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ON THE DIFFRACTION OF A PLANE WAVE BY AN INFINITE PLANE GRATING

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1. Introduction. Consider a plane electromagnetic wave of arbitrary polarization which is incident normal to a diffraction grating in the plane $x = 0$. The grating consists of an infinite set of identical perfectly conducting strips parallel to the z axis with the spacing between any two adjacent strips equal to the width of one of them. We shall determine the field at large distances from the screen, or, more precisely, the amplitude, phase and direction of propagation of the transmitted and reflected waves. (See Fig. 1 for a front view.)

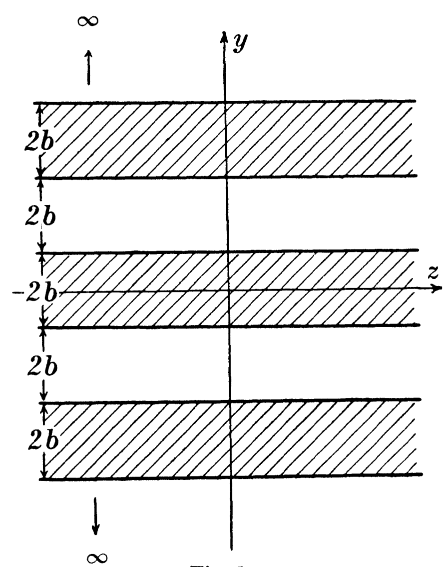

Fig. 1.

Related diffraction problems have been considered by Rayleigh [7] and Lamb [5], who discussed cases for which the ratio of the aperture width to free space wavelength is much less than unity. The interest in the particular problem we treat here is that with the restricted spacing we are able to calculate the reflection and transmission characteristics without assuming the aperture width to be small compared to the free space wave-length. Furthermore, these characteristics are calculated in closed form and this is possible because of the particular spacing which we have chosen.

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2. The problem and its mathematical formulation. Although we are not limited to a particular polarization of the incident wave, no loss of generality results in considering the case for which the magnetic vector lies parallel to the edges of the strips. Indeed, the complementary excitation, that is, the case in which the electric vector is parallel to the edges of the strips, leads to a similar boundary value problem. The responses of the two independent polarizations in the incident wave may be superimposed to give the field due to a more general excitation. Henceforth we shall take the time dependence of the field components to be monochromatic, so that all field components may be expressed as complex functions of position multiplied by $e^{-i\omega t}$. This time factor will be suppressed throughout the subsequent calculations.

If we observe that all of the components of the electric and magnetic fields are independent of the z coordinate, we find that the Maxwell equations in the steady state may be reduced to the following system, in which Δ is the Laplace operator in x and y:

$$\begin{gathered} \Delta H_z + k^2 H_z = 0\,, \\ E_z = H_x = H_y = 0\,, \\ E_y = -(\sigma - i\omega\varepsilon)^{-1}\,\partial H_z/\partial x\,, \qquad E_x = (\sigma - i\omega\varepsilon)^{-1}\,\partial H_z/\partial y\,, \\ k^2 = \omega^2\varepsilon\mu + i\omega\sigma\mu\,, \qquad k = p + iq\,, \quad p > 0, \quad q \geqq 0\,. \end{gathered} \tag{2.1}$$

We have assumed that the medium is homogeneous and isotropic and free of space charge. Since we have also made the assumption that the strips are perfect conductors, the tangential component of the electric field vanishes on each strip, or

$$\partial H_z/\partial x = 0\,, \qquad \begin{cases} x = 0, \quad (4n-1)b \leqq y \leqq (4n+1)b\,, \\ n = 0, \pm 1, \pm 2, \ldots\,. \end{cases} \tag{2.2}$$

We note in passing that equations (2.1) and (2.2) define a problem in acoustic diffraction.

We consider the case of a plane wave incident from the left and of the form e^{ikx}. In view of the periodicity of the structure, it is convenient to formulate the problem in terms of a parallel plate wave guide. By symmetry $\partial H_z/\partial y$ and E_x are zero along the planes $y = 2nb$, $n = 0, \pm 1, \ldots$, so that we may imagine these planes to be occupied by thin perfectly conducting sheets without changing the nature of the field in any manner. Furthermore, if we could find the field in the region $0 \leqq y \leqq 2b$, the field in other regions may be obtained by using the relations

$$H_z(x, y) = H_z(x, -y) = H_z(x, y+4b)\,.$$

The problem is therefore reduced to determining the effects of a conducting fin inserted in a parallel plate wave guide which is excited by a dominant mode wave from the left. The fin is perpendicular to the direction of propagation and extends from one conducting plane half way to the other (see Fig. 2).

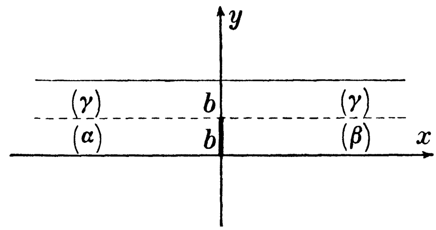

Fig. 2.

Turning now to the precise mathematical formulation, we write $\varphi(x, y)$ for $H_z(x, y)$ and consider equation (2.1), that is,

$$(2.1) \qquad \Delta\varphi + k^2\varphi = 0, \qquad k = p + iq\,,$$

in the region Ω having the two lines $y = 0$ and $y = 2b$ and the segment $x = 0,\ 0 < y \leqq b$ as its boundary.

The principal case, k non-real, $q > 0$, will be treated first. Partly for physical reasons we limit ourselves to seek solutions φ possessing the following properties:

1° The normal derivative $\partial\varphi/\partial n$ vanishes at the boundary, in particular at both sides of the segment $x = 0,\ 0 < y < b$.

2° The incident wave dominates when $x \to -\infty$. More precisely, $\varphi(x, y)$ may be written

$$(2.3) \qquad \varphi(x, y) = e^{ikx} + \psi(x, y)$$

where

$$(2.4) \qquad \psi(x, y) = O(e^{-q|x|}), \qquad \psi_x(x, y) = O(e^{-q|x|}) \qquad \text{for} \quad |x| \to \infty\,.$$

Here and henceforth partial differentiation with respect to a variable is indicated by placing it as subscript. We observe that the function ψ must satisfy equation (2.1), that is,

$$(2.5) \qquad \Delta\psi + k^2\psi = 0\,,$$

and the boundary conditions

$$(2.6) \qquad \psi_y(x, 0) = \psi_y(x, 2b) = 0 \qquad \text{for} \quad -\infty < x < \infty\,,$$

$$(2.7) \qquad \psi_x(0, y) = -ik \qquad \text{for} \quad 0 < y < b\,.$$

3° Besides the regularity conditions imposed by the equations of the problem, we also assume that $\varphi(x, y)$ is sufficiently regular to warrant the applications of Green's theorem which are made in the following. Thus, in particular, we assume $\varphi_y(x, b)$ to be integrable in the neighborhood of $x = 0^+$.

Sections 3–5 contain an analysis of our problem. In these sections φ denotes a function which is *assumed* to be a solution of (2.1) having the properties 1°–3°. The result will be that φ is uniquely determined. In fact both an integral representation and an infinite series representation (adequate for numerical computation) will be derived. Section 6 deals with the proof of the fact that the representations converge and determine a solution to (2.1) having the properties 1°–3°.

The series derived for k non-real will converge and its sum will satisfy (2.1) with the boundary condition 1° in the case k real also. This is discussed in Sections 7–8. Simple algebraic expressions are derived for its coefficients.

Returning to the principal case, k non-real, we observe that a simple proof of the uniqueness of the solution to our problem may be established, independently, as follows. Suppose $\varphi_1(x, y) = e^{ikx}+\psi_1(x, y)$ and $\varphi_2(x, y) = e^{ikx}+\psi_2(x, y)$ to be solutions of (2.1) each having the properties 1°–3°, and form the difference $\chi(x, y) = \psi_1(x, y)-\psi_2(x, y)$. Since $\Delta\chi+k^2\chi = 0$ and according to (2.6) and (2.7), the normal derivative of χ vanishes on the boundary, application of Green's theorem

$$\iint [\chi\Delta\bar{\chi} + |\text{grad}\,\chi|^2]\,dx\,dy = \int \chi \frac{\partial\bar{\chi}}{\partial n}\,ds$$

in Ω gives

$$\iint [|\text{grad}\,\chi|^2 - k^2|\chi|^2]\,dx\,dy = 0\,.$$

Since k is assumed to be non-real, this implies

$$\iint |\chi|^2\,dx\,dy = 0$$

and hence $\chi \equiv 0$.

Further we observe, if ψ is the function which occurs in (2.3), then $\chi_1(x, y) = \psi(x, y)+\psi(-x, y)$ satisfies $\Delta\chi_1+k^2\chi_1 = 0$ and has zero normal derivative on the boundary just as χ above. Hence again $\chi_1 \equiv 0$, or

$$\psi(x, y) + \psi(-x, y) = 0\,,$$

that is, ψ must be an odd function of x.

3. Derivation of an integral equation. Assuming φ to be a solution of (2.1) having the properties 1°–3°, we first seek representations φ^α, φ^β, φ^γ in the regions (cf. Fig. 2):

$$(\alpha) \qquad x < 0, \qquad 0 \leqq y \leqq b;$$

$$(\beta) \qquad x > 0, \qquad 0 \leqq y \leqq b;$$

$$(\gamma) \qquad -\infty < x < \infty, \qquad b \leqq y \leqq 2b\,.$$

The Green's function for the regions (α) and (β) which satisfies the conditions $G_y = 0$ for $y = 0$ and $y = b$ and $G_x = 0$ at $x = 0$ and which includes only terms which represent outward going radiation from the source point, is

$$G^\alpha(x, y; x', y') = G^\beta(x, y; x', y')$$

$$= -\sum_0^\infty \frac{2-\delta_{0n}}{2ib\Gamma_{2n}} \cos(n\pi y/b)\cos(n\pi y'/b)\left(e^{i\Gamma_{2n}|x-x'|} + e^{i\Gamma_{2n}|x+x'|}\right).$$

Here $\delta_{0n} = 1$ for $n = 0$ and is zero otherwise and the numbers Γ_ν are defined by

$$\Gamma_\nu^2 = k^2 - (\tfrac{1}{2}\nu\pi/b)^2, \qquad \operatorname{Im}\Gamma_\nu > 0, \qquad \nu = 0, 1, 2, \ldots .$$

They are situated on a hyperbola in the complex plane as indicated on Fig. 3a. The Green's function for region (γ) which satisfies the conditions $G_y = 0$ for $y = b$ and $y = 2b$ and the condition of outward going radiation from the source point is

$$G^\gamma(x, y; x', y') = -\sum_0^\infty \frac{2-\delta_{0n}}{2ib\Gamma_{2n}} \cos(n\pi y/b)\cos(n\pi y'/b)\, e^{i\Gamma_{2n}|x-x'|}.$$

The construction of such Green's functions has been described by many authors, see for example [1, Section 2]. In the case of region (γ) we now integrate over the rectangle

$$-L \leqq x' \leqq L, \quad b \leqq y' \leqq 2b$$

and use Green's theorem. Observing that the derivative with respect to y' of $G^\gamma(x, y; x', y')$ is zero for $y' = b$ and $y' = 2b$ and since also $\varphi_{y'}(x', 2b) = 0$ we have, in virtue of the regularity of φ,

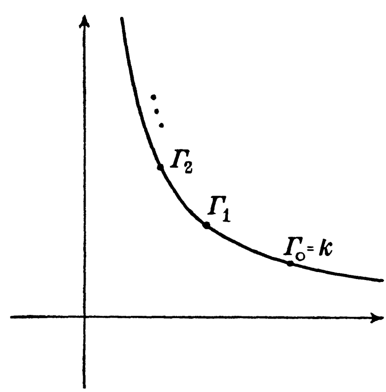

Fig. 3a.

$$(3.1)\quad \varphi^\gamma(x, y) = -\int_b^{2b} (G^\gamma\varphi_{x'} - \varphi G^\gamma_{x'})\Big|_{x'=-L}^{x'=L} dy' - \int_{-L}^{L} G^\gamma\varphi_{y'}\Big|_{y'=b} dx'.$$

For L large, we invoke, still assuming $\operatorname{Im} k = q > 0$, the asymptotic forms of the Green's function, $\varphi(x, y)$ and their derivatives. The contributions to (3.1) due to the integrals over the ends at $x' = L$ and $x' = -L$ is seen to be e^{ikx} (from the incident wave, cf. 2° of Section 2) plus a term which is $O(e^{-2qL})$. We have seen that $\varphi_{y'}(x', y')$ is an odd function of the variable x'. This enables us to rewrite (3.1) as

$$\varphi^\gamma(x, y) = e^{ikx} - \int_0^L [G^\gamma(x,y;x',b) - G^\gamma(x,y;-x',b)]\,\varphi_{y'}(x',b)\,dx' + O(e^{-2qL}). \tag{3.2}$$

Similarly, if we consider the region (β), we obtain

$$\varphi^\beta(x, y) = \int_0^L G^\beta(x, y; x', b)\,\varphi_{y'}(x', b)\,dx' + O(e^{-2qL}) .$$

The continuity of $\varphi(x, y)$ along the line $y = b$ then gives us the equation

$$\varphi^\gamma(x,b) - \varphi^\beta(x,b) = e^{ikx} - 2\int_0^L \varkappa(x-x')\,\varphi_{y'}(x',b)\,dx' + O(e^{-2qL}) = 0, \quad x > 0, \tag{3.3}$$

where the kernel, $\varkappa(x-x')$, is defined as

$$\varkappa(x-x') = G^\gamma(x, b; x', b) = -\sum_0^\infty \frac{2-\delta_{0n}}{2ib\Gamma_{2n}}\, e^{i\Gamma_{2n}|x-x'|} .$$

From (3.3) we obtain, by letting $L \to \infty$ the integral equation we seek, that is,

$$e^{ikx} - 2\int_0^\infty \varkappa(x-x')\,\varphi_{y'}(x', b)\,dx' = 0, \qquad x > 0 . \tag{3.4}$$

If we repeat this derivation for $x < 0$ we obtain the same equation, a situation which is to be expected in view of the oddness of $\varphi_{y'}(x', b)$ in the variable x'. Because the kernel is a function only of $|x-x'|$ we have an integral equation of the Wiener-Hopf type, and as such we have a method to solve it [1], [6].

4. Solution of the integral equation. Derivation of integral representations. In order to solve equation (3.4) we extend it so as to hold for all x by putting

$$g_1(x) + g_2(x) = 2\int_{-\infty}^{+\infty} \varkappa(x-x')f(x')\,dx' , \tag{4.1}$$

where we take $f(x) = \varphi_y(x, b)$, $g_1(x) = e^{ikx}$, $g_2(x) \equiv 0$ when $x > 0$ and $f(x) \equiv 0$, $g_1(x) \equiv 0$ and $g_2(x)$ is to be determined by the equation when $x < 0$.

Let us now examine the Fourier transforms of the functions $g_1(x)$, $g_2(x)$, $f(x)$ and $\varkappa(x)$, with the transformed variable being $w = u+iv$. Since $f(x)$ and $g_1(x)$ vanish for negative x, their respective Fourier transforms, $F(w)$ and $G_1(w)$, are unilateral. By the definition of $g_1(x)$, we see that the integral defining $G_1(w)$ converges to $i/(k-w)$, whenever $v < q$, where q is the imaginary part of k. For this reason, $G_1(w)$ is analytic in the same lower half plane $v < q$. According to our assumption 3° in Section 2, the function $f(x) = \varphi_y(x, b)$ is integrable in the neighborhood of the point $x = 0^+$ and since it is $O(e^{-qx})$ as $x \to +\infty$, the unilateral Fourier transform of $f(x)$ is analytic in the lower half plane $v < q$. Similarly, $G_2(w)$, the transform of $g_2(x)$, is analytic in the upper half-plane $v > -q$. The bilateral Fourier transform of $\varkappa(x)$ is (cf. [2])

$$K(w) = \int_{-\infty}^{+\infty} \varkappa(x) e^{-iwx} dx = -(k^2-w^2)^{-\frac{1}{2}} \cot[b(k^2-w^2)^{\frac{1}{2}}].$$

By means of the well known infinite product expansions of sine and cosine this may be written as

$$K(w) = \frac{1}{b(w^2-k^2)} \prod_{n=1}^{\infty} \frac{1 - \dfrac{b^2(k^2-w^2)}{(n-\frac{1}{2})^2\pi^2}}{1 - \dfrac{b^2(k^2-w^2)}{n^2\pi^2}}$$

$$= \frac{1}{b(w^2-k^2)} \prod_{n=1}^{\infty} \frac{w^2-\Gamma_{2n-1}^2}{w^2-\Gamma_{2n}^2} \left(\frac{2n}{2n-1}\right)^2 = b^{-1}K_-(w)/K_+(w),$$

where

$$K_-(w) = \frac{1}{w-k} \prod_{n=1}^{\infty} \frac{w-\Gamma_{2n-1}}{w-\Gamma_{2n}} \frac{2n}{2n-1}, \qquad 1/K_+(w) = \frac{1}{w+k} \prod_{n=1}^{\infty} \frac{w+\Gamma_{2n-1}}{w+\Gamma_{2n}} \frac{2n}{2n-1}.$$

The infinite products in $K_-(w)$ and $K_+(w)$ are absolutely convergent save for $w = \Gamma_\nu$, $w = -\Gamma_\nu$, respectively. In particular, $K_-(w)$ and its reciprocal are regular in the lower half plane $v < q$, while $K_+(w)$ and its reciprocal are regular in the half plane $v > q$. We note the formulas

$$K_+(w)K_-(-w) = -1 \tag{4.2}$$

and

$$K_-(w)(k^2-w^2)^{\frac{1}{2}} \sin[b(k^2-w^2)^{\frac{1}{2}}] = -bK_+(w)\cos[b(k^2-w^2)^{\frac{1}{2}}]. \tag{4.3}$$

In view of the fact that the transforms have a common strip of regularity, $-q < v < q$, it is permissible to apply the Fourier transform to obtain

$$G_2(w) + i/(k-w) = 2b^{-1}F(w)K_-(w)/K_+(w).$$

By considering the half planes $v > -q$ and $v < q$, we conclude from this equation (cf. [1] and [6, Chap. 4]) that

$$F(w) = -\frac{ibK_+(k)}{2K_-(w)(w-k)}. \tag{4.4}$$

We note (cf. [1, Section 3]) that

$$K_-(w) = O(w^{-\frac{1}{2}}) \qquad \text{for} \quad |w| \to \infty, \quad v < q\,,$$

and for future reference that

$$(4.5) \qquad K_+(w) = O(w^{\frac{1}{2}}) \qquad \text{for} \quad |w| \to \infty, \quad v > -q\,.$$

Thus $F(w) = O(w^{-\frac{1}{2}})$ for $|w| \to \infty, v < q$.

By letting L tend to infinity in (3.3) we obtain

$$(4.5) \quad \varphi^\gamma(x, y) = e^{ikx} - \int_0^\infty [G^\gamma(x, y; x', b) - G^\gamma(x, y; -x', b)]\, \varphi_{y'}(x', b)\, dx'.$$

The Fourier transform of ${\varphi^\gamma}_y(x, b) = f(x)$ is the function $F(w)$ given by (4.4) and the Fourier transform of $G^\gamma(x, y; 0, b)$ with respect to x is

$$h^\gamma(w; y) = -\frac{\cos[(2b-y)(k^2-w^2)^{\frac{1}{2}}]}{(k^2-w^2)^{\frac{1}{2}} \sin[b(k^2-w^2)^{\frac{1}{2}}]}\,.$$

Hence, by the convolution theorem equation (4.5) yields

$$(4.7) \qquad \varphi^\gamma(x, y) = e^{ikx} + \frac{1}{2\pi} \int_{-\infty}^{\infty} h^\gamma(w; y)[F(w)-F(-w)]e^{iwx} dw\,.$$

Similarly

$$(4.8) \qquad \varphi^\beta(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} h^\beta(w; y)[F(w)+F(-w)]e^{iwx} dw\,,$$

the Fourier transform of $G^\beta(x, y; 0, b)$ with respect to x being

$$h^\beta(w; y) = -\frac{\cos[y(k^2-w^2)^{\frac{1}{2}}]}{(k^2-w^2)^{\frac{1}{2}} \sin[b(k^2-w^2)^{\frac{1}{2}}]}\,.$$

Finally, $\varphi^\alpha(x, y)$ is determined by the oddness of the function $\psi(x,y) = \varphi(x, y)-e^{ikx}$.

It has now been proved that if the differential equation has a solution φ with the properties 1°–3° of Section 2, then it must be determined by the integral representations just derived.

5. Derivation of the infinite series. Calculation of the coefficients. First we consider, for example, the function $\varphi^\gamma(x, y)$. The only singularities of the integrand in (4.7) are simple poles. The calculus of residues enables us to write φ^γ as an infinite series. This procedure has been applied by many authors to such a situation and since the conditions of applicability are satisfied, we shall not pursue this point [3, Sections 5 and 7].

In order to put the poles into evidence, we now rewrite equation (4.7) as

$$\varphi^{\gamma}(x, y) = e^{ikx} - (\tfrac{1}{4}i/\pi) \int_{-\infty}^{\infty} K_{+}(k) \cos[(2b-y)(k^2-w^2)^{\frac{1}{2}}] \{\ldots\} e^{iwx} dw ,$$

where

$$\{\ldots\} = \frac{1}{K_{+}(w)(w-k)\cos[b(k^2-w^2)^{\frac{1}{2}}]} - \frac{bK_{+}(w)}{(w+k)(k^2-w^2)^{\frac{1}{2}}\sin[b(k^2-w^2)^{\frac{1}{2}}]}$$

and we have used equations (4.2), (4.3) and (4.4). In order to find φ^{γ} for $x > 0$, we need only calculate the residues from the poles in the upper half plane. Since $K_{+}(w)$ is regular there, we find that we have two different sets of poles to consider. These, of course, are the zeros of $\cos[b(k^2-w^2)^{\frac{1}{2}}]$ and $(k^2-w^2)^{\frac{1}{2}} \sin[b(k^2-w^2)^{\frac{1}{2}}]$, that is, $\Gamma_1, \Gamma_3, \ldots$ and $\Gamma_0 = k, \Gamma_2, \ldots,$ respectively. By a direct residue calculation, we obtain

$$\varphi^{\gamma}(x, y) = T_0 e^{ikx} + \sum_{\nu=1}^{\infty} T_\nu \cos(\tfrac{1}{2}\nu\pi y/b)\, e^{i\Gamma_\nu x}, \qquad x > 0 ,$$

where

(5.1) $$T_0 = \tfrac{1}{2}\{1 + \tfrac{1}{4}(K_{+}(k)/k)^2\}$$

while

(5.2) $$T_{2n} = (-1)^n \frac{K_{+}(k)K_{+}(\Gamma_{2n})}{2\,\Gamma_{2n}\,(k+\Gamma_{2n})}, \qquad n = 1, 2, \ldots ,$$

and

(5.3) $$T_{2n-1} = (-1)^n \frac{K_{+}(k)(k+\Gamma_{2n-1})}{(2n-1)\pi\,\Gamma_{2n-1}K_{+}(\Gamma_{2n-1})}, \qquad n = 1, 2, \ldots .$$

The same series is found for the region (β) from the integral representation (4.8) by a similar calculation. Hence

(5.4) $$\varphi(x, y) = T_0 e^{ikx} + \sum_{\nu=1}^{\infty} T_\nu \cos(\tfrac{1}{2}\nu\pi y/b)\, e^{i\Gamma_\nu x}$$

in the whole strip $0 < x < \infty$, $0 \leqq y \leqq 2b$. Finally, due to the oddness of the function $\psi(x, y) = \varphi(x, y) - e^{ikx}$, it follows immediately that

(5.5) $$\varphi(x, y) = e^{ikx} + R_0 e^{-ikx} + \sum_{\nu=1}^{\infty} R_\nu \cos(\tfrac{1}{2}\nu\pi y/b)\, e^{-i\Gamma_\nu x}$$

in the strip $-\infty < x < \infty$, $0 \leqq y \leqq 2b$ where

$$R_0 = 1 - T_0 \quad \text{and} \quad R_\nu = -T_\nu, \qquad \nu = 1, 2, \ldots .$$

6. Verification. The integral in (4.7) converges for $b \leqq y \leqq 2b$ and for all values of x, thus defining $\varphi^{\gamma}(x, y)$ in the closure of the domain (γ). In the same way $\varphi^{\alpha}(x, y)$ and $\varphi^{\beta}(x, y)$ are defined by their integral representations in the closures of (α) and (β) It is easily seen that the derivatives of these functions can be obtained by differentiating their integral representations under the integral signs. The differentiated integrals converge save for $x = 0, y = b$. It thus follows that φ^{α}, φ^{β} and φ^{γ} satisfy (2.1) in their appropriate regions.

We already observed that φ^{β} and φ^{γ} for $x > 0$ are represented by the same series (5.4) in their different regions of definition. Since it follows from (5.1–3) and (4.5) that

$$T_{\nu} = O(\nu^{-\frac{3}{2}}), \qquad \nu \to \infty ,$$

and because of the presence of the factors $e^{i\Gamma_{\nu}x}$, this series and its derivatives converge for $x \neq 0$. The derivatives can be obtained by termwise differentiation. It follows that the function $\varphi(x, y)$ which coincides with $\varphi^{\alpha}(x, y), \varphi^{\beta}(x, y), \varphi^{\gamma}(x, y)$ in (α), (β) and (γ) represents a solution of (2.1) in the whole region $(\alpha) \cup (\beta) \cup (\gamma)$ as well as on the line $y = b$ $(x \neq 0)$. Obviously it satisfies condition 2° of Section 2.

Necessary information about the boundary values of the derivatives of φ can be deduced from the integral representations of $\varphi^{\alpha}, \varphi^{\beta}, \varphi^{\gamma}$. This procedure has been indicated in [3, Section 7] and we shall not pursue these matters in detail. With this procedure we can show that the y derivative of $\varphi(x, y)$ vanishes at $y = 2b$. The method will also verify the boundary condition at $y = 0$ as well as on $x = 0, 0 < y < b$.

From the integral representations it also follows that $\varphi = O(1)$ and that its first order derivatives are $O(r^{-\frac{1}{2}})$ when $r \to 0$, where r is the distance between the points (x, y) and $(0, b)$. This verifies our earlier applications of Green's theorem in the neighborhood of the edge of the fin. In the neighborhood of the corners at (0,0) the function $\varphi(x, y)$ and its first derivatives behave even more regularly.

7. The limiting case *k* real. We now consider the resulting formulas derived for non-real k, in particular (5.1–5), for real values of k. The numbers Γ_{ν} shall be defined by

$$\Gamma_{\nu}^{2} = k^{2} - (\tfrac{1}{2}\nu\pi/b)^{2}, \qquad k \text{ real, positive},$$

and the following sign conventions. Let N denote the largest integer not exceeding $2bk/\pi$. If $\nu \leqq N$, that is $\Gamma_{\nu}^{2} \geqq 0$, the sign is chosen so that Γ_{ν} is non-negative. If $\nu > N$, that is $\Gamma_{\nu}^{2} < 0$, the sign is chosen so that Γ_{ν} is situated on the upper imaginary axis (see Fig. 3b).

An examination of the arguments used in Section 6 will prove that the series in (5.4) and (5.5) again converge and that the sum $\varphi(x, y)$ is a solution of (2.1). Of course, the paths of integration in the right hand sides of (4.7–8) should then include the poles on the real axis for $x > 0$. For $x < 0$, the poles on the negative real axis have to be included while those on the positive real axis are excluded. The integrals, now interpreted in an obvious manner in the sense of Cauchy's principal, are convergent with $\varphi(x, y)$ as their value. The boundary condition 1° of Section 2 is satisfied. The conditions 2° and 3° and the uniqueness are not discussed when k is real.

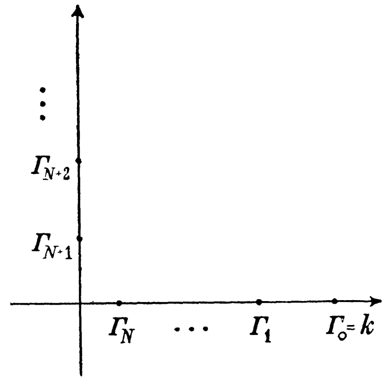

Fig. 3b.

The formulas (5.1–3) yield, in the case k real, very simple expressions for the coefficients T_ν, adequate for numerical computations, especially for the absolute values $|T_\nu|$. We are interested only in $T_0, T_1, \ldots, T_N$, that is, the amplitudes corresponding to propagating modes.

As a first example, we evaluate T_0 in the simplest case $N = 0$, that is $0 \leqq 2bk < \pi$, where only one propagating term occurs. Since

$$|k + \Gamma_\nu| = \tfrac{1}{2}\nu\pi/b, \qquad \nu = 1, 2, \ldots ,$$

it is obvious (see also Fig. 4) that the modulus of

$$K_+(k) = 2k \prod_{n=1}^{\infty} \frac{k+\Gamma_{2n}}{k+\Gamma_{2n-1}} \frac{2n-1}{2n}$$

is $2k$, and the argument $\varrho = \varrho(k) = \arg K_+(k)$ is given by

$$\varrho = \sum_{n=1}^{\infty} (-1)^{n+1} \arcsin(2bk/n\pi) .$$

From $K_+(k) = 2ke^{i\varrho}$ and (5.1) follows

$$T_0 = \tfrac{1}{2}(1 + e^{2i\varrho}) = e^{i\varrho}\cos\varrho, \qquad R_0 = 1 - T_0 = -ie^{i\varrho}\sin\varrho .$$

As a somewhat more complicated example we consider the case $\pi \leqq 2bk < 2\pi$, $N = 1$, with two propagating modes. Writing

$$K_+(k) = k\frac{k+\Gamma_2}{k+\Gamma_1} \prod_{n=2}^{\infty} \frac{k+\Gamma_{2n}}{k+\Gamma_{2n-1}} \frac{2n-1}{2n}$$

we find immediately (as above)

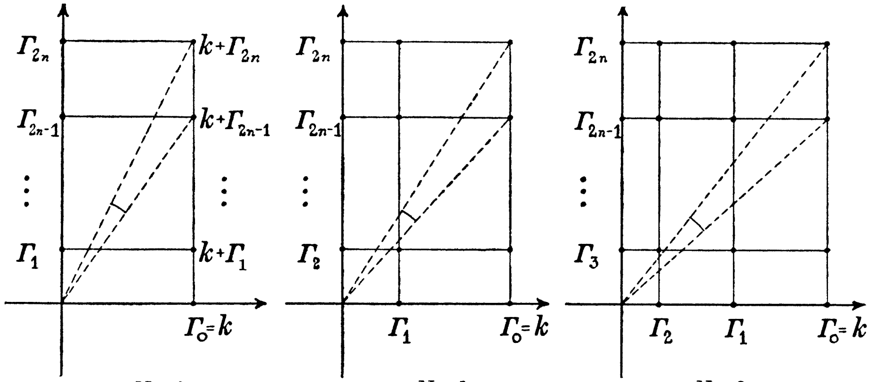

Fig. 4.

$$|K_+(k)| = k\frac{|k+\Gamma_2|}{k+\Gamma_1} = 2k\left(\frac{k-\Gamma_1}{k+\Gamma_1}\right)^{\frac{1}{2}}$$

and

$$\varrho = \arg K_+(k) = \arccos(kb/\pi) + \sum_{n=2}^{\infty}(-1)^{n+1}\arcsin(2kb/n\pi)\,.$$

Hence we have

$$T_0 = \frac{1}{2}\left\{1 + \frac{k-\Gamma_1}{k+\Gamma_1}e^{2i\varrho}\right\}.$$

In order to calculate T_1 for this case, we are required to evaluate the modulus and argument of $K_+(\Gamma_1)$. We have here that

$$K_+(\Gamma_1) = \frac{(k+\Gamma_1)(\Gamma_1+\Gamma_2)}{4\Gamma_1}\prod_{n=2}^{\infty}\frac{\Gamma_1+\Gamma_{2n}}{\Gamma_1+\Gamma_{2n-1}}\,\frac{2n-1}{2n}\,.$$

Since

$$|\Gamma_1+\Gamma_\nu|^2 = \Gamma_1{}^2+|\Gamma_\nu|^2 = (\tfrac{1}{2}\pi/b)^2(\nu^2-1), \qquad \nu = 2, 3, \ldots,$$

we get

$$|K_+(\Gamma_1)| = \frac{\pi 3^{\frac{1}{2}}}{8b}\,\frac{\Gamma_1+k}{\Gamma_1}\prod_{n=2}^{\infty}\frac{2n-1}{2n}\left(\frac{(2n-1)(2n+1)}{2n(2n-2)}\right)^{\frac{1}{2}} = \frac{k+\Gamma_1}{2^{\frac{3}{2}}\,b\,\Gamma_1}$$

as a direct application of the Wallis' product will reveal. The modulus of T_1 is therefore

$$|T_1| = \frac{|K_+(k)|\,(k+\Gamma_1)}{\pi\Gamma_1|K_+(\Gamma_1)|} = \frac{2^{\frac{5}{2}}k}{k+\Gamma_1}$$

and we shall omit the expression for the argument of T_1 since our main interest is to compare the modulii of the T's.

As a final example, we consider the case $2\pi \leqq 2bk \leqq 3\pi$, $N = 2$, where three modes are propagating. Here we have

$$K_+(k) = \frac{k(k+\Gamma_2)}{k+\Gamma_1} \prod_{n=2}^{\infty} \frac{k+\Gamma_{2n}}{k+\Gamma_{2n-1}} \frac{2n-1}{2n},$$

$$K_+(\Gamma_1) = \frac{(k+\Gamma_1)(\Gamma_1+\Gamma_2)}{4\Gamma_1} \prod_{n=2}^{\infty} \frac{\Gamma_1+\Gamma_{2n}}{\Gamma_1+\Gamma_{2n-1}} \frac{2n-1}{2n},$$

$$K_+(\Gamma_2) = \frac{(k+\Gamma_2)\Gamma_2}{\Gamma_1+\Gamma_2} \prod_{n=2}^{\infty} \frac{\Gamma_2+\Gamma_{2n}}{\Gamma_2+\Gamma_{2n-1}} \frac{2n-1}{2n},$$

$$|\Gamma_\mu + \Gamma_\nu|^2 = \Gamma_\mu^2 + |\Gamma_\nu|^2 = (\tfrac{1}{2}\pi/b)^2(\nu^2-\mu^2), \quad \mu = 0, 1, 2; \nu = 3, 4, \ldots .$$

Hence

$$|K_+(k)| = \frac{k(k+\Gamma_2)}{k+\Gamma_1},$$

$$\varrho = \arg K_+(k) = \sum_{n=2}^{\infty} (-1)^{n+1} \arcsin(2bk/n\pi),$$

$$|K_+(\Gamma_1)| = \frac{\pi}{4}\left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{(k+\Gamma_1)(\Gamma_1+\Gamma_2)}{\Gamma_1},$$

and

$$|K_+(\Gamma_2)| = \left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{(k+\Gamma_2)\Gamma_2}{\Gamma_1+\Gamma_2},$$

since

$$\prod_{n=2}^{\infty} \frac{2n-1}{2n} \left(\frac{(2n-2)(2n+2)}{(2n-3)(2n+1)}\right)^{\frac{1}{2}} = \left(\frac{3}{2}\right)^{\frac{1}{2}}.$$

It follows that

$$T_0 = \frac{1}{2}\left\{1 + \frac{(k+\Gamma_2)(k-\Gamma_1)}{(k-\Gamma_2)(k+\Gamma_1)} e^{2i\varrho}\right\},$$

$$|T_1| = 2|T_2| = \left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{k(k+\Gamma_2)}{(k+\Gamma_1)(\Gamma_1+\Gamma_2)}.$$

A convenient method for checking our results is found in a simple application of Green's theorem. Integrating along the boundary of the rectangle $-L \leqq x \leqq L$, $0 \leqq y \leqq 2b$, we have that

$$\int_0^{2b} (\varphi\bar{\varphi}_x - \bar{\varphi}\varphi_x)dy = 0 \tag{6.1}$$

since φ and therefore the conjugate $\bar{\varphi}$ are source free and φ_y vanishes at $y = 0$ and $y = 2b$, while φ_x vanishes at $x = 0$, $0 \leqq y \leqq 2b$. For L sufficiently large, φ is composed of two types of terms, that is an expononential term which vanishes for $L \to \infty$, as well as a bounded term corresponding to the real poles. Hence we need only substitute in the bounded term into (6.1) and evaluate the remaining integrals. We then get for $N\pi \leqq 2bk < (N+1)\pi$, after dividing through by $2ibk$

$$1 - |R_0|^2 - \tfrac{1}{2}k^{-1}\sum_{\nu=1}^{N} \Gamma_\nu |R_\nu|^2 = |T_0|^2 + \tfrac{1}{2}k^{-1}\sum_{\nu=1}^{N} \Gamma_\nu |T_\nu|^2 \tag{6.2}$$

in the limit $L \to \infty$. Upon using the relation that $T_\nu = -R_\nu$, we have finally that

$$|R_0|^2 + |T_0|^2 + \sum_{\nu=1}^{N} k^{-1}\Gamma_\nu |T_\nu|^2 = 1 ,$$

and this serves to relate the magnitude of the propagating modes.

While T_0 is the transmission coefficient and R_0 the reflection coefficient of the dominant mode, the $|T_\nu|$ for $\nu \geqq 1$ are not the transmission coefficients for the higher propagating modes. The factors $\frac{1}{2}k^{-1}\Gamma_\nu$ in (6.2) serve to convert the T_ν and the R_ν, $\nu \geqq 1$, from amplitudes of transmitted and reflected waves to transmission and reflection coefficients, that is, quantities which are numerically less than unity.

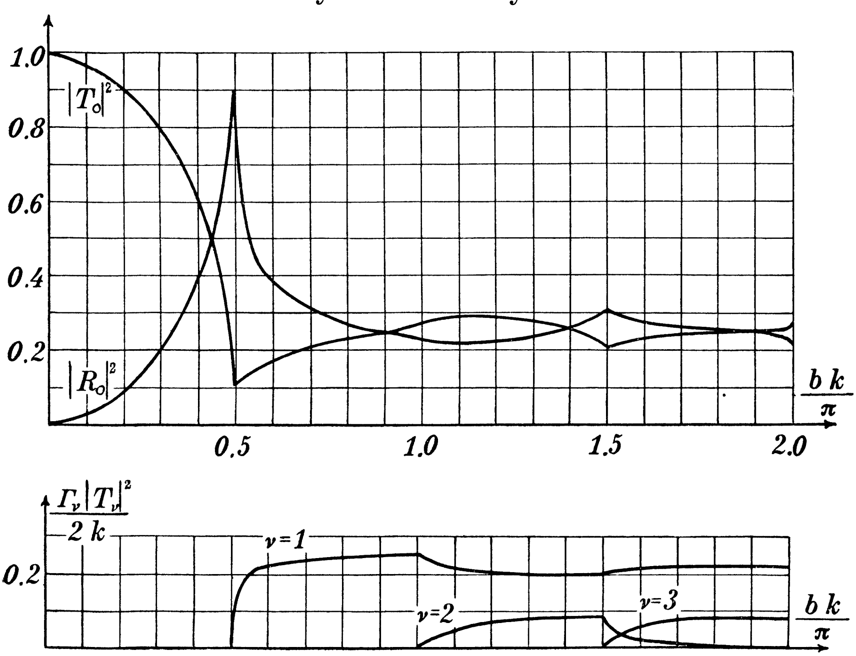

Fig. 5.

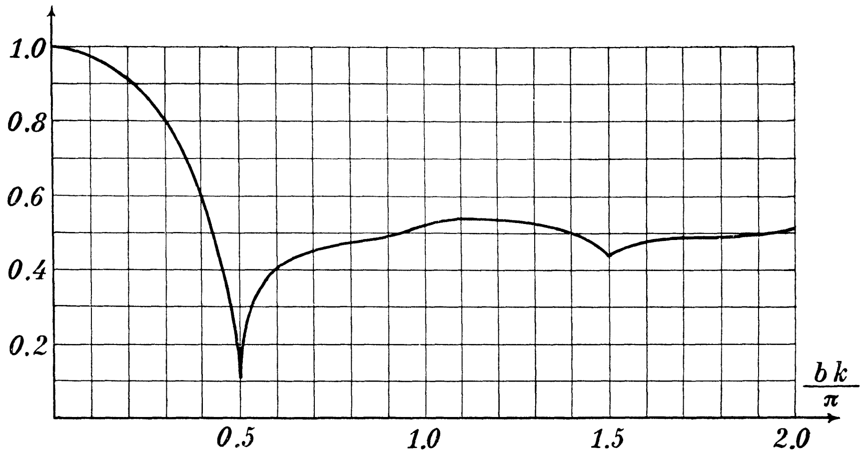

Fig. 6.

In Figure 5 the quantities $|T_0|^2$ and $\frac{1}{2}k^{-1}\Gamma_\nu|T_\nu|^2$, $\nu = 1, 2, 3$, $0 \leqq bk < 2\pi$, are plotted. Figure 6 gives a plot of the right hand side of (6.2).

8. Conversion of results to give scattered fields. In order to obtain the amplitudes and directions of propagation of the diffracted waves, we need only rewrite the wave function describing the field in a typical transmitted mode as follows:

$$\varphi_n = T_n \cos(\tfrac{1}{2}n\pi y/b)\, e^{i\Gamma_n x} = \tfrac{1}{2}T_n e^{i\Gamma_n x + \frac{1}{2}n\pi y/b} + \tfrac{1}{2}T_n e^{i\Gamma_n x - \frac{1}{2}n\pi y/b}.$$

Obviously φ_n represents two waves, each of amplitude $\frac{1}{2}T_n$, traveling in directions making angles of plus and minus θ_n with the positive x axis, where $\theta_n = \arcsin(\frac{1}{2}n\pi/bk)$. Similarly, there are two reflected waves on the left side of the screen with amplitudes equal to $-\frac{1}{2}T_n$ which travel in directions making angles of plus and minus θ_n with the negative x axis. The energy in each individual wave, expressed in terms of that in the incident wave, is given by the square of the absolute value of the amplitude, or $\frac{1}{4}|T_n|^2$.

BIBLIOGRAPHY

1. J. F. Carlson and A. E. Heins, *The reflection of an electromagnetic plane wave by an infinite set of plates* I, Quart. Appl. Math. 4 (1947), 313–329.
2. A. E. Heins, *The radiation and transmission properties of a pair of parallel plates*, Quart. Appl. Math. 6 (1948), 215–220.
3. A. E. Heins, *Water waves over a channel of infinite depth with a dock*, Amer. J. Math. 70 (1948), 730–748.

4. A. E. Heins, *The reflection of an electromagnetic plane wave by an infinite set of plates* III, Quart. Appl. Math. 8 (1950), 281–291.
5. H. Lamb, *On the reflection and transmission of electric waves by a metallic grating*, Proc. London Math. Soc. 29 (1898), 523–544.
6. R. E. A. C. Paley and N. Wiener, *The Fourier transform in the complex domain* (Amer. Math. Soc. colloquium publications 19), New York, N. Y., 1934.
7. Lord Rayleigh, *On the passage of waves through apertures in plane screens and allied problems*, Phil. Mag. 43 (1897), 259–272.

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