MATH. SCAND. 1 (1953)

A RECIPROCITY FORMULA FOR WEIGHTED QUADRATIC PARTITIONS

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1. Introduction. Let $q = p^n$, where p is an odd prime. For $\alpha \in GF(q)$, put

$$e(\alpha) = e^{2\pi i t(\alpha)/p}, \tag{1.1}$$

where

$$t(\alpha) = \alpha + \alpha^p + \ldots + \alpha^{p^{n-1}};$$

also let

$$S(\alpha, \lambda, Q) = \sum_{Q(\xi)=\alpha} e(2\lambda_1\xi_1 + \ldots + 2\lambda_r\xi_r), \tag{1.2}$$

where $\alpha, \lambda_j \in GF(q)$,

$$Q(u) = \sum_1^r \alpha_{kj} u_k u_j \qquad (\alpha_{kj} \in GF(q),\ \delta = |\alpha_{kj}| \neq 0),$$

and the summation in the right member of (1.2) is over all $\xi_i \in GF(q)$ such that $Q(\xi_1, \ldots, \xi_r) = \alpha$. It was shown incidentally in [2] that if

$$\lambda_k = \sum_{j=1}^r \alpha_{kj} \lambda_j' \qquad (k = 1, \ldots, r) \tag{1.3}$$

then $S(\alpha, \lambda, Q)$ satisfies the following reciprocity relation,

$$S(\alpha, \lambda, Q) = S(\alpha, \lambda', Q'), \tag{1.4}$$

where $Q'(u)$ denotes the quadratic form inverse to $Q(u)$. In this note we give a direct proof of (1.4) as well as of one or two extensions. We also consider the analogous formula when the coefficients are rational integers.

2. By a well-known theorem [1, p. 160, Theorem 3] the linear transformation

$$\xi_k = \sum \alpha_{kj} \xi_j' \tag{2.1}$$

carries Q into Q', that is

$$Q(\xi') = Q'(\xi). \tag{2.2}$$

We have also

$$\sum_{j=1}^r \lambda_j \xi_j' = \sum_{j=1}^r \lambda_j' \xi_j. \tag{2.3}$$

Received May 16, 1953

Now by (1.2),

$$S(\alpha, \lambda, Q) = \sum_{Q(\xi')=\alpha} e(2\lambda_1\xi_1' + \dots + 2\lambda_r\xi_r'),$$

and by (2.2) and (2.3) this becomes

$$S(\alpha, \lambda, Q) = \sum_{Q'(\xi)=\alpha} e(2\lambda_1'\xi_1 + \dots + 2\lambda_r'\xi_r) = S(\alpha, \lambda', Q'),$$

which evidently proves (1.4).

If $f(u) = f(u_1, \dots, u_r)$ denotes an arbitrary polynomial with coefficients in $GF(q)$, we define

$$S(\alpha, f, Q) = \sum_{Q(\xi)=\alpha} e(f(\xi)), \tag{2.4}$$

which clearly generalizes (1.2). Now let (2.1) carry f into f', that is,

$$f(\xi') = f'(\xi), \tag{2.5}$$

thus generalizing (2.3). Then it is clear that the previous argument may be applied to yield the formula

$$S(\alpha, f, Q) = S(\alpha, f', Q'). \tag{2.6}$$

We have thus obtained a first generalization of (1.4). However this can be carried a bit further. Let $g(u) = g(u_1, \dots, u_r)$ denote another arbitrary polynomial with coefficients in $GF(q)$ and let (2.1) carry g into g', that is,

$$g(\xi') = g'(\xi). \tag{2.7}$$

We define

$$S(\alpha, f, g) = \sum_{g(\xi)=\alpha} e(f(\xi)), \tag{2.8}$$

the summation extending over all $\xi_i \in GF(q)$ such that $g(\xi_1, \dots, \xi_r) = \alpha$. Then exactly as in the proof of (1.4) we have

$$S(\alpha, f, g) = \sum_{g(\xi')=\alpha} e(f(\xi')) = \sum_{g'(\xi)=\alpha} e(f'(\xi)),$$

which implies

$$S(\alpha, f, g) = S(\alpha, f', g'). \tag{2.9}$$

Thus (2.9) together with (2.5) and (2.7) furnish a two-fold generalization of (1.4). Note that it is no longer necessary to assume p odd.

3. We now briefly consider an analog of (1.4) involving positive quadratic forms with rational integral coefficients. Let

$$Q(u) = \sum_{1}^{r} a_{kj}u_k u_j \qquad (|a_{kj}| = 1), \tag{3.1}$$

where the a_{kj} are rational integers. We assume that $Q(u)$ is a positive definite form; thus the equation $Q(u) = m$, where m is an arbitrary positive integer, has at most a finite number of integral solutions. We define

$$S(m, \lambda, Q) = \sum_{Q(u)=m} \exp\left(2\pi i(\lambda_1 u_1 + \dots + \lambda_r u_r)\right), \tag{3.2}$$

where the λ_j denote arbitrary complex numbers. If we put

$$u_k = \sum_j a_{kj} u_j' , \tag{3.3}$$

then in view of the hypothesis $|a_{kj}| = 1$, the inverse of (3.3) also has integral coefficients; also, as in (2.2), we have now

$$Q(u') = Q'(u) , \tag{3.4}$$

where again Q' denotes the quadratic form inverse to Q. If we define λ_k' by means of

$$\lambda_k = \sum_1^r a_{kj} \lambda_j' , \tag{3.5}$$

then exactly as in § 2 we may prove the reciprocity formula

$$S(m, \lambda, Q) = S(m, \lambda', Q') . \tag{3.6}$$

Clearly (3.6) can be generalized but we shall not take the space to do so.

The following remark may be of interest. Define

$$\begin{aligned} \vartheta(t, \lambda, Q) &= \sum_{m=0}^{\infty} S(m, \lambda, Q) e^{-mt} \\ &= \sum_{u_1, \dots, u_r = -\infty}^{\infty} \exp\left(-tQ(u) + 2\pi i(\lambda_1 u_1 + \dots + \lambda_r u_r)\right) , \end{aligned} \tag{3.7}$$

where $\operatorname{Re}(t) > 0$. Applying (3.6), we see that (3.7) yields the formula

$$\vartheta(t, \lambda, Q) = \vartheta(t, \lambda', Q') , \tag{3.8}$$

subject to (3.4), (3.5) and the stated hypothesis for Q.

REFERENCES

1. M. Bôcher, *Higher algebra*, New York, 1924.
2. L. Carlitz, *Weighted quadratic partitions over a finite field*, Canadian J. Math. 5 (1953), 317–323.

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