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GENERALIZATION OF A THEOREM OF BOGOLIOÙBOFF TO TOPOLOGICAL ABELIAN GROUPS

WITH AN APPENDIX ON BANACH MEAN VALUES IN NON-ABELIAN GROUPS

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1. Introduction. In the simplest known proof of the main theorem on ordinary almost periodic functions, given in 1939 by Bogolioùboff [2], a theorem on relatively dense sets of numbers plays the decisive role. Using the same principal idea as Bogolioùboff, the author [3], [4] gave a proof of the main theorem on almost periodic functions in an arbitrary abelian group. The decisive role was played here by a theorem on relatively dense sets of elements in a denumerable abelian group (viz. a certain subgroup of the original group). The proofs of these theorems on relatively dense sets were based on the elementary "Fourier analysis" in finite abelian groups.

In the present paper these theorems will be generalized to arbitrary topological abelian groups, but the proof will no longer be elementary. It will be based on a decomposition theorem of Godement for positive definite functions in groups which generalizes a theorem of Bochner for the usual positive definite functions. The existence of an invariant Banach mean value for all bounded functions on an arbitrary abelian group will be a tool in the proof. This existence was established by Banach by way of an example.

In my efforts to generalize Bogolioùboff's theorem on relatively dense sets of numbers to a theorem on relatively dense class sets in a non-abelian group (see [5]) I have found some results on Banach mean values in non-abelian groups which may have an interest in themselves. They are set forth in an appendix. However, I have not been able to decide whether Bogolioùboff's theorem may be generalized to arbitrary non-abelian groups.

2. The generalized Bogolioùboff theorem and some related theorems. We shall now state the generalization to an arbitrary topological abelian

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group G of Bogolioùboff's theorem on relatively dense sets of numbers. We recall that a subset E of G is called relatively dense with respect to k elements $a_1, \ldots, a_k$ if $(E+a_1) \cup \ldots \cup (E+a_k) = G$. See [3, p. 360]. By $E \pm E$ (respectively $E+a$) we understand the set of all $x \pm y$ (respectively $x+a$) with $x \in E, y \in E$.

THEOREM 1. *Let G be an arbitrary topological abelian group and E a subset of G which is relatively dense with respect to k elements. Let furthermore V be an arbitrary neighbourhood of 0 in G. Then there exist q continuous characters $\chi_1(x), \ldots, \chi_q(x)$ on G where $q \leqq k^2$ such that every x which satisfies the q inequalities*

$$\operatorname{Re}\chi_1(x) \geqq 0, \ldots, \operatorname{Re}\chi_q(x) \geqq 0$$

belongs to the set $E-E+E-E+V$.

If in particular G is a discrete group the word "continuous" and the neighbourhood V can, of course, be omitted in the formulation, corresponding to the fact that V can be taken equal to $\{0\}$.

Bogolioùboff [2] proved this theorem (with some changes in details) for the case where G is the discrete additive group of all integers. From this case he passed to the case where G is the additive group of all real numbers with the usual topology. In [3], [4] I proved the theorem for the case where G is an arbitrary denumerable discrete abelian group, but I had to replace the $E-E+E-E$ in the theorem by $E-E+E-E+E-E+E-E$ (and $\geqq$ in the inequalities by $>$).

From Theorem 1 we shall deduce in a simple way two corollaries. We recall first that a topological abelian group G is called maximally almost periodic if to any two different elements a and b from G there exists a continuous almost periodic function $f(x)$ on G with $f(a) \neq f(b)$ or, equivalently, there exists to any $a \neq 0$ in G a continuous character $\chi(x)$ on G such that $\chi(a) \neq 1$. Further, G is called minimally almost periodic if the constants are the only continuous almost periodic functions on G or, equivalently, the principal character is the only continuous character on G. The closure of a subset E of G is denoted by $\overline{E}$.

COROLLARY 1. *A necessary and sufficient condition that a topological abelian group G be maximally almost periodic is that there exists to every $a \neq 0$ in G a relatively dense set E in G such that $a \notin \overline{E-E+E-E}$.*

COROLLARY 2. *A necessary and sufficient condition that a topological abelian group G be minimally almost periodic is that for any relatively dense set E in G the relation $\overline{E-E+E-E} = G$ is valid.*

Next we shall state two theorems which are related to Theorem 1. With regard to the first of these theorems we make the following remark. An almost periodic function $f(x)$ on an abelian group G may be defined as a function which for every $\varepsilon > 0$ has a relatively dense set of translation elements belonging to ε, that is, elements τ with $|f(x+\tau)-f(x)| \leqq \varepsilon$ for all x. If $f(x)$ is also continuous, it is easily seen to be uniformly continuous. Consequently, for any $\varepsilon > 0$ the set of ε-translation elements of $f(x)$ contains a set of the form $E+V_0$ where E is a relatively dense set and V_0 is a neighbourhood of 0.

The first of the following theorems is a partial converse of the simple fact that if a complex function $f(x)$ can be ε_0-approximated by a continuous almost periodic function $g(x)$, that is, $|f(x)-g(x)| \leqq \varepsilon_0$ for all x, then for every $\varepsilon > 0$ the set of $(2\varepsilon_0+\varepsilon)$-translation elements of $f(x)$ contains a set of the form $E+V_0$.

THEOREM 2. *Let $f(x)$ be a complex function on a topological abelian group G and let for a given $\varepsilon_0 > 0$ the set of translation elements of $f(x)$ belonging to ε_0 contain a set of the form $E+V_0$ where E is a relatively dense set and V_0 is a neighbourhood of* 0. *Then there exists a continuous almost periodic function $g(x)$ on G such that $|f(x)-g(x)| \leqq 2\varepsilon_0$ for all x.*

In the next theorem the notion of upper mean measure $\bar{m}E$ of a set E in G occurs. It is defined as $\{\bar{M}f(x)\}$ where $f(x)$ is the characteristic function of E, and $\bar{M}\{f(x)\}$ is defined in Section 3, (1), below.

THEOREM 3. *Let E be a relatively dense set in the topological abelian group G and V an arbitrary neighbourhood of* 0. *Then there exists a finite number of continuous characters $\chi_1(x), \ldots, \chi_m(x)$ such that all the x which satisfy the inequalities $\operatorname{Re}\chi_1(x) \geqq 0, \ldots, \operatorname{Re}\chi_m(x) \geqq 0$, with exception of a set with upper mean measure zero, belong to the set $E-E+V$.*

3. Banach mean values in an abelian group. The introduction of a translation invariant mean value for all bounded real functions on an abelian group is indicated by Banach [1, pp. 30–32] by way of an example, viz. the group of real numbers modulo 1. We shall repeat only those parts of Banach's proof which will be directly needed for the discussion of the non-abelian case in the appendix.

Let $f(x)$ be a bounded real function on the abelian group G. (In this section the topology is of no interest.) We define the upper mean value

$$\bar{M}f = \bar{M}_x\{f(x)\} = \inf_{\mathcal{A}} \sup_x \Sigma \alpha_n f(x+a_n) \tag{1}$$

where $\mathcal{A} = \{\alpha_1, \ldots, \alpha_N; a_1, \ldots, a_N\}$, $\alpha_n > 0$, $\Sigma \alpha_n = 1$, $a_n \in G$. It satisfies the relations

$$\inf_x f(x) \leqq \overline{M} f \leqq \sup_x f(x) \tag{2}$$

$$\overline{M}_x\{f(x+a)\} = \overline{M}\{f(x)\} \tag{3}$$

$$\overline{M}\{\lambda f\} = \lambda \overline{M} f \qquad (\lambda \geqq 0) \tag{4}$$

$$\overline{M}\{f+g\} \leqq \overline{M} f + \overline{M} g \tag{5}$$

$$\overline{M}_x\{f(x)-f(x+a)\} = 0\,. \tag{6}$$

For a non-abelian group

$$\overline{M}_1 f = \inf_{\mathcal{A}} \sup_{x,y} \Sigma\, \alpha_n f(x a_n y) \tag{1*}$$

satisfies (2), (4), (5), and the analogue of (3), but in general not the analogue of (6). (Without the y in the expression (1*) the fundamental relation (5) would not be valid in general.)

Banach now obtains a mean value Mf with the desired properties by using the following theorem (cf. [1, pp. 27–28]), the proof of which is based on transfinite induction.

BANACH'S THEOREM. *Let $\overline{M} f$ be a real functional defined on a real linear space L and satisfying* (4) *and* (5). *Then there exists a linear functional Mf on L for which $Mf \leqq \overline{M} f$.*

This theorem is applied to the space L of all bounded real functions on G and the functional $\overline{M} f$ defined in (1). The resulting Mf satisfies the following relations

$$\inf_x f(x) \leqq \underline{M} f \leqq M f \leqq \overline{M} f \leqq \sup_x f(x) \tag{a}$$

$$M_x\{f(x+a)\} = M\{f(x)\} \tag{b}$$

$$M\{\lambda f\} = \lambda M f \qquad (\lambda \text{ real}) \tag{c}$$

$$M\{f+g\} = Mf + Mg \tag{d}$$

where $\underline{M} f = -\overline{M}(-f)$. Only $\inf f \leqq \underline{M} f \leqq Mf$ and (b) need proofs. From (c) and $Mf \leqq \overline{M} f \leqq \sup f$ follows $Mf = -M\{-f\} \geqq -\overline{M}\{-f\} \geqq -\sup(-f) = \inf f$. As to (b) we have $M\{f(x)\} \leqq M\{f(x+a)\}$ since from (d), (a), and (6) it follows that

$$M\{f(x)\} - M\{f(x+a)\} = M\{f(x)-f(x+a)\} \leqq \overline{M}\{f(x)-f(x+a)\} = 0\,.$$

Replacing $f(x)$ by $f(x+a)$ and a by $-a$ we get also $M\{f(x)\} \geqq M\{f(x+a)\}$.

We extend the functional Mf to all bounded complex functions on G by putting $M\{u(x)+iv(x)\}$, where $u(x)$ and $v(x)$ are bounded real functions, equal to $M\{u(x)\}+iM\{v(x)\}$. Obviously (b), (c), (d) are still valid, (c) also for complex λ. Furthermore $|Mf| \leqq M\{|f|\}$. The proof is known: For a certain real θ we have

$$|Mf| = e^{i\theta}Mf = M\{e^{i\theta}f\} = M\{\mathrm{Re}(e^{i\theta}f)\} \leqq M\{|e^{i\theta}f|\} = M\{|f|\}\,.$$

Remark. If $f(x)$ is almost periodic or positive definite we have necessarily $Mf = \mathfrak{M}f$ where $\mathfrak{M}f$ denotes the usual mean value of $f(x)$. See [8, p. 451] and [6, p. 59].

4. Proofs of Theorem 1 and Corollaries 1 and 2. In the following Mf denotes a Banach mean value defined as indicated in Section 3 on all bounded complex functions on G. If f is almost periodic or positive definite we may write it $\mathfrak{M}f$.

Proof of Theorem 1. Let V_0 be a neighbourhood of 0 chosen so that $V_0-V_0+V_0-V_0 \subset V$ (see [9], p. 11). There exists a uniformly continuous non-negative function $h_0(x) \leqq 1$ on G which is equal to 1 at the point 0 and equal to 0 outside of V_0 (see [9, pp. 13–14]). The function

$$j(x) = \sup_{y\in E} h_0(-y+x)$$

is easily seen to be a uniformly continuous, non-negative function not exceeding 1 which is positive only in $E+V_0$; furthermore $j(x) \geqq c(x)$ where $c(x)$ is the characteristic function of E.

Next, we form the function

$$\mu(x) = \underset{t}{M}\{j(t)j(t+x)\}\,.$$

Obviously $\mu(x)$ is a non-negative, uniformly continuous function. Furthermore $\mu(x)$ can be positive only if $x \in E-E+V_0-V_0$; for $\mu(x) > 0$ implies the existence of a t such that $j(t) > 0$, $j(t+x) > 0$, that is, $t \in E+V_0$, $t+x \in E+V_0$.

We shall show now that $\mu(x)$ is a positive definite function (cf. Khintchine [7, p. 568]), that is,

$$\sum_{n=1}^{N}\sum_{m=1}^{N} d_n \bar{d}_m \mu(x_n-x_m) \geqq 0$$

for arbitrary complex numbers d and elements x. Indeed, we have

$$\sum_{n=1}^{N}\sum_{m=1}^{N} d_n \bar{d}_m \mu(x_n - x_m) = \sum_{n=1}^{N}\sum_{m=1}^{N} d_n \bar{d}_m \underset{t}{M}\{j(t)j(t+x_n-x_m)\}$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} d_n \bar{d}_m \underset{t}{M}\{j(t-x_n)j(t-x_m)\} = \underset{t}{M}\left\{\sum_{n=1}^{N}\sum_{m=1}^{N} d_n \bar{d}_m j(t-x_n)\overline{j(t-x_m)}\right\}$$

$$= \underset{t}{M}\left\{\left|\sum_{n=1}^{N} d_n j(t-x_n)\right|^2\right\} \geqq 0 .$$

Next, we shall obtain a lower estimate for $\mathfrak{M}\mu$. From the relation $(E+a_1) \cup \ldots \cup (E+a_k) = G$ we get

$$\sum_{p=1}^{k} \mu(x-a_p) = \sum_{p=1}^{k} \underset{t}{M}\{j(t)j(t+x-a_p)\}$$

$$= \sum_{p=1}^{k} \underset{t}{M}\{j(t-x)j(t-a_p)\} \geqq \sum_{p=1}^{k} \underset{t}{M}\{j(t-x)c(t-a_p)\}$$

$$= \underset{t}{M}\left\{j(t-x) \sum_{p=1}^{k} c(t-a_p)\right\} \geqq \underset{t}{M}\{j(t-x)\} = Mj ,$$

since $c(t-a_p)$ is the characteristic function of $E+a_p$. Hence

$$\mu(x-a_1) + \ldots + \mu(x-a_k) \geqq Mj$$

and by taking mean values on both sides of the inequality we obtain the desired estimate

$$\mathfrak{M}\mu \geqq k^{-1} Mj . \tag{1}$$

Here $Mj > 0$, for from $(E+a_1) \cup \ldots \cup (E+a_k) = G$ we get $kMc \geqq 1$, so that $Mj \geqq Mc \geqq k^{-1} > 0$. For $\mu(0)$ we shall later use the simple estimate

$$\mu(0) = M\{j(t)^2\} \leqq Mj , \tag{2}$$

which follows from $0 \leqq j(t) \leqq 1$.

Since Godement has only formulated the result to be used, in the special case of a locally compact group, we consider for a moment G with the discrete topology instead of the given one. Then G is a locally compact group and $\mu(x)$ a continuous positive definite function on G. Hence, in consequence of Godement's theorem [6, p. 64],

$$\mu(x) = a(x) + z(x) \tag{3}$$

where $a(x)$ is a positive definite, almost periodic function,

$$a(x) = \sum_{n=0}^{\infty} A_n \chi_n(x) , \tag{4}$$

$A_n \geqq 0$, $\Sigma A_n < +\infty$, and $z(x)$ is a positive definite function with $\mathfrak{M}\{|z|^2\} = 0$. For any n with $A_n > 0$ we consider the known relation

$$(5) \qquad A_n\chi_n(x) = a*\chi_n(x) = \mu*\chi_n(x) ,$$

the last sign of equality being valid since

$$0 \leqq |z*\chi_n(x)| = |\mathfrak{M}\{z(t)\chi_n(-t+x)\}| \leqq \left(\mathfrak{M}\{|z|^2\}\right)^{\frac{1}{2}} = 0 .$$

When we reconsider the given topology in G for which, as we have mentioned, $\mu(x)$ is a uniformly continuous function, we see from (5) that $\chi_n(x)$ is also a continuous function. Since $\mu(x)$ is real, it follows also from (5) that

$$A_n\bar{\chi}_n(x) = \mu*\bar{\chi}_n(x) = a*\bar{\chi}_n(x)$$

so that $\bar{\chi}_n(x)$ appears in (4) with the same coefficient A_n as $\chi_n(x)$. Thus $a(x)$ and $z(x)$ are real continuous functions.

Let $\chi_0(x)$ be the principal character. It follows from (1) that

$$(6) \qquad A_0 = \mathfrak{M}\mu \geqq k^{-1}Mj \qquad (> 0) .$$

For $n \geqq 1$ we have $A_n \leqq A_0$ since

$$A_n = \mathfrak{M}\{\mu(x)\mathrm{Re}\chi_n(x)\} \leqq \mathfrak{M}\{\mu(x)\} = A_0 .$$

We now arrange all of the positive A_n in a descending sequence $A_0 = A_{\nu_0} \geqq A_{\nu_1} \geqq \dots \geqq A_{\nu_l} \geqq \dots$ (this sequence may be finite). Since $z(x)$ is a positive definite function we have $z(0) \geqq 0$. Hence from (2) and (3) we get $a(0) \leqq Mj$ and this together with (4) implies that

$$\sum_{l=0}^{\infty} A_{\nu_l} \leqq Mj$$

and hence

$$(7) \qquad A_{\nu_l} \leqq (l+1)^{-1}Mj .$$

Finally, we form the function

$$(8) \quad \mu_1(x) = \mu*\mu(x) = \mathfrak{M}\{\mu(t)\mu(-t+x)\} = a*a(x) = \sum_{l=0}^{\infty} A_{\nu_l}{}^2\chi_{\nu_l}(x) .$$

Obviously $\mu_1(x)$ is non-negative. Furthermore $\mu_1(x)$ can be positive only if $x \in E-E+E-E+V$; in fact $\mu_1(x) > 0$ implies the existence of a t such that $\mu(t) > 0$ and $\mu(-t+x) > 0$, that is, t, $-t+x$ lie both in $E-E+V_0-V_0$, and V_0 is chosen so that $V_0-V_0+V_0-V_0 \subset V$. From (8) we get

$$\mu_1(x) = A_{\nu_0}{}^2 + \sum_{l=1}^{q} A_{\nu_l}{}^2\mathrm{Re}\chi_{\nu_l}(x) + \sum_{l=q+1}^{\infty} A_{\nu_l}{}^2\mathrm{Re}\chi_{\nu_l}(x) .$$

We shall show that $\mu_1(x) > 0$ for every x which satisfies the q inequalities $\mathrm{Re}\chi_{\nu_l}(x) \geqq 0$, $l = 1, 2, \ldots, q$, when only q is chosen $\geqq k^2$. From (6) and (7) we have

$$\mu_1(x) \geqq \frac{(Mj)^2}{k^2} - (Mj)^2 \left(\frac{1}{(q+2)^2} + \frac{1}{(q+3)^2} + \cdots \right) \geqq \frac{(Mj)^2}{k^2} - \frac{(Mj)^2}{q+1}$$

which, since $Mj > 0$, is positive for every $q \geqq k^2$. Hence in Theorem 1 we can actually use a number $q \leqq k^2$. This finishes the proof of Theorem 1.

Proof of Corollary 1. We shall first show that the condition is necessary. Let $\chi(x)$ be a continuous character such that $\chi(a) \neq 1$. By E we denote the set of x for which $|\mathrm{Arg}\,\chi(x)| \leqq \frac{1}{5}|\mathrm{Arg}\,\chi(a)|$. As is well known (see for instance [4, p. 568]), this is a relatively dense set. For every $x \in E-E+E-E$ we have $|\mathrm{Arg}\,\chi(x)| \leqq \frac{4}{5}|\mathrm{Arg}\,\chi(a)|$. Since the set of x for which this last inequality is satisfied, is a closed set which does not contain a, the set $\overline{E-E+E-E}$ does not contain a either.

That the condition is sufficient is a simple consequence of Theorem 1. Given an $a \neq 0$ we shall find a continuous character $\chi(x)$ with $\chi(a) \neq 1$. We choose a relatively dense set E such that $\overline{E-E+E-E}$ does not contain a. Then there exists a neighbourhood V of 0 such that $E-E+E-E+V$ does not contain a. It follows from Theorem 1 that there exist q continuous characters $\chi_1(x), \ldots, \chi_q(x)$ such that every x which satisfies the inequalities $\mathrm{Re}\chi_1(x) \geqq 0, \ldots, \mathrm{Re}\chi_q(x) \geqq 0$ belongs to $E-E+E-E+V$. Since a does not belong to the latter set, at least one of the characters $\chi_1(x), \ldots, \chi_q(x)$ must be different from 1 at a.

Proof of Corollary 2. Analogously.

5. Proof of Theorem 2. We shall now let the previous E and V_0 indicate the E and V_0 appearing in Theorem 2. From (4) and (6) in Section 4 we have

$$A = \sup_x a(x) = a(0) > 0 .$$

We consider the non-identically vanishing, continuous, almost periodic function $a_1(x)$ which is equal to $a(x)-A/2$ for $a(x) \geqq A/2$ and equal to 0 elsewhere (it has obviously the same translation elements as $a(x)$). Then all x for which the non-negative function $a_1(x)$ is positive will be $2\varepsilon_0$-translation elements of $f(x)$ with the exception of the x in a set Z with $\bar{m}Z = 0$. In fact, $a_1(x) > 0$ implies $a(x) > A/2$ and hence by (3) we get $\mu(x) > 0$ with the exception of the x in a set Z where $z(x) < -A/2$; but $\mathfrak{M}\{z^2\} = 0$ and hence $\bar{m}Z = 0$. To complete the reasoning we have only to remark that an x with $\mu(x) > 0$ belongs to $(E+V_0)-(E+V_0)$ and is therefore a $2\varepsilon_0$-translation element of $f(x)$.

We now form the kernel

$$K(x) = a_1(x)/\mathfrak{M}a_1$$

which has the following properties. $K(x)$ is continuous and almost periodic, $K(x) \geqq 0$, $\mathfrak{M}K = 1$, and the set of x where $K(x) > 0$ has the form $T \cup Z$ where T and Z are disjoint sets and T consists of $2\varepsilon_0$-translation elements of $f(x)$ while $\bar{m}Z = 0$.

Let E be relatively dense with respect to the elements $a_1, \ldots, a_k$. Since all elements in E are ε_0-translation elements of $f(x)$ we get

$$|f(x)| \leqq \varepsilon_0 + \max\{|f(a_1)|, \ldots, |f(a_k)|\}$$

so that $f(x)$ is bounded. Hence we can form the function

$$g(x) = \underset{t}{M}\{f(t+x)K(t)\} = \underset{t}{M}\{f(t)K(t-x)\}\,.$$

It is continuous and almost periodic since

$$|g(x+\tau)-g(x)| = |\underset{t}{M}\{f(t)\big(K(t-x-\tau)-K(t-x)\big)\}| \leqq \varepsilon M\{|f(t)|\}$$

when τ is any ε-translation element of $K(x)$. Finally, we have

$$|f(x)-g(x)| \leqq 2\varepsilon_0$$

since

$$|f(x)-g(x)| = |\underset{t}{M}\{\big(f(x)-f(t+x)\big)K(t)\}| \leqq \underset{t}{M}\{|f(x)-f(t+x)|\,K(t)\}$$
$$= \underset{t\in T}{M} + \underset{t\in Z}{M} + \underset{t\in[K=0]}{M} \leqq 2\varepsilon_0 + 0 + 0\,.$$

This completes the proof of Theorem 2.

6. Proof of Theorem 3. This time we choose the previous V_0 as a neighbourhood of 0 with $V_0 - V_0 \subset V$. Since $A_0 > 0$ in consequence of (6) in Section 4, we can choose m so large that

$$\sum_{m+1}^{\infty} A_n < A_0\,.$$

When x satisfies the inequalities $\mathrm{Re}\chi_1(x) \geqq 0, \ldots, \mathrm{Re}\chi_m(x) \geqq 0$, we get from (4) in Section 4

$$a(x) = \sum_{n=0}^{\infty} A_n \mathrm{Re}\chi_n(x) \geqq A_0 - \sum_{m+1}^{\infty} A_n > 0$$

and since $\mathfrak{M}\{z^2\} = 0$ we get from (3) in Section 4 that $\mu(x) > 0$ for all such x with the exception of a set Z of x with $\bar{m}Z = 0$. When $\mu(x) > 0$ we have $x \in E-E+V_0-V_0 \subset E-E+V$. This proves Theorem 3.

Appendix. Banach mean values in non-abelian groups.

THEOREM 4. *Let L be any right-translation invariant linear space of bounded real functions on a group G. A necessary and sufficient condition that there exist a real functional Mf on L with the properties*

$$\text{(a)} \qquad \inf_x f(x) \leqq Mf \leqq \sup_x f(x)$$

$$\text{(b)} \qquad \underset{x}{M}\{f(xa)\} = M\{f(x)\}$$

$$\text{(c)} \qquad M\{\lambda f\} = \lambda Mf \qquad (\lambda \text{ real})$$

$$\text{(d)} \qquad M\{f+g\} = Mf + Mg$$

is that $\sup_x H(x) \geqq 0$ *for every function $H(x)$ of the form*

$$H(x) = h_1(x) - h_1(xa_1) + \ldots + h_n(x) - h_n(xa_n)$$

where $h_1, \ldots, h_n$ are arbitrary functions from L and $a_1, \ldots, a_n$ arbitrary elements from G. If L is bi-translation invariant and the property (b) *is replaced by the stronger property*

$$\text{(b}'\text{)} \qquad \underset{x}{M}\{f(bxa)\} = M\{f(x)\},$$

in the above condition one has to consider functions $H(x)$ of the form

$$H(x) = h_1(x) - h_1(b_1xa_1) + \ldots + h_n(x) - h_n(b_nxa_n).$$

In the two cases the functional Mf is called a right-invariant Banach mean value or a bi-invariant Banach mean value on L, respectively.

Of course, if there exists a bi-invariant mean value Mf, the functional $M^*f = \frac{1}{2}(M\{f(x)\} + M\{f(x^{-1})\})$ will have all the above properties and furthermore the property

$$\text{(e)} \qquad M^*\{f(x)\} = \underset{x}{M^*}\{f(x^{-1})\}.$$

PROOF. It is plain that in both cases the condition $\sup_x \dot{H}(x) \geqq 0$ is necessary, for if there exists an M with the properties mentioned above we must have

$$\sup_x H(x) \geqq MH = 0.$$

In order to show that the condition (in the two different cases) is sufficient we introduce the functional

$$\text{(1)} \qquad \bar{M}f = \inf_H \sup_x \big(f(x) + H(x)\big).$$

We have $\sup_x (f(x)+H(x)) \geqq \inf_x f(x) + \sup_x H(x) \geqq \inf_x f(x)$

since $\sup_x H(x) \geqq 0$ and hence

(2) $$\sup_x f(x) \geqq \overline{M}f \geqq \inf_x f(x) \, ,$$

the first sign of inequality being due to the fact that the constant 0 is an $H(x)$. In the first of our two cases we have

(3) $$\overline{M}_x\{f(xa)\} = \overline{M}f$$

because, together with $H(x)$ every $H(xa)$ is also an $H(x)$ (and conversely). Analogously we have in the second case

(3′) $$\overline{M}_x\{f(bxa)\} = \overline{M}f \, .$$

Similarly we get $\overline{M}\{\lambda f\} = \lambda \overline{M}f$ for $\lambda > 0$. This is true also for $\lambda = 0$ since $\overline{M}0 = \inf_H \sup_x H(x) = 0 = 0\,\overline{M}f$. Thus we have proved

(4) $$\overline{M}\{\lambda f\} = \lambda \overline{M}f \quad \text{for} \quad \lambda \geqq 0 \, .$$

Next, we shall prove

(5) $$\overline{M}\{f+g\} \leqq \overline{M}f + \overline{M}g \, .$$

For any $\varepsilon > 0$ we can find $H_1(x)$ and $H_2(x)$ such that

$$\overline{M}f + \varepsilon \geqq f(x) + H_1(x) \quad \text{and} \quad \overline{M}g + \varepsilon \geqq g(x) + H_2(x)$$

for all x. This gives

$$\overline{M}f + \overline{M}g + 2\varepsilon \geqq f(x) + g(x) + H_1(x) + H_2(x)$$

for all x. Since $H_1(x) + H_2(x)$ is also an $H(x)$ we see that

$$\overline{M}f + \overline{M}g + 2\varepsilon \geqq \overline{M}\{f+g\} \, .$$

This proves (5). Finally we shall prove that in our first case

(6) $$\overline{M}\{f(x) - f(xa)\} = 0$$

and in the second case

(6′) $$\overline{M}\{f(x) - f(bxa)\} = 0 \, .$$

Let us prove (6), for example. On the one hand we have from (5) and (3)

$$\overline{M}\{f(x) - f(xa)\} \geqq \overline{M}\{f(x)\} - \overline{M}\{f(xa)\} = 0;$$

on the other hand, since $f(xa) - f(x)$ is an $H(x)$, we have

$$\overline{M}\{f(x) - f(xa)\} \leqq \sup_x \{f(x) - f(xa) + f(xa) - f(x)\} = 0 \, .$$

This proves (6).

We now proceed as in Section 3 by applying Banach's Theorem to the above L and $\bar{M}$ and prove as in Section 3 that the Mf obtained satisfies (a), (b) or (b′) respectively, (c), (d).

This completes the proof of Theorem 4.

THEOREM 5. *Besides the finite non-abelian groups, there exist also infinite non-abelian groups G with a bi-invariant Banach mean value defined on all bounded real functions on G.*

PROOF. This is shown by the following example. We consider as our group G the group of all those permutations of the numbers 1, 2, 3, ... each of which involves only a finite set of these numbers and leaves the remaining numbers fixed. The reason that this group possesses an Mf with the properties mentioned is that it can be exhausted by finite groups, viz. the groups G_n where G_n denotes the subgroup of G consisting of all permutations of the numbers $1, 2, \ldots, n$. In the proof we may, for instance, use Theorem 4. On account of Theorem 4 it is sufficient to prove that $\sup_x H(x) \geqq 0$ for every function of the form

$$H(x) = h_1(x) - h_1(b_1 x a_1) + \ldots + h_n(x) - h_n(b_n x a_n) .$$

Let F be a finite subgroup of G which contains $a_1, \ldots, a_n;\ b_1, \ldots, b_n$. Then we have by Theorem 4

$$\sup_{x \in G} H(x) \geqq \sup_{x \in F} H(x) \geqq 0$$

since there exists a bi-invariant mean value in F.

THEOREM 6. *There exists a non-abelian group G and a subset E of G such that there does not exist a right-invariant Banach mean value on the right-translation invariant linear space which is spanned by the characteristic function $f(x)$ of E and its right-translates.*

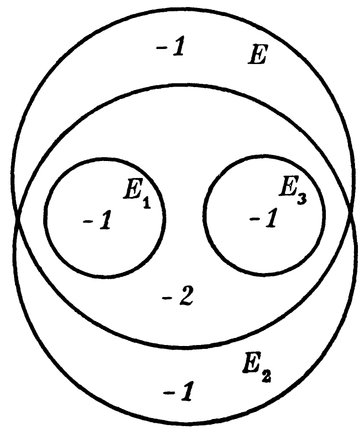
PROOF. We choose as our group G the free group with the two generators a and b. Except for the unit element, every element in G may in one and only one way be written as a product of powers alternatingly of a and b (with integral exponents $\neq 0$). As the set E we consider the set of all elements which end on a power of a. In consequence of (the trivial part of) Theorem 4 our theorem will be proved when we have shown that $\sup_x H(x) < 0$ for the function

$$H(x) = f(xa^2b) - f(xa) + f(xa^2b^2) - f(x) .$$

The four functions on the right-hand side are the characteristic functions of the four sets

$$E_1 = E\,b^{-1}a^{-2}, \quad E_2 = E\,a^{-1}, \quad E_3 = E\,b^{-2}a^{-2}, \quad \text{and} \quad E ,$$

respectively. The set E_2 consists partly of all elements which end on a^r ($r \neq 0$ and $\neq -1$), partly of all elements which end on b^s ($s \neq 0$), and finally of the unit element. Thus $E \cup E_2 = G$. Further $E \cap E_2$ consists of all elements which end on a^r ($r \neq 0$ and $\neq -1$). Since all elements in E_1 end on $b^{-1}a^{-2}$ and all elements in E_3 end on $b^{-2}a^{-2}$ we see that both E_1 and E_3 are contained in $E \cap E_2$ and that E_1 and E_3 are disjoint sets. The figure illustrates the situation. The value of $H(x)$ can immediately be found in any of the five subsets into which G is divided by the sets E_1, E_2, E_3, and E. The values are indicated on the figure; they are either -1 or -2. Thus $\sup_x H(x)$ is not $\geqq 0$, as was to be proved.



$E \cup E_2 = G$

Finally, we shall prove the following theorem which goes in "positive" direction. By a class function on G we understand a function $f(x)$ which satisfies $f(xy) = f(yx)$ for all x and y.

Theorem 7. *On the bi-invariant linear space which consists of all functions which can be uniformly approximated by linear combinations of the bi-translates of a given (real) class function $f(x)$ it is possible to introduce a bi-invariant Banach mean value.*

Proof. We apply Theorem 4 and have to show that if $f_1(x), \ldots, f_n(x)$ are arbitrary functions from the linear space in question and $c_1, \ldots, c_n$; $d_1, \ldots, d_n$ are arbitrary elements, then

$$\sup_x [f_1(x) - f_1(d_1xc_1) + \ldots + f_n(x) - f_n(d_nxc_n)] \geqq 0 . \tag{7}$$

Evidently it suffices to prove (7) for functions $f_1, \ldots, f_n$ from the linear space which is spanned by the bi-translates of $f(x)$. And in order to show (7) for such functions it is clear that we need only show

$$\sup_x [\alpha_1(f(xa_1) - f(xb_1)) + \ldots + \alpha_n(f(xa_n) - f(xb_n))] \geqq 0 \tag{8}$$

for arbitrary $a_1, \ldots a_n$; $b_1, \ldots, b_n$ and all $\alpha_r > 0$ (the translations to the left are made to disappear by using $f(xy) = f(yx)$). For convenience we may assume that $\Sigma \alpha_r = 1$.

In order to prove (8) indirectly we assume that with an $\varepsilon > 0$ we have

$$\alpha_1 f(xa_1) + \ldots + \alpha_n f(xa_n) \leqq \alpha_1 f(xb_1) + \ldots + \alpha_n f(xb_n) - \varepsilon \tag{9}$$

for all x. We replace x in this inequality by xa_r and multiply by α_r which gives

$$\alpha_r \sum_s \alpha_s f(xa_r a_s) \leqq \alpha_r \sum_s \alpha_s f(xa_r b_s) - \alpha_r \varepsilon .$$

Summing over r we obtain

$$\sum_{r,s} \alpha_r \alpha_s f(xa_r a_s) \leqq \sum_{r,s} \alpha_r \alpha_s f(xa_r b_s) - \varepsilon .$$

Since $f(x)$ is a class function we may write the right-hand side of this inequality as

$$\sum_{r,s} \alpha_r \alpha_s f(b_s x a_r) - \varepsilon = \sum_{s=1}^{n} \alpha_s \sum_{r=1}^{n} \alpha_r f(b_s x a_r) - \varepsilon$$

and applying (9) with x replaced successively by $b_s x$, $s = 1, 2, \ldots, n$, we see that this quantity is

$$\leqq \sum_{s=1}^{n} \alpha_s \sum_{r=1}^{n} \alpha_r f(b_s x b_r) - 2\varepsilon = \sum_{r,s} \alpha_r \alpha_s f(x b_r b_s) - 2\varepsilon .$$

Thus we have arrived at the inequality

$$\sum_{r,s} \alpha_r \alpha_s f(xa_r a_s) \leqq \sum_{r,s} \alpha_r \alpha_s f(x b_r b_s) - 2\varepsilon . \tag{9'}$$

This is an inequality of the same type as (9) since $\Sigma \alpha_r \alpha_s = 1$ and $\alpha_r \alpha_s > 0$, but on the right-hand side we have 2ε in place of ε. The process may now be repeated as long as necessary and we obtain finally a contradiction since all sums occurring in our chain of inequalities lie between the infimum and the supremum of $f(x)$ while $2^m \varepsilon \to \infty$ for $m \to \infty$. This completes the proof of Theorem 7.

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