MATH. SCAND. 2 (1954), 46–48

A GENERALIZATION OF DIXON'S FORMULA

JONAS EKMAN FJELDSTAD

The following formula was found by Dixon [1]:

$$\sum_{p=0}^{2n} (-1)^p \binom{2n}{p}^3 = (-1)^n \frac{(3n)!}{(n!)^3} .$$

We are going to prove a generalization of it, namely:

$$\sum_{s=0}^{2m} (-1)^s \binom{2m}{s} \binom{2n}{n-m+s} \binom{2p}{p-m+s}$$
$$= (-1)^m \frac{(m+n+p)!\,(2m)!\,(2n)!\,(2p)!}{(m+n)!\,(m+p)!\,(n+p)!\,m!\,n!\,p!} .$$

For $m = n = p$ we get the formula of Dixon.

Starting from the well-known formula

$$(2\cos\varphi)^n = \sum_{r=0}^{n} \binom{n}{r} \cos\,(n-2r)\varphi ,$$

we obtain immediately

$$\text{(1)} \qquad \frac{1}{\pi}\int_0^\pi (2\cos\varphi)^n \cos\,(n-2p)\varphi\, d\varphi = \binom{n}{p} .$$

We now consider the integral

$$\text{(2)} \qquad I = \frac{1}{\pi^2}\int_0^\pi dx \int_0^\pi \big(2\sin\,(x+y)\big)^{2m} (2\cos x)^{2n} (2\cos y)^{2p}\, dy ,$$

which can be calculated in two different ways. Substituting first

$$\big(2\sin(x+y)\big)^{2m} = (-1)^m \sum_{r=0}^{2m} (-1)^r \binom{2m}{r} \cos\big((2m-2r)(x+y)\big) ,$$

we get

$$I = \frac{(-1)^m}{\pi^2} \sum_{r=0}^{2m} (-1)^r \binom{2m}{r} \int_0^\pi (2\cos x)^{2n} \cos\,(2m-2r)\,x\, dx \cdot$$
$$\cdot \int_0^\pi (2\cos y)^{2p} \cos\,(2m-2r)\,y\, dy ,$$

Received March 5, 1954.

because the integrals containing $\sin(2m-2r)x$ and $\sin(2m-2r)y$ vanish. Hence, by aid of (1) we find the result

$$(3) \qquad I = (-1)^m \sum_{r=0}^{2m} (-1)^r \binom{2m}{r} \binom{2n}{n-m+r} \binom{2p}{p-m+r} .$$

On the other hand, we have

$$\sin^{2m}(x+y) = (\sin x \cos y + \cos x \sin y)^{2m}$$
$$= \sum_{s=0}^{2m} \binom{2m}{s} \sin^{2m-s} x \cos^s x \cos^{2m-s} y \sin^s y \ ;$$

and substituting this expression in (2), we obtain

$$I = \frac{2^{2m+2n+2p}}{\pi^2} \sum_{s=0}^{2m} \binom{2m}{s} \int_0^\pi \sin^{2m-s} x \cos^{2n+s} x \, dx \int_0^\pi \sin^s y \cos^{2m+2p-s} y \, dy .$$

If s is an odd number the integrals vanish, but for even values of s the integrals can be expressed by beta functions. Writing $2s$ for s, we have

$$I = \frac{2^{2m+2n+2p}}{\pi^2} \sum_{s=0}^{m} \binom{2m}{2s} B(m-s+\tfrac{1}{2},\ n+s+\tfrac{1}{2}) \, B(2m+p-s+\tfrac{1}{2},\ s+\tfrac{1}{2}) =$$
$$\frac{2^{2m+2n+2p}}{\pi^2} \sum_{s=0}^{m} \frac{(2m)!}{(2s)!\,(2m-2s)!} \cdot \frac{\Gamma(m-s+\frac{1}{2})\Gamma(n+s+\frac{1}{2})\Gamma(m+p-s+\frac{1}{2})\Gamma(s+\frac{1}{2})}{(m+n)!\,(m+p)!} .$$

From a well-known formula for the gamma function we get

$$\Gamma(m-s+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-2s}} \frac{(2m-2s)!}{(m-s)!} ,$$

$$\Gamma(s+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2s}} \frac{(2s)!}{s!} .$$

The two remaining gamma expressions can be replaced by a beta expression:

$$\Gamma(n+s+\tfrac{1}{2}) \, \Gamma(m+p-s+\tfrac{1}{2}) = (m+n+p)! \, B(n+s+\tfrac{1}{2}, m+p-s+\tfrac{1}{2}) .$$

Substituting these expressions, we obtain

$$I = \frac{2^{2n+2p}}{\pi} \sum_{s=0}^{m} \frac{(m+n+p)!}{(m-s)!\,s!} \frac{(2m)!}{(m+n)!\,(m+p)!} B(n+s+\tfrac{1}{2}, m+p-s+\tfrac{1}{2})$$
$$= \frac{2^{2n+2p}}{\pi} \frac{(m+n+p)!\,(2m)!}{(m+n)!\,(m+p)!\,m!} \sum_{s=0}^{m} \frac{m!}{s!\,(m-s)!} \int_0^1 x^{n+s-\frac{1}{2}} (1-x)^{m+p-s-\frac{1}{2}} dx$$
$$= \frac{2^{2n+2p}}{\pi} \frac{(m+n+p)!\,(2m)!}{(m+n)!\,(m+p)!\,m!} \int_0^1 x^{n-\frac{1}{2}} (1-x)^{p-\frac{1}{2}} \sum_{s=0}^{m} \binom{m}{s} x^s (1-x)^{m-s} dx .$$

However,

$$\sum_{s=0}^{m} \binom{m}{s} x^s (1-x)^{m-s} = 1,$$

and

$$\int_0^1 x^{n-\frac{1}{2}}(1-x)^{p-\frac{1}{2}}dx = B(n+\tfrac{1}{2},\ p+\tfrac{1}{2})$$

$$= \frac{\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(n+p)!} = \pi \frac{(2n)!}{n!} \frac{(2p)!}{p!} \frac{1}{(n+p)!} \frac{1}{2^{2n+2p}}.$$

Hence we have

$$I = \frac{(m+n+p)!\,(2m)!\,(2n)!\,(2p)!}{(m+n)!\,(m+p)!\,(n+p)!\,m!\,n!\,p!}. \tag{4}$$

Finally, equating the two expressions (3) and (4) for I, we obtain the announced generalization of Dixon's formula.

Remark. I have been informed that Th. Bang has noticed that the generalized formula can be written in a simpler and more symmetric form, namely

$$\sum (-1)^s \binom{m+n}{m+s} \binom{n+p}{n+s} \binom{p+m}{p+s} = \frac{(m+n+p)!}{m!\,n!\,p!},$$

where the summation is extended over all integers s yielding summands $\neq 0$.

This can be seen by expressing the binomial coefficients by factorials and some rearrangement of the factors. Replacing the letter of summation s by $m+s$, we obtain

$$\binom{2m}{m+s}\binom{2n}{n+s}\binom{2p}{p+s}$$

$$= \frac{(2m)!\,(2n)!\,(2p)!}{(m+n)!\,(n+p)!\,(p+m)!} \binom{m+n}{m+s}\binom{n+p}{n+s}\binom{p+m}{p+s}$$

from which the new version of the formula immediately follows.

REFERENCE

1. A. C. Dixon, *On the sum of the cubes of the coefficients in a certain expansion by the binomial theorem*, Messenger of Mathematics 20 (1891), 79–80.

UNIVERSITY OF OSLO, NORWAY