigcap_{i=1}^{n-1} [S_i < 0][S_n = 0]\right\} \leqq 1 .$$

We may therefore for $s \neq 0$ divide by $s\big(1-F(s)\big)$ in (4.16) and obtain

$$(4.17) \qquad s^{-1}C(s) = \big(1-F(s)\big)^{-1}F'(s) , \qquad 0 < |s| < 1 .$$

Integration of (4.17) gives (4.11). The equation (4.12) is derived from (4.9) and (4.17).

We still need to establish equations (4.7) and (4.8). They are derived from (4.11) and (4.12) respectively. In (4.11) we introduce $C(t) = \sum_{\nu=1}^{\infty} c_\nu t^\nu$ and obtain

$$\begin{aligned} F(s) &= 1 - \exp\left(-\int_0^s t^{-1}C(t)\,dt\right) = 1 - \exp\left(-\int_0^s \sum_{\nu=1}^{\infty} c_\nu t^{\nu-1}dt\right) \\ &= 1 - \exp\left(-\sum_{\nu=1}^{\infty} c_\nu \nu^{-1} s^\nu\right) \\ &= 1 - \prod_{\nu=1}^{\infty} \exp\left(-c_\nu \nu^{-1} s^\nu\right) = 1 - \prod_{\nu=1}^{\infty} \sum_{\alpha_\nu=0}^{\infty} (\alpha_\nu !)^{-1}(-c_\nu \nu^{-1} s^\nu)^{\alpha_\nu} \\ &= 1 - \sum_{n=0}^{\infty} s^n \sum_{\alpha_1, \ldots, \alpha_n}^{*} \prod_{\nu=1}^{n} (\alpha_\nu !)^{-1}(-c_\nu)^{\alpha_\nu} \nu^{-\alpha_\nu} . \end{aligned}$$

These formal operations are valid for $|s| < 1$. First, the series used are absolutely convergent when $|s| < 1$, since $0 < c_\nu < 1$. Furthermore, the infinite products are absolutely convergent since

$$\left|\sum_{\alpha_\nu=0}^{\infty} (\alpha_\nu !)^{-1}(-c_\nu \nu^{-1} s^\nu)^{\alpha_\nu} - 1\right| = |\exp(-c_\nu \nu^{-1} s^\nu) - 1| \leqq 2c_\nu \nu^{-1} |s|^\nu < 2|s|^\nu .$$

In the last operation we have collected terms where the exponent of s is $n = \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n$. The definition of $F(s)$ now gives (4.7). Relation (4.8) is proved analogously.

From Theorems 4 and 5 it follows that we may calculate

$$\Pr\{[K_n = m][S_{n+1} = 0]\}, \qquad m = 0, 1, \ldots, n,$$

when $c_i = \Pr\{S_i = 0\}$ is known for $i = 1, \ldots, n+1$. We have

$$(4.18) \qquad \Pr\{[K_n = m][S_{n+1} = 0]\} = \sum_{k=m}^{n} f_{k+1} u_{n-k}, \qquad \begin{aligned} n &= 0, 1, \ldots, \\ m &= 0, \ldots, n , \end{aligned}$$

and

$$(4.19)\quad \Pr\{[K_n \leqq m][S_{n+1}=0]\} = \sum_{i=0}^{m}\sum_{k=i}^{n} f_{k+1}u_{n-k} = \sum_{k=0}^{m-1}(k+1)f_{k+1}u_{n-k} + (m+1)\sum_{k=m}^{n} f_{k+1}u_{n-k}\,, \qquad n = 0, 1, \ldots,\quad m = 0, \ldots, n\,.$$

It may at first seem surprising that the conditional distribution of the number of positive sums S_i is independent of the probabilities $\Pr\{S_i > 0\}$ and $\Pr\{S_i < 0\}$. We have, however, in a certain sense symmetrized the distribution by the assumptions $S_{n+1} = 0$.

5. We shall now consider the special case where the independent and identically distributed random variables $X_1, X_2, \ldots$ represent Bernoulli trials. We have then for $i = 1, 2, \ldots$

$$(5.1)\quad \left.\begin{aligned} \Pr\{X_i = 1\} &= p \\ \Pr\{X_i = -1\} &= q \end{aligned}\right\} \quad \text{where} \quad p+q = 1\,.$$

The sums $S_i = X_1 + \ldots + X_i$ have binomial distributions and

$$\left.\begin{aligned} c_{2n-1} &= \Pr\{S_{2n-1} = 0\} = 0 \\ c_{2n} &= \Pr\{S_{2n} = 0\} = \binom{2n}{n} p^n q^n \end{aligned}\right\}, \qquad n = 1, 2, \ldots.$$

Since $\binom{2n}{n} = (-4)^n \binom{-\frac{1}{2}}{n}$, we obtain

$$(5.2)\quad c_{2n} = \binom{-\frac{1}{2}}{n}(-4pq)^n$$

and

$$(5.3)\quad C(s) = \sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}(-4pq)^n s^{2n} = (1-4pqs^2)^{-\frac{1}{2}} - 1\,.$$

If we introduce this expression in (4.11) we obtain

$$(5.4)\quad F(s) = 1 - \exp\left(-\int_0^s t^{-1}\big((1-4pqt^2)^{-\frac{1}{2}}-1\big)dt\right) = \tfrac{1}{2}\big(1-(1-4pqs^2)^{\frac{1}{2}}\big) = -\tfrac{1}{2}\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-4pq)^n s^{2n}\,.$$

From (4.9) we now obtain

$$(5.5)\quad U(s) = (2pqs^2)^{-1}\big(1-(1-4pqs^2)^{\frac{1}{2}}\big) = -(2pqs^2)^{-1}\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-4pq)^n s^{2n}\,.$$

The equations (5.4) and (5.5) yield

$$(5.6)\qquad f_{2n-1} = u_{2n-1} = 0, \qquad n = 1, 2, \ldots,$$

$$(5.7)\qquad f_{2n} = -\tfrac{1}{2}\binom{\frac{1}{2}}{n}(-4pq)^n = c_{2n}(4n-2)^{-1}, \qquad n = 1, 2, \ldots,$$

$$(5.8)\qquad u_{2n} = 2\binom{\frac{1}{2}}{n+1}(-4pq)^n = c_{2n}(n+1)^{-1}, \qquad n = 1, 2, \ldots.$$

From Theorem 4 follows, when we introduce these expressions,

$$(5.9)\quad \Pr\{[K_{2n-1} = 2m+1][S_{2n} = 0]\} = \Pr\{[K_{2n-1} = 2m][S_{2n} = 0]\},$$
$$m = 0, \ldots, n-1\,,$$

and

$$\begin{aligned}(5.10)\quad \Pr\{[K_{2n-1} = 2m][S_{2n} = 0]\} &= \sum_{k=m+1}^{n} f_{2k}u_{2n-2k} \\ &= \sum_{k=m+1}^{n}(-\tfrac{1}{2})\binom{\frac{1}{2}}{k}(-4pq)^k 2\binom{\frac{1}{2}}{n-k+1}(-4pq)^{n-k} \\ &= -\sum_{k=m+1}^{n}\binom{\frac{1}{2}}{k}\binom{\frac{1}{2}}{n-k+1}(-4pq)^n \\ &= -c_{2n}\binom{-\frac{1}{2}}{n}^{-1}\sum_{k=m+1}^{n}\binom{\frac{1}{2}}{k}\binom{\frac{1}{2}}{n-k+1}, \qquad m = 0, 1, \ldots, n-1\,.\end{aligned}$$

We now apply the formula

$$(5.11)\qquad \sum_{i=0}^{k}\binom{a}{i}\binom{1-a}{n-i} = \frac{(n-1)(1-a)-k}{n(n-1)}\binom{a-1}{k}\binom{-a}{n-k-1},$$
$$n = 2, 3, \ldots, \qquad k = 0, 1, \ldots, n-1,$$

which may easily be proved by induction. (For a detailed proof see [2].) Using $a = \frac{1}{2}$ and applying the formula with $n+1$ in place of n and for $k = m$ and n we obtain

$$\begin{aligned}\sum_{i=m+1}^{n} -\binom{\frac{1}{2}}{i}\binom{\frac{1}{2}}{n-i+1} &= -\sum_{i=0}^{n}\binom{\frac{1}{2}}{i}\binom{\frac{1}{2}}{n-i+1} + \sum_{i=0}^{m}\binom{\frac{1}{2}}{i}\binom{\frac{1}{2}}{n-i+1} \\ &= -\frac{n\cdot\frac{1}{2}-n}{n(n+1)}\binom{-\frac{1}{2}}{n}\binom{-\frac{1}{2}}{0} + \frac{n\cdot\frac{1}{2}-m}{n(n+1)}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m} \\ &= \frac{1}{2(n+1)}\binom{-\frac{1}{2}}{n} + \frac{n-2m}{2n(n+1)}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}.\end{aligned}$$

When we use this we get from (5.9) and (5.10) the results

$$(5.12)\quad \Pr\{[K_{2n-1}=2m][S_{2n}=0]\}=\Pr\{[K_{2n-1}=2m+1][S_{2n}=0]\}$$
$$=\tfrac{1}{2}c_{2n}(n+1)^{-1}\left(1+(n-2m)n^{-1}\binom{-\frac12}{m}\binom{-\frac12}{n-m}\binom{-\frac12}{n}^{-1}\right),$$
$$m=0,\ldots,n-1\,,$$

and

$$(5.13)\quad \Pr\{K_{2n-1}=2m\mid S_{2n}=0\}=\Pr\{K_{2n-1}=2m+1\mid S_{2n}=0\}$$
$$=\tfrac{1}{2}(n+1)^{-1}\left(1+(n-2m)n^{-1}\binom{-\frac12}{m}\binom{-\frac12}{n-m}\binom{-\frac12}{n}^{-1}\right),$$
$$m=0,\ldots,n-1\,.$$

We shall also derive formulae for $\Pr\{[K_{2n-1}\leqq 2m+1][S_{2n}=0]\}$ and $\Pr\{K_{2n-1}\leqq 2m+1\mid S_{2n}=0\}$. These formulae might be derived from (5.12) and (5.13), but then we should have to sum the unpleasant expression

$$\sum(n-2m)\binom{-\frac12}{m}\binom{-\frac12}{n-m}.$$

We shall therefore use formula (4.19)

$$\Pr\{[K_{2n-1}\leqq 2m+1][S_{2n}=0]\}$$
$$=\sum_{k=0}^{2m}(k+1)f_{k+1}u_{2n-1-k}+2(m+1)\sum_{k=2m+1}^{2n-1}f_{k+1}u_{2n-1-k}\,.$$

Since $f_{2n+1}=u_{2n+1}=0$ in the case of Bernoulli trials, this formula reduces to

$$(5.14)\quad \Pr\{[K_{2n-1}\leqq 2m+1][S_{2n}=0]\}$$
$$=2\sum_{k=1}^{m}kf_{2k}u_{2n-2k}+2(m+1)\sum_{k=m+1}^{m}f_{2k}u_{2n-2k}$$
$$=2\sum_{k=1}^{m}kf_{2k}u_{2n-2k}+2(m+1)\Pr\{[K_{2n-1}=2m+1][S_{2n}=0]\}.$$

In the sum $\Sigma_{k+1}^{m}kf_{2k}u_{2n-2k}$ we introduce the expressions

$$f_{2n}=-\tfrac12\binom{\frac12}{n}(-4pq)^n \qquad\text{and}\qquad u_{2n}=2\binom{\frac12}{n+1}(-4pq)^n$$

and obtain

$$(5.15)\quad 2\sum_{k=1}^{m} kf_{2k}u_{2n-2k} = 2\sum_{k=1}^{m} k(-\tfrac{1}{2})\binom{\frac{1}{2}}{k}(-4pq)^k 2\binom{\frac{1}{2}}{n-k+1}(-4pq)^{n-k}$$
$$= -(-4pq)^n\sum_{k=1}^{m}\binom{-\frac{1}{2}}{k-1}\binom{\frac{1}{2}}{n-k+1}.$$

We now apply the formula

$$\sum_{i=0}^{k}\binom{a}{i}\binom{-a}{n-i} = \frac{n-k}{n}\binom{a-1}{k}\binom{-a}{n-k} = -\frac{k+1}{n}\binom{a}{k+1}\binom{-a-1}{n-k-1},$$
$$n = 1, 2, \ldots, \qquad k = 0, 1, \ldots, n,$$

which may be proved by induction. (For a detailed proof see [2].) We shall use $a = -\frac{1}{2}$ and obtain, when we apply the formula for $k = m-1$,

$$(5.16)\qquad \sum_{i=0}^{m-1}\binom{-\frac{1}{2}}{i}\binom{\frac{1}{2}}{n-i} = -mn^{-1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}.$$

From (5.15) and 5.16) we obtain

$$2\sum_{k=1}^{m} kf_{2k}u_{2n-2k} = (-4pq)^n mn^{-1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}$$
$$= c_{2n}mn^{-1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}\binom{-\frac{1}{2}}{n}^{-1}.$$

When we use this and (5.12) in (5.14), we obtain

$$(5.16)\qquad \Pr\{[K_{2n-1} \leqq 2m+1][S_{2n} = 0]\}$$
$$= c_{2n}mn^{-1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}\binom{-\frac{1}{2}}{n}^{-1}$$
$$+ 2(m+1)c_{2n}\tfrac{1}{2}(n+1)^{-1}\left(1 + (n-2m)n^{-1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}\binom{-\frac{1}{2}}{n}^{-1}\right)$$
$$= c_{2n}(n+1)^{-1}\left(m+1 + (n-m)(2m+1)n^{-1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}\binom{-\frac{1}{2}}{n}^{-1}\right).$$

When we change to conditional probabilities, we obtain, for $m = 0, 1, \ldots, n-1$,

$$(5.17)\qquad \Pr\{K_{2n-1} \leqq 2m+1 \mid S_{2n} = 0\}$$
$$= (n+1)^{-1}\left(m + 1 + (n-m)(2m+1)n^{-1}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n-m}\binom{-\frac{1}{2}}{n}^{-1}\right).$$

If we use that $(-1)^n\binom{-\frac{1}{2}}{n} \sim (\pi n)^{-\frac{1}{2}}$ for $n \to \infty$, then we obtain from (5.13) and (5.17) the formulae

(5.18) $\Pr\{K_{2n-1} = [2n\alpha] \mid S_{2n} = 0\} = (2n)^{-1} + R(n, \alpha), \qquad 0 < \alpha < 1,$

where $R(n, \alpha) \sim (1-2\alpha)\big(\pi\alpha(1-\alpha)\big)^{-\frac{1}{2}} n^{-3/2}$ if $\alpha \neq \frac{1}{2}$ and $R(n, \frac{1}{2}) = O(n^{-2})$, and

(5.19) $\Pr\{K_{2n-1} \leqq [2n\alpha] \mid S_{2n} = 0\} = \alpha + T(n, \alpha), \qquad 0 < \alpha < 1,$

where $T(n, \alpha) \sim 2\alpha^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}(\pi n)^{-\frac{1}{2}}$. The computations, which lead to (5.18) and (5.19), are rather trivial and are omitted here.

6. Under the condition $S_{n+1} = 0$ the limiting uniform distribution of L_n, M_n and N_n found above in the case of Bernoulli trials is also the limiting distribution in case of more general random variables. Before we can show this we need some preparation.

LEMMA 3. *If $F(s)$ is the generating function defined in Theorem 5, then $\sum_{n=1}^{\infty} f_n = F(1) < 1$ and $\sum_{n=0}^{\infty} u_n = U(1) < \infty$, unless* $\Pr\{X_1 = 0\} = 1$.

PROOF: We have either $\Pr\{X_1 > 0\} > 0$ or $\Pr\{X_1 > 0\} = 0$ and $\Pr\{X_1 < 0\} > 0$. If $\Pr\{X_1 > 0\} > 0$, then we have, since the events $[X_1 > 0]$ and $\bigcap_{i=1}^{n-1}[S_i < 0][S_n = 0]$, $n = 1, 2, \ldots$, are non-overlapping, that

$$1 \geqq \Pr\{X_1 > 0\} + \sum_{n=1}^{\infty} \Pr\left\{\bigcap_{i=1}^{n-1}[S_i < 0][S_n = 0]\right\}$$

$$= \Pr\{X_1 > 0\} + \sum_{n=1}^{\infty} f_n = \Pr\{X_1 > 0\} + F(1);$$

hence $F(1) < 1$. If $\Pr\{X_1 > 0\} = 0$ and $\Pr\{X_1 < 0\} > 0$, then we have for $n = 2, 3, \ldots$

$$f_n = \Pr\left\{\bigcap_{i=1}^{n-1}[S_i < 0][S_n = 0]\right\} \leqq \Pr\{X_n > 0\} = 0,$$

and $f_1 = \Pr\{X_1 = 0\} < 1$, so that $F(1) < 1$.

Since we have $U(s) = \big(1-F(s)\big)^{-1}$, it follows that $\sum_{n=0}^{\infty} u_n = U(1) < \infty$, unless $\Pr\{X_1 = 0\} = 1$.

LEMMA 4. *If $X_1, X_2, \ldots$ are independent and identically distributed random variables, such that* $\Pr\{X_1 = 0\} < 1$, *and if there exist positive constants K and a such that*

(6.1) $$f_n < Kn^{-a-1}, \qquad n = 1, 2, \ldots,$$

then there exists a positive constant M such that

(6.2) $$u_n < Mn^{-a-1}, \qquad n = 1, 2, \ldots.$$

PROOF: We choose α such that $0 < \alpha < 1$ and $F(1)(1-\alpha)^{-a-1} < 1$; this is possible since, from Lemma 3, $F(1) < 1$. We shall show that if we choose M, such that

$$M = KU(1)\alpha^{-a-1}\big(1 - F(1)(1-\alpha)^{-a-1}\big)^{-1}$$

or

$$M = KU(1)\alpha^{-a-1} + MF(1)(1-\alpha)^{-a-1}, \tag{6.3}$$

then $u_n < Mn^{-a-1}$ for $n = 1, 2, \ldots$. We first observe that (6.2) holds for $n = 1$ since $f_n = u_n$ and $M > K$. We then proceed by induction and assume that $u_n < Mn^{-a-1}$ for $n = 1, 2, \ldots, N-1$, where $N > 1$, and shall show that $u_N < MN^{-a-1}$. We use equation (4.5) and get

$$u_N = \sum_{k=1}^{N} f_k u_{N-k} = \sum_{k=1}^{[N\alpha]} f_k u_{N-k} + \sum_{k=[N\alpha]+1}^{N} f_k u_{N-k}\,. \tag{6.4}$$

We estimate the first term on the right-hand side using

$$u_{N-k} < M(N-k)^{-a-1} \leqq M\big(N-[N\alpha]\big)^{-a-1} \leqq MN^{-a-1}(1-\alpha)^{-a-1},$$
$$k = 1, 2, \ldots, [N\alpha] < N.$$

We then obtain

$$\sum_{k=1}^{[N\alpha]} f_k u_{N-k} < \sum_{k=1}^{[N\alpha]} f_k MN^{-a-1}(1-\alpha)^{-a-1} < F(1)MN^{-a-1}(1-\alpha)^{-a-1}. \tag{6.5}$$

In the second term we use

$$f_k < Kk^{-a-1} \leqq K\big([N\alpha]+1\big)^{-a-1} < KN^{-a-1}\alpha^{-a-1}, \qquad k = [N\alpha]+1, \ldots, N,$$

and obtain

$$\sum_{k=[N\alpha]+1}^{N} f_k u_{N-k} < \sum_{k=[N\alpha]+1}^{N} KN^{-a-1}\alpha^{-a-1}u_{N-k} < KN^{-a-1}\alpha^{-a-1}U(1)\,. \tag{6.6}$$

From (6.4), (6.5) and (6.6) we now obtain

$$\begin{aligned} u_N &< F(1)MN^{-a-1}(1-\alpha)^{-a-1} + KN^{-a-1}\alpha^{-a-1}U(1) \\ &= \big(F(1)M(1-\alpha)^{-a-1} + K\alpha^{-a-1}U(1)\big)N^{-a-1} = MN^{-a-1}, \end{aligned} \tag{6.7}$$

since we have (6.3). The proof is finished by complete induction.

We are now able to prove the following:

THEOREM 6. *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables. Let there exist positive constants A, B ($\leqq A$) and a so that*

$$c_n = \Pr\{S_n = 0\} < An^{-a}, \qquad n = 1, 2, \ldots, \tag{6.8}$$

$$c_{n_\nu+1} > B(n_\nu+1)^{-a} \qquad \textit{for a subsequence } n_1, n_2, \ldots. \tag{6.9}$$

Let the function $f_a(x)$ be defined in the following way:

$$(6.10)\qquad f_a(x) = \begin{cases} x^{-a} & \text{for } 0 < a < 1\,, \\ x^{-1}\log x & \text{for } a = 1\,, \\ x^{-1} & \text{for } a > 1\,. \end{cases}$$

Let K_n be one of the variables L_n, M_n, or N_n. Then there exist two constants K and K' depending on the distribution of the variables $X_1, X_2, \ldots$ such that

$$(6.11)\qquad \Pr\{K_{n_\nu} \leqq m \mid S_{n_\nu+1} = 0\} = (m+1)(n_\nu+1)^{-1} + C(n_\nu),$$
$$m = 0, 1, \ldots, n_\nu\,,$$

where

$$(6.12)\qquad 0 < C(n_\nu) < Kf_a(n_\nu+1)\,,$$

and

$$(6.13)\qquad \Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\} = (n_\nu+1)^{-1} + D(n_\nu, m),$$
$$m = 1, \ldots, n_\nu - 1,$$

where

$$(6.14)\qquad -K'(n_\nu+1)^{-1}f_a(n_\nu - m) < D(n_\nu, m) < K'(n_\nu+1)^{-1}f_a(m)\,.$$

PROOF: We use the relations (4.19) and (4.6), the last one with $n+1$ in place of n, and obtain

$$(6.15)\quad \Pr\{[K_n \leqq m][S_{n+1} = 0]\} - (m+1)(n+1)^{-1}\Pr\{S_{n+1} = 0\}$$
$$= \sum_{k=0}^{m-1}(k+1)f_{k+1}u_{n-k} + (m+1)\sum_{k=m}^{n} f_{k+1}u_{n-k}$$
$$- (m+1)(n+1)^{-1}\sum_{k=0}^{n}(k+1)f_{k+1}u_{n-k}$$
$$= (n-m)(n+1)^{-1}\sum_{k=0}^{m-1}(k+1)f_{k+1}u_{n-k}$$
$$+ (m+1)(n+1)^{-1}\sum_{k=m}^{n-1}(n-k)f_{k+1}u_{n-k} \geqq 0\,.$$

Since $f_i \geqq 0$ and $u_i \geqq 0$ for $i = 1, 2, \ldots$, we obtain from (4.6) the inequality $c_n \geqq nf_nu_0 = nf_n$. Therefore, from the assumption $c_n < An^{-a}$ we obtain $f_n < An^{-a-1}$ for $n = 1, 2, \ldots$. The existence of a positive constant M, such that $u_n < Mn^{-a-1}$ then follows from Lemma 4. If we use these inequalities we get

$$(6.16)\quad \begin{aligned} 0 &\leqq \Pr\{[K_n \leqq m][S_{n+1}=0]\} - (m+1)(n+1)^{-1}\Pr\{S_{n+1}=0\} \\ &\leqq (n-m)(n+1)^{-1}\sum_{k=0}^{m-1} A(k+1)^{-a}M(n-k)^{-a-1} \\ &\qquad + (m+1)(n+1)^{-1}\sum_{k=m}^{n-1} A(k+1)^{-a-1}M(n-k)^{-a} \\ &= AM(n+1)^{-1}\Big((n-m)\sum_{k=0}^{m-1}(k+1)^{-a}(n-k)^{-a-1} \\ &\qquad + (m+1)\sum_{k=m}^{n-1}(k+1)^{-a-1}(n-k)^{-a}\Big) \\ &< AM(n+1)^{-1}\sum_{k=0}^{n-1}(k+1)^{-a}(n-k)^{-a} \\ &\qquad \leqq AM(n+1)^{-1}2^{1+a}(n+1)^{-a}\sum_{k=0}^{[\frac{1}{2}n]}(k+1)^{-a}. \end{aligned}$$

If we consider only the sequence $n_1, n_2, \ldots$ we obtain, when we change to conditional probabilities and use $c_{n_\nu+1} > B(n_\nu+1)^{-a}$, the relation

$$(6.17)\quad \begin{aligned} 0 &\leqq \Pr\{K_{n_\nu} \leqq m \mid S_{n_\nu+1}=0\} - (m+1)(n_\nu+1)^{-1} \\ &\leqq c_{n_\nu+1}{}^{-1}AM(n_\nu+1)^{-1}2^{1+a}(n_\nu+1)^{-a}\sum_{k=0}^{[\frac{1}{2}n_\nu]}(k+1)^{-a} \\ &< AMB^{-1}(n_\nu+1)^{-1}2^{1+a}\sum_{k=0}^{[\frac{1}{2}n_\nu]}(k+1)^{-a}. \end{aligned}$$

We now have to consider the three different cases $0 < a < 1$, $a = 1$ and $a > 1$.

1° In case $0 < a < 1$ we have $\sum_{k=1}^{m} k^{-a} < (1-a)^{-1}m^{1-a}$, since

$$\begin{aligned} &(1-a)^{-1}m^{1-a} + (m+1)^{-a} \\ &= (1-a)(m+1)^{1-a}\big((1-(m+1)^{-1})^{1-a} + (1-a)(m+1)^{-1}\big) \\ &< (1-a)^{-1}(m+1)^{1-a}. \end{aligned}$$

Hence

$$(6.18)\quad \begin{aligned} 0 &\leqq \Pr\{K_{n_\nu} \leqq m \mid S_{n_\nu+1}=0\} - (m+1)(n_\nu+1)^{-1} \\ &< AMB^{-1}(n_\nu+1)^{-1}2^{1+a}(1-a)^{-1}\big([\tfrac{1}{2}n_\nu]+1\big)^{1-a} \\ &< AMB^{-1}2^{1+a}(1-a)^{-1}(n_\nu+1)^{-a} = K(n_\nu+1)^{-a}. \end{aligned}$$

2° In case $a = 1$ we have $\sum_{k=1}^{m} k^{-1} < \int_{\frac{1}{2}}^{m+\frac{1}{2}} t^{-1}dt = \log(2m+1)$. Hence

$$(6.19)\quad \begin{aligned} 0 &\leqq \Pr\{K_n \leqq m \mid S_{n_\nu+1}=0\} - (m+1)(n_\nu+1)^{-1} \\ &< AMB^{-1}4(n_\nu+1)^{-1}\log(n_\nu+1) = K(n_\nu+1)^{-1}\log(n_\nu+1). \end{aligned}$$

3° In case $a > 1$ we have $\sum_{k=1}^{m} k^{-a} < \sum_{k=1}^{\infty} k^{-a} = C_a < \infty$. Hence

$$(6.20)\quad \begin{aligned} 0 &\leqq \Pr\{K_{n_\nu} \leqq m \mid S_{n_\nu+1}=0\} - (m+1)(n+1)^{-1} \\ &< AMB^{-1}2^{1+a}C_a(n_\nu+1)^{-1} = K(n_\nu+1)^{-1}. \end{aligned}$$

The relations (6.18) – (6.20) just found prove the statements of Theorem 6 about[2] $\Pr\{K_{n_\nu} \leqq m \mid S_{n_\nu+1} = 0\}$.

We now pass to the proof of the rest of the theorem. It is evident from equation (4.18) that

$$\text{(6.21)} \qquad \Pr\{K_{n_\nu} = 0 \mid S_{n_\nu+1} = 0\}$$
$$\geqq \Pr\{K_{n_\nu} = 1 \mid S_{n_\nu+1} = 0\} \geqq \ldots \geqq \Pr\{K_{n_\nu} = n_\nu \mid S_{n_\nu+1} = 0\} \geqq 0 .$$

We therefore have

$$\begin{aligned}\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\} &\leqq (m+1)^{-1} \Pr\{K_{n_\nu} \leqq m \mid S_{n_\nu+1} = 0\}\\ &\leqq (m+1)^{-1}\big((m+1)(n_\nu+1)^{-1} + Kf_a(n_\nu+1)\big)\\ &= (n_\nu+1)^{-1} + K(m+1)^{-1}f_a(n_\nu+1) .\end{aligned}$$

If $m \geqq [\frac{1}{2}n_\nu]$, we have

$$\begin{aligned}\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\} &\leqq (n_\nu+1)^{-1} + K(\tfrac{1}{2}n_\nu+\tfrac{1}{2})^{-1}f_a(n_\nu+1)\\ &= (n_\nu+1)^{-1} + 2K(n_\nu+1)^{-1}f_a(n_\nu+1)\\ &\leqq (n_\nu+1)^{-1} + 2K(n_\nu+1)^{-1}f_a(m) .\end{aligned}$$

If $m < [\frac{1}{2}n_\nu]$, we obtain

$$\begin{aligned}&\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\}\\ &\quad\leqq \Pr\{K_{n_\nu} = [\tfrac{1}{2}n_\nu] \mid S_{n_\nu+1} = 0\} + c_{n_\nu+1}{}^{-1}\sum_{k=m}^{[\frac{1}{2}n_\nu]-1} f_{k+1}u_{n_\nu-k}\\ &\quad\leqq (n_\nu+1)^{-1} + 2K(n_\nu+1)f_a(m) + c_{n_\nu+1}{}^{-1}\sum_{k=m}^{[\frac{1}{2}n_\nu]-1} f_{k+1}u_{n_\nu-k} .\end{aligned}$$

Since

$$\begin{aligned}c_{n_\nu+1}{}^{-1}\sum_{k=m}^{[\frac{1}{2}n_\nu]-1} f_{k+1}u_{n_\nu-k} &\leqq B^{-1}(n_\nu+1)^a \sum_{k=m}^{[\frac{1}{2}n_\nu]-1} A(k+1)^{-a-1}M(n_\nu-k)^{-a-1}\\ &\leqq AMB^{-1}(n_\nu+1)^a(n_\nu-[\tfrac{1}{2}n_\nu]+1)^{-a-1}\sum_{k=m}^{[\frac{1}{2}n_\nu]-1}(k+1)^{-a-1}\\ &\leqq AMB^{-1}2^{1+a}(n_\nu+1)^{-1}\int_m^{[\frac{1}{2}n_\nu]} x^{-a-1}dx\\ &\leqq AMB^{-1}2^{1+a}a^{-1}(n_\nu+1)^{-1}m^{-a} \leqq AMB^{-1}2^{1+a}a^{-1}(n_\nu+1)^{-1}f_a(m),\end{aligned}$$

we obtain for $m = 1, \ldots, n_\nu$,

$$\begin{aligned}&\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\}\\ &\quad\leqq (n_\nu+1)^{-1} + 2K(n_\nu+1)^{-1}f_a(m) + AMB^{-1}2^{1+a}a^{-1}(n_\nu+1)^{-1}f_a(m)\\ &\quad= (n_\nu+1)^{-1} + K'(n_\nu+1)^{-1}f_a(m) ,\end{aligned}$$

if we let $K' = 2K + AMB^{-1}2^{1+a}a^{-1}$.

[2] It follows from the proof of Lemma 4 and the arguments above that an upper bound for the constant K in Theorem 6 may be calculated explicitly if the distribution of the variables X_i is known.

The other half of (6.14) is proved analogously. First we obtain from (6.21)

$$\begin{aligned}\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\} &\geqq (n_\nu - m+1)^{-1}\Pr\{K_{n_\nu} \geqq m \mid S_{n_\nu+1} = 0\}\\ &\geqq (n_\nu+1)^{-1} - K(n_\nu-m+1)^{-1}f_a(n_\nu+1)\,.\end{aligned}$$

If $m \leqq [\frac{1}{2}n_\nu]$, we have

$$\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\} \geqq (n_\nu+1)^{-1} - 2K(n_\nu+1)^{-1}f_a(n_\nu-m)\,.$$

If $m > [\frac{1}{2}n_\nu]$, we obtain

$$\begin{aligned}&\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\}\\ &\geqq \Pr\{K_{n_\nu} = [\tfrac{1}{2}n_\nu] \mid S_{n_\nu+1} = 0\} - c_{n_\nu+1}{}^{-1}\sum_{k=[\frac{1}{2}n_\nu]}^{m-1} f_{k+1}u_{n_\nu-k}\\ &\geqq (n_\nu+1)^{-1} - 2K(n_\nu+1)^{-1}f_a(n_\nu-m) - AMB^{-1}2^{1+a}a^{-1}(n_\nu+1)^{-1}f_a(n_\nu-m)\,.\end{aligned}$$

We therefore have for $m = 0, 1, \ldots, n_\nu - 1$,

$$\Pr\{K_{n_\nu} = m \mid S_{n_\nu+1} = 0\} \geqq (n_\nu+1)^{-1} - K'(n_\nu+1)^{-1}f_a(n_\nu-m)\,.$$

This completes the proof of Theorem 6.

7. Applications of Theorem 6. Let the variables $X_1, X_2, \ldots$ be independent and have a common distribution, which is a lattice distribution, that is, X_i assumes only integer values. Let us furthermore assume that $\Pr\{X_i = 0\} < 1$. It then follows from results of Gnedenko [5] that if

$$E(X_i) = 0, \qquad E(X_i^2) < \infty \tag{7.1}$$

and furthermore the greatest common divisior of the values which X_i assume with positive probability is one, then $\Pr\{S_n = 0\} \sim An^{-\frac{1}{2}}$ for $n \to \infty$ and some positive constant A. We may therefore apply Theorem 6 and obtain, for $m = 0, 1, \ldots, n$,

$$(m+1)(n+1)^{-1} < \Pr\{N_n \leqq m \mid S_{n+1} = 0\} < (m+1)(n+1)^{-1} + K(n+1)^{-\frac{1}{2}},$$

for some positive constant K depending on the common distribution of the random variables X_i.

It is easy to see that if we consider N_n' (see Section 1), then we obtain, for $m = 0, 1, \ldots, n$,

$$\begin{aligned}(m+1)(n+1)^{-1} - K(n+1)^{-\frac{1}{2}} &< \Pr\{N_n' \leqq m \mid S_{n+1} = 0\}\\ &< (m+1)(n+1)^{-1} + K(n+1)^{-\frac{1}{2}}\,,\end{aligned}$$

where K is the same constant as above. This result evidently contains the theorem of M. Lipschutz in [7].

From the results in another paper [6] of Gnedenko it follows that if (7.1) is replaced by the condition that X_i belongs to the domain of attraction of a symmetric stable law with exponent $\alpha < 2$, then $\Pr\{S_n = 0\} \sim A n^{-1/\alpha}$, for $n \to \infty$, so that Theorem 6 may be applied with $a = \alpha^{-1}$.

Theorem 6 may, however, also be used for distributions which are not lattice distributions. As a simple example, we shall consider the following distribution:

$$\Pr\{X_i = -1\} = 1/4, \qquad \Pr\{X_i = -b\} = 1/4\,,$$
$$\Pr\{X_i = +1\} = 1/4, \qquad \Pr\{X_i = +b\} = 1/4\,,$$

where b is an irrational number. It is easy to see that in this case $\Pr\{S_n = 0\}$ equals the probability of return to the origin after n steps in a random walk with unit steps parallel to the x-axis and y-axis in a plane. This probability is $\sim K n^{-1}$ for n even and $n \to \infty$, see for example Feller [4, pp. 297–8]. We may therefore apply Theorem 6 with $a = 1$, and the sequence $n = 2, 4, \ldots$.

Finally it may be noted that application of Theorem 6 to Bernoulli trials shows that, for $a = \frac{1}{2}$, the order of magnitude in the remainder terms cannot be improved.

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