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PLANE CONTINUA ADMITTING NON-PERIODIC AUTOHOMEOMORPHISMS WITH EQUICONTINUOUS ITERATES

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Introduction. The purpose of this paper is to treat, for compact subsets of the plane, the following problem posed to the author by A. Edrei: Does there exist a simple necessary and sufficient condition which a continuum X must fulfill in order that it possess a homeomorphism onto itself whose powers form an equicontinuous, non-periodic family; and how is such a homeomorphism characterized.

If X is a simple closed curve, the problem is a special case of that treated by Kneser [9, pp. 141–144], who studied the nature of homeomorphisms of a simple closed curve upon itself. See also J. Nielsen [12] and van Kampen [14]. Kerékjártó [6;7;8] has treated aspects of the problem in case X is a two dimensional, closed, orientable manifold without boundary. The procedure of the present paper is, in part, similar to that of Kerékjártó [6]. However, in many cases the author has preferred to restate and reprove statements made there, since the assumptions here are different and since some of Kerékjártó's arguments seem to require clarification.

The first section contains, for the convenience of the reader, some more or less well-known lemmas of a general nature which are used subsequently to establish that a certain open set has a simple closed curve as a boundary. In particular it includes a condition that a bounded, open, connected plane set be uniformly locally connected. The second section deals with some properties of uniformly equicontinuous families of functions, and with the properties of images of open spheres under the elements of such families.

The remainder of the paper deals with a plane continuum M and a homeomorphism f of M onto itself whose iterates $\{f^n\}$, $n = 0, \pm 1, \pm 2, \ldots$, form an equicontinuous family. The homeomorphism f will be called periodic if there is an $m \neq 0$ such that f^m is the identity f^0. Points p of M will be called periodic if there is an $m \neq 0$ such that $f^m(p) = p$.

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The third section establishes, in case M has interior points, the existence of a simple closed curve in M which is invariant under an iterate of f. The fourth section deals with the case in which such an invariant simple closed curve contains a periodic point; it is proved that any component of the interior of M containing an invariant curve with a periodic point must consist of fixed points under some iterate of f. The fifth section considers the case in which there is a non-periodic interior point of M. Under this condition it is proved that there exists a homeomorphism h of M onto a circular disc or annulus and that there exists an irrational rotation φ of this disc or annulus such that, in the case of the disc, $f = h^{-1}\varphi h$ and in the case of the annulus either $f = h^{-1}\varphi h$ or else $f^2 = h^{-1}\varphi h$, the latter occurring when f permutes the boundary curves of M. Thus, if one component of the interior of M consists entirely of periodic points, all do. The sixth section gives an example which shows that all interior points of M can be periodic under a non periodic function f of the desired type and that in this case the structure of M can be exceedingly complicated. The seventh section deals with the orbit of a point $p \in M$, that is, the set of points $f^n(p)$, $n = 0, \pm 1, \pm 2, \ldots$. It is proved that if the closure of such an orbit is connected, then M must be topologically either a disc, an annulus or a simple closed curve and either f or f^2 must be $h^{-1}\varphi h$ as already described. The concluding section describes some examples in space.

1. General preparatory remarks. Although a number of the lemmas proved are obviously valid in more general spaces, they will all be proved for plane sets. The closure, boundary, interior, and complement in the plane of a point set E will be denoted by $\overline{E}$, $\mathcal{F}E$, $\operatorname{Int} E$, and $\mathcal{C}E$, respectively. The open sphere with center at x and radius r will be denoted by $S_{x,r}$. The distance between two points, x and y, will be denoted by $\varrho(x, y)$.

In this section, let V be a bounded, open, connected point set and let R be the unbounded component of $\mathcal{C}\overline{V}$. Its boundary $\mathcal{F}R$ shall be termed the *outer boundary* of V.

LEMMA 1.1. *The outer boundary of V is connected.*

Since V is connected, the closure $\overline{V}$ is also connected; and hence, according to a well-known theorem of Brouwer (see for example [3, p. 345]), every component of $\mathcal{C}\,\overline{V}$ has a connected boundary.

The next lemma is essentially due to R. L. Moore [11, Theorem 3].

LEMMA 1.2. *Let $U_1, U_2, \ldots$ denote those bounded components of $\mathcal{C}\,\overline{V}$*

whose boundaries are not contained in the outer boundary $\mathcal{F}R$ of V. Then the set

$$B_V = [\mathcal{F}\overline{V} - \mathcal{F}R] \cup \operatorname{Int}\overline{V} \cup U_1 \cup U_2 \cup \dots \tag{1.1}$$

is bounded, open, and connected, and its boundary is the outer boundary of V, that is $\mathcal{F}B_V = \mathcal{F}R$.

It is observed at once that the boundary of every component of $\mathcal{C}\overline{V}$ is contained in $\mathcal{F}\overline{V} = \mathcal{F}\operatorname{Int}\overline{V}$; in particular, $\mathcal{F}R \subset \mathcal{F}\overline{V}$.

Since $B_V \subset \mathcal{C}R$ and $\mathcal{C}R$ is bounded, B_V is bounded. Obviously

$$\mathcal{C}B_V = \mathcal{F}R \cup R \cup W_1 \cup W_2 \cup \dots$$

where $W_1, W_2, \dots$ are those components of $\mathcal{C}\overline{V}$, whose boundaries are contained in $\mathcal{F}R$. Hence no sphere $S = S_{x, r(x)}$, for which $S \cap \mathcal{F}R = 0$, contains points from both B_V and $\mathcal{C}B_V$. Chosing $x \in B_V$, one sees that B_V is open. Chosing in particular $x \in \mathcal{F}\overline{V} - \mathcal{F}R \subset B_V$ as a boundary point of U_μ, one sees that B_V is connected since $\operatorname{Int}\overline{V}$ and all U_μ are connected.

COROLLARY. *The closure of B_V and the complement of $\overline{B}_V$ are, respectively,*

$$\overline{B}_V = \overline{V} \cup U_1 \cup U_2 \dots, \tag{1.2}$$

$$\mathcal{C}\overline{B}_V = R \cup W_1 \cup W_2 \cup \dots,$$

where $W_1, W_2, \dots$ denote those bounded components of $\mathcal{C}\overline{V}$ whose boundaries belong to $\mathcal{F}R$. The (outer) boundary of B_V is the same as the outer boundary of V.

LEMMA 1.3. *Let A be any bounded, open, connected set in the plane, whose boundary $\mathcal{F}A$ is contained in $\overline{B}_V$ but not entirely in $\mathcal{F}R = \mathcal{F}B_V$. Then $A \subset B_V$.*

If a point of R belongs to a bounded open set, then this set has a boundary point in R since R is open, connected, and unbounded. Hence the set A is contained in $\mathcal{C}\overline{R}$, and thus

$$A \subset \mathcal{C}\overline{R} = [\mathcal{F}\overline{V} - \mathcal{F}R] \cup \operatorname{Int}\overline{V} \cup [U_1 \cup U_2 \cup \dots] \cup [W_1 \cup W_2 \cup \dots]$$
$$= B_V \cup W_1 \cup W_2 \cup \dots .$$

Since there is a boundary point of A in $\overline{B}_V - \mathcal{F}R = B_V$, there is also a point $a \in A$ in B_V. Assume that a point $a' \in A$ belongs to one of the sets $W_1, W_2, \dots$, say W_1. Since A is connected, there is an arc in A connecting a and a'. Such an arc would contain a point $p \in \mathcal{F}W_1 \subset \mathcal{F}R$. Since $p \in A$, this contradicts $A \subset \mathcal{C}\overline{R}$, and the lemma is proved.

As usual, a set A will be called locally connected at a point $x \in A$ if to every $\varepsilon > 0$ there exists a $\gamma = \gamma(\varepsilon) > 0$ such that whenever $\varrho(x, y) < \gamma$ then there is a connected set in $A \cap S_{x,\varepsilon}$ containing x and y. If a set is locally connected at each of its points, it is called locally connected. If to every ε the same $\gamma = \gamma(\varepsilon)$ may be used for every $x \in A$, then A is called uniformly locally connected. The following lemma yields a criterion for uniform local connectivity.

LEMMA 1.4. *Let B be a bounded, open, connected point set in the plane having the following properties:*

1° B contains every bounded, open, connected set in the plane whose boundary is contained in $\overline{B}$ but not entirely in the boundary $\mathscr{F}B$;

2° $\overline{B}$ is locally connected.

Then B is uniformly locally connected.

Since $\overline{B}$ is bounded and closed, it is uniformly locally connected; let $\varepsilon > 0$ be given and let $\gamma = \gamma(\varepsilon) > 0$ be chosen to correspond. Then, to any given $x \in B$, $y \in B$, with $\varrho(x, y) < \gamma$ let a connected set $N \subset \overline{B} \cap S_{x,\varepsilon}$ be chosen such that $x \in N$, $y \in N$. Without loss of generality it may be assumed, that N is closed. It will be proved that there exists a curve in $B \cap \overline{S_{x,\varepsilon}}$ connecting x and y; this will clearly prove the lemma.

Since the set B is open and connected, there is an arc in B connecting x and y, which consists of straight line segments. It can be chosen in such a way that none of its vertices lies in $\mathscr{F}S_{x,\varepsilon}$ and none of its segments is tangent to $\mathscr{F}S_{x,\varepsilon}$. Let K be such an arc. If K belongs to $\overline{S_{x,\varepsilon}}$, no further argument is needed. If K contains points outside $\overline{S_{x,\varepsilon}}$, it is easily seen that the union $K \cup N$ separates the plane, and thus that there is an unbounded component and at least one bounded component of $\mathscr{C}(K \cup N)$. At a point p, where the circle $\mathscr{F}S_{x,\varepsilon}$ crosses the polygonal line K, it must pass from the unbounded component of $\mathscr{C}(K \cup N)$ to a bounded one, say Q. Obviously $\mathscr{F}Q \subset \overline{B}$. Moreover, since $\mathscr{F}Q$ contains points of K, $\mathscr{F}Q$ is not contained in $\mathscr{F}B$. Hence, according to 1°, $Q \subset B$. Thus one may proceed from p along an arc of $\mathscr{F}S_{x,\varepsilon}$ which lies in B and whose second endpoint lies on K. Using such circular arcs, it is easy to obtain an arc K_1 consisting of straight line segments and circular arcs of $\mathscr{F}S_{x,\varepsilon}$ which lies in $B \cap \overline{S_{x,\varepsilon}}$ and which connects x and y.

2. Uniform equicontinuity. Mapping of spheres. Let M be a point set of a metric space with distance function ϱ and let Φ denote a family of homeomorphisms of M onto M. As usual, the family Φ is called equicontinuous at $x \in M$ if to any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$y \in M$, $\varrho(x, y) < \delta$ imply $\varrho(f(x), f(y)) < \varepsilon$ for every $f \in \Phi$. The family is called equicontinuous on M if it is equicontinuous at every $x \in M$. Equicontinuity on M is called uniform if the $\delta(\varepsilon)$ is independent of x; thus Φ is called uniformly equicontinuous on M if there exists a real function $\delta = \delta(\varepsilon) = \delta_\Phi(\varepsilon) > 0$ such that

$$(2.1) \quad x \in M, y \in M, f \in \Phi, \varrho(x, y) < \delta(\varepsilon) \quad \text{imply} \quad \varrho(f(x), f(y)) < \varepsilon .$$

Equicontinuity on a compact set is uniform.

Throughout the following *Φ denotes a uniformly equicontinuous family.*

LEMMA 2.1. *Assume M compact, and let Φ be an infinite, uniformly equicontinuous family of homeomorphisms of M onto M having the following property: If $\varphi_1 \in \Phi$, $\varphi_2 \in \Phi$ then either $\varphi_1 \varphi_2^{-1} \in \Phi$ or $\varphi_2 \varphi_1^{-1} \in \Phi$.*

Then, to every $\varepsilon > 0$ there are infinitely many $f \in \Phi$ such that $\varrho(x, f(x)) < \varepsilon$ for every $x \in M$.

Since Φ is infinite, it contains a sequence $g_1, g_2, \ldots$ such that $g_\nu g_\mu^{-1}$ is different from both $g_\varkappa g_\lambda^{-1}$ and $g_\lambda g_\varkappa^{-1}$ for $(\nu, \mu) \neq (\varkappa, \lambda)$. Let $c_1, c_2, \ldots, c_r$ be the centers of a finite covering of M by spheres of radius $\delta = \delta_\Phi(\frac{1}{3}\varepsilon)$. The sequence $g_1, g_2, \ldots$ has a subsequence which converges on all the centers. In order to avoid complication of the notation, it will be assumed that $g_n(c_j)$ converges for every j, $j = 1, 2, \ldots, r$. Then there exists a number $N = N(\varepsilon)$ such that

$$(2.2) \qquad \varrho(g_n(c_j), g_q(c_j)) < \tfrac{1}{3}\varepsilon , \qquad j = 1, 2, \ldots, r ,$$

for $n > N, q > N$. Without restricting the generality, it may be assumed that $f = f_{nq} = g_n g_q^{-1} \in \Phi$. To each $x \in M$ let $y = g_q^{-1}(x)$ and denote by c_j a center such that $\varrho(y, c_j) < \delta(\frac{1}{3}\varepsilon)$. Then

$$(2.3) \qquad \varrho(g_n(y), g_n(c_j)) < \tfrac{1}{3}\varepsilon \quad \text{and} \quad \varrho(g_q(y), g_q(c_j)) < \tfrac{1}{3}\varepsilon .$$

From (2.2) and (2.3) it follows that $\varrho(g_n(y), g_q(y)) < \varepsilon$. Since $g_q(y) = x$ and $g_n(y) = g_n g_q^{-1}(x)$, the homeomorphisms $f = f_{nq}$ have the desired properties, and the lemma is proved. This lemma is essentially the same as one of Edrei [2, corollary 1].

COROLLARY. *If Φ is the semi-group consisting of a homeomorphism f of a compact set M onto itself and the iterates f^q, $q = 1, 2, \ldots$, then there is a sequence $0 < q_1 < q_2 < \ldots$ such that $\varrho(x, f^{q_\nu}(x)) < \varepsilon$ for every $x \in M$ and $\nu = 1, 2, \ldots$.*

The results of sections 3 to 7 all involve the case in which Φ is the group of iterates $\{f^n\}$, $n = 0, \pm 1, \pm 2, \ldots$. It should be noted that the equicontinuity of the negative powers is a consequence of that of the positive powers. This has been related orally to the author by both A. Edrei (see also [2]) and W. Gottschalk and may be stated as follows:

If a semi-group Φ consisting of a homeomorphism f of a compact set M onto M and its iterates f^q, $q = 1, 2, \ldots$, is a uniformly equicontinuous family, then the entire group generated by f is also equicontinuous.

Let $\varepsilon > 0$ be given. To any $n > 0$ determine according to the preceding corollary a number $q > n$ such that

$$\varrho\big(f^q(x), x\big) < \delta_\Phi(\tfrac{1}{3}\varepsilon) \tag{2.4}$$

for every $x \in M$. If $\varrho(y, z) < \delta_\Phi(\frac{1}{3}\varepsilon)$, we have

$$\varrho\big(f^{q-n}(y), f^{q-n}(z)\big) < \tfrac{1}{3}\varepsilon \tag{2.5}$$

since $q-n > 0$. Further, according to (2.4) with $f^{-n}(y)$ and $f^{-n}(z)$ in place of x,

$$\varrho\big(f^{q-n}(y), f^{-n}(y)\big) < \tfrac{1}{3}\varepsilon, \qquad \varrho\big(f^{q-n}(z), f^{-n}(z)\big) < \tfrac{1}{3}\varepsilon\,. \tag{2.6}$$

From (2.5–6) it follows that $\varrho(y,z) < \delta_\Phi(\frac{1}{3}\varepsilon)$ implies $\varrho\big(f^{-n}(y), f^{-n}(z)\big) < \varepsilon$. This holds for $n = 1, 2, \ldots$, which proves the theorem.

Throughout the following we *assume that Φ has the property*,

$$f \in \Phi \quad \textit{implies} \quad f^{-1} \in \Phi\,. \tag{2.7}$$

Then the statement (2.1) is equivalent to

$$x \in M, y \in M, f \in \Phi, \varrho(x, y) \geqq \varepsilon \quad \text{imply} \quad \varrho\big(f(x), f(y)\big) \geqq \delta(\varepsilon) = \delta_\Phi(\varepsilon) \tag{2.8}$$

It follows immediately, that if $A \subset M$, $p \in M$, and $\varrho(p, A) \geqq \varepsilon$, then $\varrho\big(f(p), f(A)\big) \geqq \delta(\varepsilon)$ for all $f \in \Phi$. Moreover it shall be assumed that M *is a bounded set in the plane having interior points.*

As a basis for later considerations, a study will be made of the images of spheres in M and their boundaries under homeomorphisms belonging to Φ. In all that follows, D denotes an arbitrary open sphere with $\overline{D}$ in M. Its boundary and diameter are denoted by C and ε, respectively.

LEMMA 2.2. *If p is a boundary point of the set $\bigcup_{f \in \Phi} f(D)$; then there is a point a of C and a sequence $f_1, f_2, \ldots$ of homeomorphisms in Φ such that*

$$\lim f_n(a) = p \quad \textit{and} \quad a = \lim f_n^{-1}(p)\,.$$

If p is fixed under every $f \in \Phi$, then $a = p$.

The point p is a limit point of the set $\bigcup_{f \in \Phi} f(C)$; and, consequently, there exists a sequence $\{a_i\}$ of points of C, together with a sequence $\{f_i\}$ of homeomorphisms in Φ, such that $p = \lim f_i(a_i)$. Without loss of generality, it may be assumed that the sequence $\{a_i\}$ converges to a point a of C. The lemma now follows from the equicontinuity.

LEMMA 2.3. *The set* $\bigcup_{f \in \Phi} f(D)$ *has only a finite number* $N = N(D, \Phi)$ *of components.*

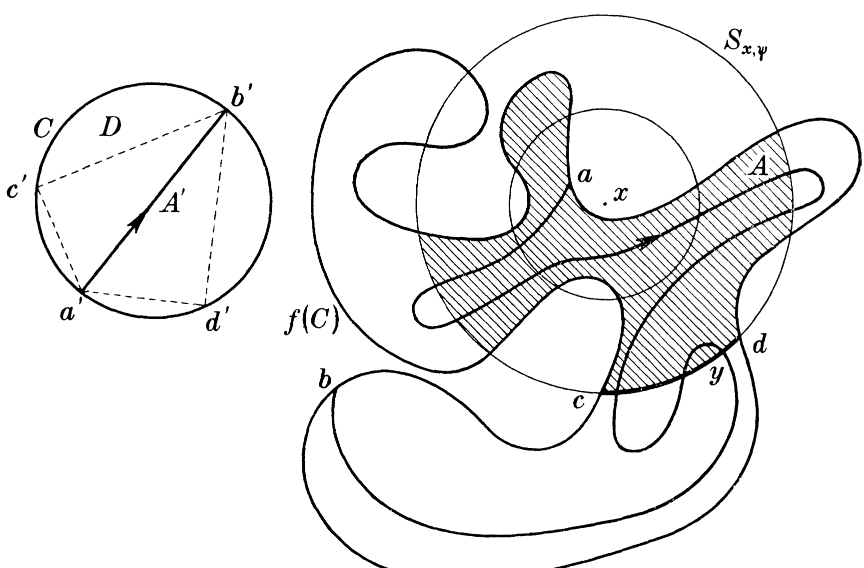
According to (2.8) every image $f(D)$, $f \in \Phi$, must contain a sphere with radius $\delta_\Phi(\frac{1}{2}\varepsilon)$. Hence, since M is bounded, there are only finitely many disjoint images $f(D), f \in \Phi$. Obviously, this proves the lemma.

If Φ is a group of homeomorphisms, then every $f \in \Phi$ maps the set $\bigcup_{f \in \Phi} f(D)$ onto itself. From lemma 2.3 there follows immediately

LEMMA 2.4. *Let* Φ *be a group of homeomorphisms, and let* Φ_1 *be the normal subgroup of* Φ *consisting of those* $f \in \Phi$ *which map every component of the set* $\bigcup_{f \in \Phi} f(D)$ *onto itself. Then the factor group* Φ/Φ_1 *is isomorphic to a group of permutations of* $N = N(D, \Phi)$ *elements, thus being at most of the order* $N!$.

In particular, if Φ *is the cyclic group generated by a homeomorphism* f, *then there is an iterate* f^q, $q > 0$, *of* f, *which maps every component of* $\bigcup_{-\infty}^{+\infty} f^n(D)$ *onto itself. Obviously, if* W *is the component of* $\operatorname{Int} M$, *which contains* D, *then* f^q *maps every component of* $\bigcup_{-\infty}^{+\infty} f^n(W)$ *onto itself.*

The next two lemmas are of quite a different character. They will serve as a tool on several occasions.



LEMMA 2.5. *Let* ψ *denote some positive number not exceeding* $\frac{1}{2}\delta(\varepsilon)$, *where* ε *is the diameter of a sphere* D $(\overline{D} \subset M)$ *with boundary* C *and* $\delta = \delta_\Phi$ *is the function of* (2.1) *and* (2.8). *Let* $S_{x, \frac{1}{2}\psi}$ *be a sphere whose closure has a*

point a in common with an image $f(C), f \in \Phi$. Then the component N of $f(D) \cap S_{x,\psi}$ whose closure contains a has in its boundary an arc H of the boundary of $S_{x,\psi}$ whose length is greater than $\eta = \delta(\delta(\frac{1}{2}\psi))$.

Let $b = f(b')$, where b' is the second end point of that diameter A' of C which has $a' = f^{-1}(a)$ as its first end point. Since $\varrho(a', b') = \varepsilon$, it follows that $\varrho(a, b) \geqq \delta(\varepsilon) \geqq 2\psi$ and hence $\varrho(b, S_{x,\psi}) \geqq \frac{1}{2}\psi$. Proceed along $A = f(A')$ from a to b and denote by y the last point which belongs to $\overline{N}$. Obviously y is situated on an arc H of the boundary of $S_{x,\psi}$ which is a part of the boundary of N and whose end points, c and d, lie on $f(C)$. Clearly there is an arc connecting a with y and which, except for the points a and y, belongs to N. Let A_1 be the curve obtained by prolonging this arc with the piece of A connecting y with b. Since A_1 crosses H exactly once, the pair c, d must separate the pair a, b on $f(C)$. Hence $c' = f^{-1}(c), d' = f^{-1}(d)$ separates a', b' on C. From the fact that the four distances from a and b to c and d are all at least $\frac{1}{2}\psi$ and that $f^{-1} \in \Phi$ it follows from (2.8) that the four distances from a' and b' to c' and d', are at least $\delta(\frac{1}{2}\psi)$. Therefore, $\varrho(c', d') \geqq \delta(\frac{1}{2}\psi)$; and hence

$$\text{length of } H > \varrho(c, d) \geqq \delta(\delta(\tfrac{1}{2}\psi)).$$

LEMMA 2.6. *Let D, ε, ψ, and η be as in lemma 2.5. Then the number of components of the set*

$$\bigcup_{f \in \Phi} f(D) \cap S_{x,\psi}$$

which have a distance from x less than $\frac{1}{2}\psi$ does not exceed $2\pi\psi/\eta$, and is thus finite.

Two distinct ones of these components contain disjoint components, say N and N', of one or two sets $f(D) \cap S_{x,\psi}$, such that both $\overline{N}$ and $\overline{N}'$ have a point in $S_{x,\frac{1}{2}\psi}$. According to lemma 2.5, each of them has in its boundary an arc whose length is greater than $\eta = \delta(\delta(\frac{1}{2}\psi))$. Since the two arcs can have no sub-arc in common, the lemma is proved.

3. Construction of an invariant simple closed curve in M. This section establishes the existence in $\operatorname{Int} M$ of a simple closed curve which is in-invariant under some iterate of f; if there is no need to specify the iterate of f, such a curve will simply be termed invariant. A process by which such invariant simple closed curves may be obtained will also be described. This process is used again and again in sections 4 and 5.

Here, and in all the subsequent sections, the equicontinuous family Φ shall consist of the iterates $f^n, n = 0, \pm 1, \pm 2, \ldots$, of a homeomorphism f of the compact, connected, plane set M upon itself. Except in section seven, M will be assumed to have interior points.

Let D be an open sphere in M with radius ε, and let V be one of the components of $\bigcup_{-\infty}^{+\infty} f^n(D)$. For this set V form the set B_V as in lemma 1.2 and denote B_V by B.

LEMMA 3.1. *B is uniformly locally connected.*

On account of lemmas 1.3 and 1.4, it is sufficient to prove that $\overline{B}$ is locally connected. This is obviously the case at all its interior points.

Let x be a point of $\mathscr{F}B \subset \overline{V}$, and let ψ be any positive number not exceeding $\frac{1}{2}\delta(\varepsilon)$ as in lemma 2.5. By lemma 2.6, there are just a finite number of components N of

$$\bigcup_{n=-\infty}^{+\infty} f^n(D) \cap S_{x,\psi}$$

which have points within $\frac{1}{2}\psi$ of x. Let γ be a positive number less than $\frac{1}{2}\psi$, and also less than the least distance from x to the finitely many sets N whose closures do not contain x, provided that such sets exist. It is clear that there may be several sets N whose closures do contain x. Let $z \in \overline{B}$ and $\varrho(x, z) < \gamma$. From the relation $\overline{B} = \overline{V} \cup U_1 \cup U_2 \cup \ldots$ in (1.2) it can be seen that either z is in a set N containing x and the lemma is proved, or else z is in one of the bounded open sets U_i. In the latter case, the straight line segment joining z to x must meet a boundary point of U_i that can only lie in a set $\overline{N}$ containing x. The closure of the portion of this line segment from z to the first such boundary point together with the set N in question is a connected subset of $\overline{B} \cup S_{x,\psi}$. This completes the proof of the local connectivity.

THEOREM 3.1. *$\mathscr{F}B$ is a simple closed curve.*

According to a theorem of R. L. Moore, cf. [15, p. 161 and 164], if the boundary of a connected, uniformly locally connected, open set is connected, then it must be a simple closed curve. Lemmas 1.1 and 1.2 prove that $\mathscr{F}B$ is connected, lemma 1.2 proves that B is open and connected, and lemma 3.1 proves that B is uniformly locally connected. Hence $\mathscr{F}B$ is a simple closed curve.

REMARK. The points of V lie in the inner region of the curve $\mathscr{F}B$. The points of $\mathscr{F}V$ lie in the inner region of or on $\mathscr{F}B$.

THEOREM 3.2. *There exists an integer $m \neq 0$ such that $\mathscr{F}B$ is invariant under f^m.*

By lemma 2.4 there is an integer $q \neq 0$ such that $f_1 = f^q$ maps the chosen component V of $\bigcup_{-\infty}^{+\infty} f^n(D)$ onto itself. Hence $f_1{}^r(V) = V$ and $f_1{}^r(\mathcal{F} V) = \mathcal{F} V$ for $r = 1, 2, \ldots$. Thus $\mathcal{F} B \subset \mathcal{F} \overline{V} \subset \mathcal{F} V$ implies $f_1{}^r(\mathcal{F} B) \subset \mathcal{F} V$. From the preceding remark it therefore follows that the points of $f_1{}^r(\mathcal{F} B)$ lie in the inner region of or on the curve $\mathcal{F} B$.

Let a point $x \in V$ be chosen; and let, in accordance with lemma 2.1, an integer $r \neq 0$ be determined such that $f_1{}^r$ is closer to the identity than $\varrho(x, \mathcal{F} B)$. From Rouché's theorem (cf. [1, p. 459]) and the fact that x is inside the curve $\mathcal{F} B$, it then follows that x lies inside $f_1{}^r(\mathcal{F} B)$. Thus, since the connected set V has no point in common with $f_1{}^r(\mathcal{F} B)$, V lies completely inside $f_1{}^r(\mathcal{F} B)$. Hence the points of $\mathcal{F} B \subset \mathcal{F} V$ lie in the inner region of or on the curve $f_1{}^r(\mathcal{F} B)$.

From these statements it can be concluded that $f_1{}^r(\mathcal{F} B) = \mathcal{F} B$ when r is determined in the indicated manner. The integer $m = rq$ meets the requirements of the theorem.

In the early pages of [6] Kerékjártó uses a procedure, analogous to the one of this section, to establish the existence of a simple closed curve which is invariant under f. However, he has the advantage of having a fixed point in the set D, and he also treats very lightly [6, p. 237] the connectedness of the set which becomes his invariant simple closed curve.

LEMMA 3.2. *If $B = B_V \subset M$ and V is invariant under some homeomorphism g of M onto M, then $\mathcal{F} B$ and B are invariant under g.*

The relation $g(V) = V$ implies $g(\mathcal{F} V) = \mathcal{F} V$ which, together with $\mathcal{F} B \subset \mathcal{F} V$, implies $g(\mathcal{F} B) \subset \mathcal{F} V$. Hence, according to the remark before theorem 3.2, the points of $g(\mathcal{F} B)$ must lie in the inner region of or on $\mathcal{F} B$. Thus the inner region of $g(\mathcal{F} B)$ belongs to M; and by the theorem on "Gebietstreue" it must be the image under g of the inner region of $\mathcal{F} B$, that is, B. From this it follows that $g(\mathcal{F} B) \subset \mathcal{F} B$; for if there were a point of $g(\mathcal{F} B)$ inside $\mathcal{F} B$, there would be a point of $\mathcal{F} B$, and hence also of $V = g(V) \subset g(B)$, outside $g(\mathcal{F} B)$. However, from $g(\mathcal{F} B) \subset \mathcal{F} B$ it follows that $g(\mathcal{F} B) = \mathcal{F} B$ since no proper subset of a simple closed curve is a simple closed curve. This, in fact, proves the lemma.

4. The case of periodic points on a invariant simple closed curve. In section 3 it was proved that in any component W of $\operatorname{Int} M$ there is a simple closed curve which is invariant under some member of the equicontinuous family $\Phi = \{f^n\}$. If f has a periodic point on such a curve, then, as will be proved in the next theorem, there is an iterate of f

which leaves every point of W fixed. However, it is very easy to give examples where no invariant curve contains a periodic point.

THEOREM 4.1. *If K is a simple closed curve in* $\operatorname{Int} M$ *which is invariant under some iterate of f and on which there is a point periodic under f, then there is an iterate of f which leaves fixed all points of W, the component of* $\operatorname{Int} M$ *containing K.*

Under somewhat different assumptions, Kerékjártó has stated and sketched a proof of a related theorem [6, p. 344]. He assumes the homeomorphisms to be order preserving. This will not be assumed here. For the proof of theorem 4.1 the following lemma is needed.

LEMMA 4.1. *Let k denote a homeomorphism of a simple closed curve K onto itself having the property that the family $\{k^n\}$ is equicontinuous on K, and let $g \in \{k^n\}$.*

If g is order preserving and K has a point fixed under g, then g is the identity on K.

If g is order preserving and K has no periodic point under g, then g is topologically an irrational rotation of K; that is, there exists a homeomorphism h of K onto a circle and an irrational rotation φ of the circle such that $g = h^{-1} \varphi h$.

If g is not order preserving, then g^2 is the identity on K.

For the first and second parts of the lemma, see Kerékjártó [6, pp. 243–245]; the second part is a special case of a theorem of Kneser [9, pp. 141–144]. The third part follows from the first and the fact, that the square of a non-orderpreserving homeomorphism is order preserving and has two fixed points.

PROOF OF THEOREM 4.1. Assume K to be invariant under f^m and some point of K to be fixed under f^l. Then both the point and K are invariant under $f_2 = f^{ml}$. Hence, according to the first and third parts of lemma 4.1, every point of K is fixed under $f_3 = f_2{}^2$; and since W is connected, there are just two components of $W \cap \mathscr{C} K$, each of which is invariant under f_3. Let q be a point of W so close to K that it lies on the boundary C of a sphere D which is centered on K, and which is so small that its images under all iterates of f_3 are contained in an open sphere with diameter less than that of K and lying in W.

The set

$$V = \bigcup_{-\infty}^{+\infty} f_3{}^n(D)$$

is connected since every $f_3{}^n(D)$ contains the center of D. By theorem 3.1

its outer boundary $\mathcal{F}B$ is a simple closed curve which, by lemma 3.2, is invariant under f_3; for V is invariant, under f_3, and $B \subset M$. Since $\mathcal{F}B$ must cross K, there is an arc L of $\mathcal{F}B$ which, except for its end points, lies in the inner region of K, while the end points are on K and thus fixed under f_3. Hence, since $\mathcal{F}B$ and the inner region of K are both invariant under f_3, L is invariant under f_3. Consequently, f_3 preserves the order of rotation on $\mathcal{F}B$. Thus by lemma 4.1 all points of K_1 are fixed under f_3, and by lemma 2.2 each point of K_1 is a point of $C = \mathcal{F}D$. Hence $K_1 = C$ and q is a fixed point under f_3.

The set E of all such points q sufficiently close to K forms an open set in W of points fixed under f_3. Their limit points are also fixed under f_3. If $E \neq W$, points sufficiently close to the set $\overline{E}$ can likewise be proved to be fixed under f_3. The points x of W that can be proved fixed by a finite repetition of this process form a set which is both open and closed in W, and hence include all of W.

5. The case in which Int M contains a non-periodic point.

THEOREM 5.1. *If a component W of* Int M *contains a non-periodic point, then $\overline{W} = M$; and there exist a homeomorphism h of M onto a disc or an annulus and an irrational rotation φ of this disc or annulus such that either $f = h^{-1}\varphi h$ or $f^2 = h^{-1}\varphi h$, the second power being necessary only in case M is topologically an annulus and f permutes its boundary curves.*

The proof of this theorem falls naturally into two parts: In the first part, Theorem 5.1a on p. 136, is proved the existence of the required homeomorphism between $\overline{W}$ and a disc or an annulus; and in the second part, theorem 5.1b on p. 136, the fact that $M = \overline{W}$. The first part is proved by means of lemmas 5.1 to 5.7; the second, directly. It should be noted that, even where it is not explicitly stated, the first part of the proof of theorem 5.1 makes frequent use of ideas contained in [6].

Since W contains a non-periodic point, it follows from theorem 4.1 that no simple closed curve in W which is invariant under an iterate of f can have a periodic point. It is also clear from the third part of lemma 4.1 that the iterates of f are order preserving on the simple closed curves which they leave invariant. These remarks will be used without further comment throughout the proof of theorem 5.1.

LEMMA 5.1. *$\overline{W}$ contains a one parameter family $\mathcal{K}$ of disjoint, invariant, simple closed curves K_θ, where θ is the parameter, such that if $\alpha < \beta$ then K_α is contained in the inner region of K_β and such that each curve of $\mathcal{K}$ either lies entirely in W or else is a boundary curve of W.*

It follows from section 3 that there is a simple closed curve $K \subset W$ which is invariant under an iterate of f, say f_k. It follows from the Brouwer fixed-point theorem that if the inner region of K belongs to M, then f_k has a fixed point in the inner region of K. In this case, let such a fixed point be chosen and let it be called p. Otherwise let p designate a point of K. Let $\{D_\theta\}$ be the family of all open spheres in W having p as center. The index θ indicates the radius of D_θ. The set

$$V_\theta = \bigcup_{n=-\infty}^{\infty} f_k{}^n(D_\theta)$$

is connected since the sphere D_θ is centered on a fixed point or else on a simple closed curve on which f_k behaves, according to lemma 4.1, topologically as an irrational rotation. The outer boundary K_θ of V_θ is, by theorem 3.2, a simple closed curve which is invariant under some iterate of f_k. Thus $\{K_\theta\}$ is a family of simple closed curves each of which is invariant under some (eventually to be proved the same) iterate of f.

Either all the inner regions of the curves K_θ contain the fixed point p, or else they contain the curve K. The symbol K_0 shall designate either p or the curve K, depending on the case considered.

The distance d from the center p to the nearest boundary point of W is the largest possible radius for the spheres D_θ centered at p. Let f_d be an iterate of f leaving K_d invariant. If there are points of W in the outer region of K_d, the curve K_d will obviously lie entirely in W, for the set of all images under iterates of f_d of a small sphere in W centered on K_d will contain K_d. If there are no points of W in the outer region of K_d, then K_d is the outer boundary of W.

In case K_d is not the outer boundary of W, it is possible to extend the family $\{K_\theta\}$, $0 < \theta \leqq d$ of invariant simple closed curves by using a second family of open spheres centered at a point q of K_d. Each of these spheres is to be indexed by the sum of its radius and the number d. The additional invariant simple closed curves are the outer boundaries of the sets

$$V_{d+\theta} = \bigcup_{n=-\infty}^{\infty} f_d{}^n(D_{d+\theta}) \, .$$

They are to be given the index of the sphere from which they arose. If the largest of this second family of spheres does not yield a curve K_{d+d_1} which is the outer boundary of W, the process of extending the family of curves can be repeated.

If the inner region of K contains a point of $\mathscr{C}M$, it contains an open set in $\mathscr{C}M$. Let z be a point in this open set; let the plane be compactified to a sphere; and let the sphere be punctured by the removal of the point z. The *inner boundary* of a connected open set in M shall be defined as its outer boundary in the punctured sphere. Clearly it is possible to carry over to this situation all the previous results proving that certain outer boundaries are invariant simple closed curves. All further considerations will be carried out in the original plane.

The inner boundary of V_θ, where the sphere D_θ is centered at $p \in K_0$, is thus well defined. Each of these inner boundaries is to be indexed by the negative of the radius of the open sphere from which it arose; and, if it is necessary to consider further families of spheres as above, they shall be indexed by the negative of the numbers described there.

The family $\mathcal{K}$ shall consist of all the simple closed curves obtained in the manner just described. It is clear that they form a one parameter family, and that K_α is contained in the inner region of K_β if $\alpha < \beta$. Thus the curves of $\mathcal{K}$ are disjoint.

The next lemma is analogous to statements made by Kerékjártó [6, p. 247 and p. 249] for a family of curves formed in somewhat the same manner as $\mathcal{K}$. It involves the rotation constant k of an iterate f_1 of f leaving a curve $K \in \mathcal{K}$ invariant. In the lemmas to follow two well-known determinations of the rotation constants are used. It will be clear that these determinations are equivalent and that it is meaningful to speak of a function f_1 inducing the same irrational rotation on two curves of $\mathcal{K}$.

LEMMA 5.2. *To each $K_\alpha \in \mathcal{K}$ and each $\varepsilon > 0$ there is a number $\eta = \eta(\varepsilon, \alpha)$ with the following property: to each index β with $|\beta - \alpha| < \eta$ there is a homeomorphism $f_1 \in \Phi = \{f^n\}$ leaving K_α and K_β invariant and a homeomorphism g of K_α onto K_β such that*

$$(5.1) \qquad \varrho(x, g(x)) < \varepsilon \quad \textit{and} \quad f_1(x) = g^{-1} f_1 g(x) \quad \textit{for} \quad x \in K_\alpha,$$

and f_1 induces the same irrational rotation on K_α and K_β.

Without loss of generality the discussion may be restricted to the two cases:

$$(1) \quad d < \alpha < \beta \leqq d_1, \qquad (2) \quad d = \alpha < \beta \leqq d_1,$$

where d and d_1 are successive indices at which a new family of concentric open spheres entered into the construction of $\mathcal{K}$. The iterate of f leaving K_d invariant which was used in the formation of K_β from D_β will be denoted by f_d.

Let $S_{q,r}$ denote a sphere contained in the inner region of a given curve K_α and let ψ be any positive number less than both $r = r(\alpha)$ and the given number ε. Then $\eta = \delta(\frac{1}{2}\delta(\psi))$, where $\delta = \delta_\Phi$ is the function of (2.1), meets the requirements of the lemma; the proof of this follows.

Let β be any index with $0 < \beta - \alpha < \eta$.

To each point $b \in K_\beta$ there is, by lemma 2.2, a point c of $C_\beta = \mathcal{F} D_\beta$ and an integer m such that $\varrho(f_d^m(c), b) < \frac{1}{2}\delta(\psi)$. In case (1), since $\beta - \alpha < \delta(\frac{1}{2}\delta(\psi))$, there is a point y of D_α so close to c that $\varrho(f_d^m(y), f_d^m(c)) < \frac{1}{2}\delta(\psi)$. Thus $\varrho(b, f_d^m(y)) < \delta(\psi)$. The point y, however, lies in the inner region of K_α. Thus, there is a point a of K_α with $\varrho(b, a) < \delta(\psi)$. In case (2), the point c is so close to the center x of D_β, that $\varrho(f_d^m(x), f_d^m(c)) < \frac{1}{2}\delta(\psi)$. Thus, $a = f_d^m(x)$ is a point of K_α with $\varrho(b, a) < \delta(\psi)$. Hence in both cases

$$\varrho(f^n(a), f^n(b)) < \psi, \qquad n = 1, 2, \ldots . \tag{5.2}$$

Obviously there is an iterate f_2 of f which leaves both K_α and K_β invariant. By lemma 2.1 and the continuity of the functions h and φ of lemma 4.1 there is an iterate f_1 of f_2 such that the minor arcs from $f_1^n(a)$ to $f_1^{n+1}(a)$ on K_α and from $f_1^n(b)$ to $f_1^{n+1}(b)$ on K_β, respectively, have diameter less than ψ for all integers n. It will be proved that the rotation constants k_α and k_β which f_1 has on K_α and K_β, respectively, are equal.

Let an orientation of the plane be chosen and denote by σ_n the angle from the first to the second of the two vectors pointing from q to the points $f_1^{n-1}(a)$ and $f_1^n(a)$, respectively. The angle σ_n is chosen such that $|\sigma_n| < \frac{1}{6}\pi$, which is possible on account of the choice of f_1. Then

$$k_\alpha = \lim_{n\to\infty} n^{-1}(\sigma_1 + \ldots + \sigma_n) .$$

Similarly,

$$k_\beta = \lim_{n\to\infty} n^{-1}(\tau_1 + \ldots + \tau_n) ,$$

where τ_n is defined analogously to σ_n. According to (5.2),

$$|(\sigma_1 + \ldots + \sigma_n) - (\tau_1 + \ldots + \tau_n)| < \tfrac{1}{6}\pi .$$

Hence $k_\alpha = k_\beta$; and thus f_1 may be said to induce the same irrational rotation on K_α and K_β.

From this follows, by a standard argument, that there is a unique homeomorphism g of K_α onto K_β such that $g(f_1^n(a)) = f_1^n(b)$ for all n. Obviously g satisfies (5.1) and thus the lemma is proved.

LEMMA 5.3. *To each α there is a positive number γ such that if $0 < \alpha - \beta < \gamma$, and if f_2 is an iterate of f leaving K_α and K_β invariant, then f_2 induces the same irrational rotation on K_α and K_β.*

Let f_3 be the least iterate of f leaving K_α invariant. It follows from lemma 4.1 that $f_3 = h_\alpha^{-1}\varphi_\alpha h_\alpha$ on K_α, where h_α is a homeomorphism of

K_α onto a unit circle C and φ_α is an irrational rotation of C. Obviously $h_\alpha f_3^n = \varphi_\alpha{}^n h_\alpha$ for all integers n. Let $\omega(z_1, z_2)$ denote the length of the smallest arc on C joining the points z_1 and z_2 of C; and let ε be a positive number such that if $\omega(z_1, z_2) > \frac{1}{3}\pi$ then $\varrho(h_\alpha{}^{-1}(z_1), h_\alpha{}^{-1}(z_2)) > 3\varepsilon$. To this ε choose ψ as in the proof of lemma 5.2 so that $\psi < \varepsilon$; and let $\gamma = \delta(\frac{1}{2}\delta(\psi))$. Choose β such that $0 < \alpha - \beta < \gamma$ and denote by f_1 a homeomorphism having the properties stated in lemma 5.2. There is no loss of generality in supposing that f_2 is the least iterate of f leaving K_α and K_β invariant. Then f_1 is an iterate of f_2, say $f_2{}^s = f_1$; and from the definition of f_3 it follows that f_2 is an iterate of f_3. As before, $f_2 = h_\beta{}^{-1}\varphi_\beta h_\beta$ on K_β, where $h_\beta(K_\beta) = C$, and $f_2 = h_\alpha{}^{-1}\varphi_\alpha{}' h_\alpha$ on K_α. In the proof of lemma 5.2, points $a \in K_\alpha$ and $b \in K_\beta$ satisfying (5.2) are found. It may be assumed that $\varrho(a, b) < \psi$ and $h_\alpha(a) = h_\beta(b) = y \in C$.

By lemma 5.2 the function f_1 has the same rotation constants on K_α and K_β. Thus k_α and k_β, the respective rotation constants of f_2 on K_α and K_β, satisfy the relation: $sk_\alpha \equiv sk_\beta$, $\bmod 2\pi$; and hence $k_\alpha - k_\beta = 2\pi s^{-1} n_s$ for some integer n_s. If n_s is not a multiple of s, there is an integer p such that $|p(k_\beta - k_\alpha) - \pi| < \frac{1}{3}\pi$, $\bmod 2\pi$. From the continuity of $h_\beta{}^{-1}$ there can be found a positive number μ, less than $\frac{1}{3}\pi$, such that if $y_1 \in C$, and $\omega(y_1, y) < \mu$, then $\varrho(h_\beta{}^{-1}(y_1), b) < \psi$. Since k_β is an irrational multiple of 2π, there exists a number q with $\omega(\varphi_\beta{}^{sq+p}(y), y) < \mu$. Thus $\varrho(f_2{}^{sq+p}(b), b) < \psi$. From the choice of p it follows that $\omega(\varphi_\alpha{}'^{sq+p}(y), y) > \frac{1}{3}\pi$; this implies that $\varrho(f_2{}^{sq+p}(a), a) > 3\varepsilon > 3\psi$, which leads to a contradiction since $\varrho(f_2{}^{sq+p}(a), f_2{}^{sq+p}(b)) < \psi$ and $\varrho(a, b) < \psi$.

LEMMA 5.4. *Each point of the annular region between two curves of the family $\mathcal{K}$ lies on a member of $\mathcal{K}$ and thus belongs to M.*

Let x be a point between K_μ and K_ν, two curves of $\mathcal{K}$. Let α be the supremum of the indices θ such that K_θ has x in its outer region, and let β be the infimum of the indices θ such that K_θ has x in its inner region. It is clear that $\alpha = \beta$, since otherwise an index between α and β would furnish a curve contradicting their definitions. It follows from lemma 5.2 and Rouché's theorem that if x is not on K_α then there is an index θ such that K_θ separates x from K_α, in contradiction to the definitions of α and β.

LEMMA 5.5. *The process of lemma 5.1 which defined the family $\mathcal{K}$ required just a finite number of steps before reaching and including in $\mathcal{K}$ the outer boundary of W. If W has an inner boundary, this is unique and also in $\mathcal{K}$. Thus W is either a disc on which f has a fixed point or else an annulus.*

If K_d is not the outer boundary of W and if the spheres $D_{d+\theta}$ centered at the chosen point q of K_d do not encounter an outer boundary point of W, then, by lemma 5.4, their maximum radius is at least $r = \varrho(K_d, K_0)$. Thus, if D_e is the largest open sphere in W at q, $\varrho(K_e, K_d) \geqq \delta_\Phi(r)$, where δ_Φ is defined in (2.1). If the next family of open spheres, centered on K_e, does not meet an outer boundary point of W, its maximum radius will be at least $r+\delta_\Phi(r)$. Thus in a finite number of steps a point y of the outer boundary of W will lie on the boundary of one of the chosen spheres, say D_μ. On account of lemma 5.4 the point y cannot lie in the inner region of K_μ. Thus, since y is a limit point of points of the inner region, it must lie on K_μ. As observed in the proof of lemma 5.1, this is sufficient to prove that K_μ is the outer boundary of W.

In case there are inner boundary points of W, a similar argument shows that they lie on a unique curve of $\mathcal{K}$.

LEMMA 5.6. *With the possible exception of a single fixed point, each point of* $\overline{W}$ *lies on a curve of the family* $\mathcal{K}$. *All invariant curves of* $\overline{W}$ *are in* $\mathcal{K}$.

Let x be a point of W which is not the fixed point $p = K_0$. Since the outer boundary of W belongs to $\mathcal{K}$, x is in the inner region of at least one member of $\mathcal{K}$. It is also in the outer region of at least one member of $\mathcal{K}$: indeed, there is either an inner boundary of W which becomes the required member of $\mathcal{K}$, or else there is a fixed point K_0 about which are to be found small curves of $\mathcal{K}$ having x in their outer regions. Thus lemma 5.4 applies and $x \in K_\theta$ for some $K_\theta \in \mathcal{K}$.

Since all points of $\overline{W}$ other than the possibly existing fixed point K_0, lie on a curve of $\mathcal{K}$, there is at most one fixed point in $\overline{W}$.

If L is an invariant curve in $\overline{W}$, let K_θ be a curve of $\mathcal{K}$ through a point of L. Since there is an iterate of f leaving them both invariant which is topologically an irrational rotation of each, $L = K_\theta$.

LEMMA 5.7. *The curves of* $\mathcal{K}$ *are invariant under either* f *or* f^2; *and, if they are not all invariant under* f, *then* W *is an annulus whose two boundary curves are permuted by* f.

Let K_θ be any curve of $\mathcal{K}$ and let n_θ be the least positive value of ν for which f^ν leaves K_θ invariant. Then the set $\{f^n(K_\theta)\}$, $n = 0, \pm 1, \pm 2, \ldots,$ consists only of the finitely many curves

$$f^n(K_\theta), \qquad n = 0, 1, 2, \ldots, n_\theta - 1 .$$

Since each of these curves is invariant under f^{n_θ}, they belong to $\mathcal{K}$ according to lemma 5.6. Thus, according to lemma 5.1, they may be ordered according to the inclusion of their inner regions. Obviously they are permuted among themselves by every iterate of f, in particular by f and f^2.

If W is topologically a disc, f must preserve the ordering of the curves, and hence maps each of them, K_θ in particular, onto itself. Thus $n_\theta = 1$.

If W is topologically an annulus, there are two cases. If f does not permute the boundary curves of W, f must preserve the ordering of the curves, and thus again $n_\theta = 1$. If f permutes the boundaries of W, then f^2 does not, and hence f^2 preserves the ordering. Thus, in this case, f^2 maps K_θ onto itself. It follows that $n_\theta = 1$ or 2.

THEOREM 5.1 a. *There is a homeomorphism h of W onto either a disc or an annulus and an irrational rotation φ of the disc or annulus such that either $f = h^{-1}\varphi h$, or else $f^2 = h^{-1}\varphi h$, the second power being necessary only in case W is an annulus and f permutes the boundary curves.*

Lemmas 5.3 and 5.7 imply that f, or f^2 as the case may be, is topologically the same irrational rotation φ of all the members of $\mathcal{K}$.

According to [5, p. 244 III], which presupposes lemmas 5.2 and 5.4, there is an arc Q in $\overline{W}$ which crosses all the curves of $\mathcal{K}$ precisely once. Let h_1 be a homeomorphism of Q onto a straight line segment L. L be so placed on the radius of a circle C that the image of the point of Q on the outer boundary of W falls on C and that the other end of L falls either on the center of C or else on a point half way from C to the center, depending, respectively, upon whether W has one or two boundary curves. Let φ be the irrational rotation to which f or f^2 is equivalent on each curve of $\mathcal{K}$. It is clear that the points of $\{\varphi^n(L)\}$, $n = 1, 2, \ldots$, are dense in a disc or annulus bounded by C and that mapping the point $f^n(Q \cap K_\theta)$ on $\varphi^n\big(h_1(Q \cap K_\theta)\big)$ with a suitable extension to limit points yields the required h.

THEOREM 5.1 b. $\overline{W} = M$.

It is well known, see for instance [5, p. 73], that the homeomorphism h of theorem 5.1a can be extended to a homeomorphism (still to be called h) defined on P, the plane of W, in such a manner that $h(P)$ is the plane of $h(W)$. The function hfh^{-1} (or hf^2h^{-1}, if the second power of f is required), defined in $h(M)$, coincides with φ on $\overline{W}$, and the designation φ will be retained for the extended function. Since M is compact, the function φ will have equicontinuous iterates $\varphi^n = hf^nh^{-1}$ on $h(M)$.

It may thus be assumed without loss of generality that $\overline{W}$, itself, is a circular disc or annulus on which the function f is an irrational rotation.

Let it be assumed that there is a point x of M outside the outer boundary circle K of W. A contradiction will be obtained by proving that there must then be an annulus in $M-W$ having K as one of its boundary curves. (The case in which there are points of M in the inner region of W can be handled by an inversion and a similar treatment.)

Let A be an annulus with boundary circles K and K^*, where K^* is outside K and so large that M is in its inner region. Let x be chosen in $M \cap A$ and let A' be a smaller annulus in A with boundary circles K and K' so close that $f^n(x)$ is outside K' for all n. It will be shown that any sphere $S_{p,r}$ in A' must contain a point of M, which proves the theorem.

Consider, as usual, the universal covering space B of A as an infinite strip having covering mapping π and boundary lines K_B and $K_B{}^*$ such that $\pi(K_B) = K$ and $\pi(K_B{}^*) = K^*$. Points y of B such that $\pi(y)$ is outside K' will be said to be outside the line $K_B{}'$ defined by $\pi(K_B{}') = K'$. The strip between K_B and $K_B{}'$ is mapped on A' by π. For convenience the metric in A will be so changed that π is a local isometry. Let η be a number such that π is one to one and isometric on every sphere $S_{y,\eta}$ of the space B.

Let $\varepsilon < \min(\frac{1}{2}r, \frac{1}{2}\eta)$. Further, let

$$\varepsilon_1 = \min\big(\varepsilon, \delta(\varepsilon)\big) , \tag{5.3}$$

where $\delta = \delta_\Phi$ is the function of (2.1); and choose in $M \cap A$ an ε_1-chain C_0,

$$C_0\colon\ a_0 = x_{01}, x_{02}, \ldots, x_{0m} = x ,$$

connecting x with some point $a_0 \in K$. Such a chain exists due to the connectedness of M. It will be proved that, for some n, a point of $f^n(C_0)$ will enter the sphere $S_{p,r}$ in A'.

Let s be a number such that f^s is closer to the identity than ε_1 and consider the chains $C_1 = f^s(C_0)$, $C_2 = f^{2s}(C_0)$, $\ldots$,

$$C_\nu\colon\ f^{\nu s}(a_0) = a_\nu = x_{\nu 1}, x_{\nu 2}, \ldots, x_{\nu m} = f^{\nu s}(x) ,$$

where $x_{\nu\mu} = f^{\nu s}(x_{0\mu})$. The ε-chain C_ν lies in M and connects the point $f^{\nu s}(x)$ outside K' with the point a_ν of K.

Now let b_0 be a point on K_B such that $\pi(b_0) = a_0$; let b_1 be the first point of K_B to the right of b_0 such that $\pi(b_1) = a_1$; let b_2 be the first point of K_B to the right of b_1 such that $\pi(b_2) = a_2$; etc. Then $b_0, b_1, b_2, \ldots$ is a sequence of equidistant points on K_B which proceeds to the right from b_0; the constant distance between two consecutive points is less than ε_1. Since by (5.3) $\varepsilon_1 \leqq \varepsilon < \eta$, there is exactly one ε-chain in B,

$$D_\nu\colon\ b_\nu = y_{\nu 1}, y_{\nu 2}, \ldots, y_{\nu m} ,$$

such that $\pi(y_{\nu\mu}) = x_{\nu\mu}$, $\mu = 1, 2, \ldots, m$. D_ν connects the point b_ν of K_B with the point $y_{\nu m}$ outside the line K_B' and every point of D_ν is mapped on a point of M by π. Moreover, it can be seen from (5.3) that

$$\varrho(y_{\nu+1\,\mu}, y_{\nu\mu}) = \varrho(x_{\nu+1\,\mu}, x_{\nu\mu}) < \varepsilon < \tfrac{1}{2}r.$$

Let q be a point such that $\pi(q) = p$ and such that the sphere $S_{q,r}$ lies to the right of all points of D_0. Choose n so large that all points of D_n are to the right of $S_{q,r}$. Consider the cycle Z_ν made up of straight line segments joining the successive points of the following ε-chains in the order given:

1° D_ν: $y_{\nu 1}, y_{\nu 2}, \ldots, y_{\nu m}$,

2° $y_{\nu m}, y_{\nu+1\,m}, \ldots, y_{\nu+n\,m}$,

3° $-D_{\nu+n}$: $y_{\nu+n\,m}, y_{\nu+n\,m-1}, \ldots, y_{\nu+n\,1} = b_{\nu+n}$,

4° $b_{\nu+n}, b_{\nu+n-1}, \ldots, b_\nu = y_{\nu 1}$.

If there are no points of $\pi^{-1}(M)$ in $S_{q,r}$, it can be seen from (5.3) that each pair of cycles Z_ν, $Z_{\nu+1}$, $\nu = 0, 1, 2, \ldots, n$, satisfies the conditions of Rouché's theorem with respect to q, and thus that Z_ν and $Z_{\nu+1}$ have the same linking coefficient with respect to q. However, it is clear that the cycle Z_0 has linking coefficient 1, whereas Z_n has linking coefficient 0. Hence there must be points of $\pi^{-1}(M)$ within $S_{q,r}$. This completes the proof of theorem 5.1b and thus of theorem 5.1.

Since $\overline{W} = M$ in case there is a non-periodic interior point of M, theorem 5.1 implies

THEOREM 5.2. *If one component of* $\operatorname{Int} M$ *consists of periodic points, then all interior points of* M *are periodic.*

It should be noticed that this theorem does not imply that different components of $\operatorname{Int} M$ have the same period. In fact, there exist examples in the plane of sets M and functions f such that there is no iterate of f leaving all the components of $\operatorname{Int} M$ fixed.

6. A set M whose interior points are periodic but on which the family $\{f^n\}$ is not periodic. Let I be the closed unit interval from (0,0) to (1,0) in the cartesian plane. Let $p_1, p_2, \ldots, p_n, \ldots$ be the successive prime numbers, ordered according to increasing size. Let I be divided into 3 equally long subintervals and let the interior of the middle third be deleted. In each of the remaining make a division into 5 equal intervals and let the interior of the second and fourth be deleted. The process will be continued in such a manner that if p_i subintervals of a given interval have been retained at a certain step, then at the next step p_{i+1} subintervals of each of these will be retained. If A_i is the set of points remaining at the end of the ith step, then

$$Q = \bigcap_{i=1}^{\infty} A_i$$

is obviously a Cantor type set. The points x of Q may be described arithmetically for the purpose of this paper by sequences of numbers $a_1, \ldots, a_n, \ldots$, where $1 \leqq a_n \leqq p_n$. The number a_1 is either one or two depending respectively upon whether the abscissa of x is in the interval $[0, \frac{1}{3}]$ or in $[\frac{2}{3}, 1]$. The element a_n refers to the a_nth subinterval, counting from the left in the nth step, of the interval defined by the sequence $a_1, a_2, \ldots, a_{n-1}$. Clearly these sequences describe the points of Q uniquely.

The operation T will be defined as follows:

$$T(a_n) = \begin{cases} a_{n+1}, & \text{if} \quad a_n < p_n; \\ 1 \quad , & \text{if} \quad a_n = p_n . \end{cases}$$

The function φ shall be defined to be the one which sends the point $(a_1, \ldots, a_n, \ldots)$ onto the point $\big(T(a_1), \ldots, T(a_n), \ldots\big)$. Clearly φ is one to one and non-periodic. The collection, taken over all subintervals at all steps of the subdivision process, of sets consisting of the points of Q in a subinterval retained at the ith step, forms a basis for the open sets of Q. All points in such a basis neighborhood have the same coordinates in the first i positions. The function φ transforms each open set of this type into another of the same diameter. The iterates of φ thus form an equicontinuous family of homeomorphisms.

Let E be the cone on Q from the point $(0, 1)$ and let the function f be defined to be the identity on $(0, 1)$, φ on Q and the obvious thing on the lines of the cone. Let N be a compact set in the plane having interior points and arbitrary complexity. Let the function g be defined to be the identity on N. Let homeomorphs of the brush E be distributed in the plane in such a manner that they have just their vertex (the image of the point $(0, 1)$) in common with N. Let g be extended over these brushes so that it behaves on them as f did on E. The function g is clearly a non periodic homeomorphism of the resulting set onto itself, and the iterates of g form an equicontinuous family. In this example the interior points were all fixed; it is not hard to construct examples in which the interior is not connected and each component of the interior consists of periodic points with non-trivial periods. It is also possible for different periods to occur, and for f to be periodic on each point of M without being periodic.

7. The case in which the closures of the orbits are connected. If it is desired to introduce additional hypotheses which exclude unpleasant sets having the character of the last example, one may of course require that there be a non-periodic interior point of M. However, since the unpleasantness of the example arose from the fact that there were points the closures of whose orbits were zero dimensional, it is natural to ask what happens if no point p of M has an orbit whose closure is separated.

In this case it is useful to know that the closure of the orbit of each point forms a topological group. The author has obtained this result from a conversation with A. Edrei and has found that it is also known to Gottschalk and Hedlund. It can be proved by defining the operation $f^n(p)f^m(p)$ to be $f^{n+m}(p)$, and by extending the operation to limit points so that

$$\lim f^{n_i}(p) \lim f^{m_i}(p) = \lim f^{n_i+m_i}(p) .$$

The existence, uniqueness, and continuity of this product follow from the compactness of M together with the equicontinuity. It is not hard to prove the existence and continuity of an inverse. The group G, so obtained, is compact and abelian. The additional hypothesis of connectedness permits the use of very strong group theoretical results.

As a subset of the plane, G is at most two dimensional. It cannot, however, have dimension two, for then it would contain an open subset of the plane [4, page 44] to which the results of section 5 would apply. There the closures of the orbits were simple closed curves. Since G must be one-dimensional, a theorem of Kodaira and Abe applies [10, p. 172]: *The only n-dimensional, connected, compact, separable, abelian group which may be imbedded in euclidean* $(n+1)$*-space is the torus group.* Hence G is a simple closed curve or a point. If G contains elements other than the identity, it follows that G has no fixed points under f. If $M \neq G$, from the proof of theorem 5.1b it follows, if G is used in place of W, that M contains an open set bounded on one side by G; and hence, by theorem 5.1, *M is either a disc or an annulus and f or f^2 is topologically an irrational rotation. Otherwise M is a simple closed curve and f topologically an irrational rotation.*

8. The situation in space. This is considerably more complex as can be seen from the following examples:

(1) Let N be a solid spherical ball from the interior of which have been removed a large number of non-overlapping, open spherical balls having their centers on a chosen diameter of N. Let f be an irrational rotation about this diameter. (Toroidal holes in N are clearly also pos-

sible.) This set is clearly locally connected and the closures of the nontrivial orbits are simple closed curves.

(2) Let N be a solid torus (the product of a circle and a disc). Let f be the mapping obtained by composing an irrational rotation of the circle and one of the disc. Here the closures of all the orbits are toroidal or circular.

(3) Let N be a solenoid and f the "rotation" to which van Dantzig refers in his paper on solenoids [13, page 107, footnote 16]. This continuum is compact and connected but not locally connected.

There are reasons to believe that the existence of an open set of nonperiodic points around a point p of a space continuum is sufficient to guarantee that the closure of the orbit of p will be toroidal; however, the author does not know a proof of this.

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