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A NOTE ON FOURIER–STIELTJES TRANSFORMS AND ABSOLUTELY CONTINUOUS FUNCTIONS

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1. Let $F(\alpha)$ be a real or complex-valued function of the real variable α and of bounded variation on the whole real axis $-\infty < \alpha < \infty$:

$$\int_{-\infty}^{\infty} |dF(\alpha)| = V < \infty .$$

Let $f(u)$ be the Fourier-Stieltjes transform of $F(\alpha)$:

$$f(u) = \int_{-\infty}^{\infty} e^{iu\alpha} dF(\alpha) , \tag{1}$$

u being a real variable. In connection with a problem concerning the unique determination of certain Fourier-Stieltjes transforms the author [1, p. 19] has earlier proved the following theorem: *If $f(u)$ is equal to zero on an interval of infinite range, then $F(\alpha)$ is absolutely continuous.* The condition that $f(u)$ is equal to zero on an infinite interval ω may, however, be replaced by the weaker one that $f(u)$ belongs to the Lebesgue class L^2 on ω. In fact, in this paper we shall prove the following theorem:

THEOREM. *If $f(u)$ is the Fourier-Stieltjes transform of a function $F(\alpha)$ of bounded variation on $(-\infty, \infty)$ and if $f(u)$ belongs to L^2 on an interval of infinite range, then $F(\alpha)$ is absolutely continuous.*

Before proving the theorem we remark that if $f(u) \in L^2(-\infty, \infty)$ the theorem is an immediate consequence of well-known properties of Fourier integrals. Further, if $F(\alpha)$ is a real-valued function, then $f(-t) = \overline{f(t)}$ and $f(t) \in L^2(-\infty, \infty)$ if $f(t)$ belongs to L^2 on an infinite interval ω. Thus, in the particular cases where $\omega \equiv (-\infty, \infty)$ or $F(\alpha)$ is real, the theorem is trivial.

Further we note that if $\int_\omega |f(u)|^p du < \infty$, $p \leqq 2$, then $\int_\omega |f(u)|^2 du < \infty$ and hence $F(\alpha)$ is absolutely continuous. If, however, $\int_\omega |f(u)|^p du < \infty$,

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$p > 2$, then $F(\alpha)$ is not necessarily absolutely continuous as may be shown by examples (see [3, Section 8]).

2. Proof of the theorem. Without loss of generality we may suppose $\omega \equiv (-\infty, 0)$, that is,

$$\int_{-\infty}^{0} |f(u)|^2 du = K < \infty . \tag{2}$$

In order to prove the absolute continuity of $F(\alpha)$ we start from (1) and the Fourier inversion formula and consider

$$(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\alpha u} f(u) du .$$

This integral, however, does not necessarily exist and instead of it we form the convergent integrals

$$f_1(z) = (2\pi)^{-1} \int_{-\infty}^{0} e^{-izu} f(u) du , \qquad \operatorname{Im} z > 0 , \tag{3}$$

$$f_2(z) = (2\pi)^{-1} \int_{0}^{\infty} e^{-izu} f(u) du , \qquad \operatorname{Im} z < 0 . \tag{4}$$

Here $z = x + iy$ is a complex variable. The function $f_1(z)$ is analytic in the upper half-plane $y > 0$, $f_2(z)$ is analytic in the lower half-plane $y < 0$. Observing that

$$-i(\alpha - z)^{-1} = \int_{-\infty}^{0} e^{-izu} e^{i\alpha u} du , \qquad \operatorname{Im} z > 0 ,$$

$$-i(\alpha - z)^{-1} = \int_{0}^{\infty} e^{-izu} e^{i\alpha u} du , \qquad \operatorname{Im} z < 0 ,$$

we easily obtain from (1), (3), and (4) the following expression:

$$(2\pi i)^{-1} \int_{-\infty}^{\infty} (\alpha - z)^{-1} dF(\alpha) = \begin{cases} f_1(z) & \text{if} \quad \operatorname{Im} z > 0 , \\ -f_2(z) & \text{if} \quad \operatorname{Im} z < 0 . \end{cases} \tag{5}$$

Let us first consider the function $f_1(z)$. From (3) we get

$$f_1(x + iy) = (2\pi)^{-1} \int_{-\infty}^{0} e^{-ixu} e^{yu} f(u) du , \qquad y > 0 .$$

Hence, by (2) and the Parseval relation it follows that

$$\int_{-\infty}^{+\infty} |f_1(x+iy)|^2 dx = (2\pi)^{-1} \int_{-\infty}^{0} e^{2yu} |f(u)|^2 du \leqq (2\pi)^{-1} K \tag{6}$$

for all $y > 0$.

Then, according to a well-known theorem [2, Theorem 2.1], there exists a function $f_1{}^*(x)$ with the properties:

1° $\displaystyle\int_{-\infty}^{\infty} |f_1{}^*(x)|^2 dx \leqq K(2\pi)^{-1}$;

2° $\displaystyle\lim_{y\to+0} f_1(x+iy) = f_1{}^*(x)$ almost everywhere on the real axis;

3° $\displaystyle\lim_{y\to+0} \int_{-\infty}^{\infty} |f_1(x+iy) - f_1{}^*(x)|^2 dx = 0$;

(7) 4° $\displaystyle\lim_{y\to+0} \int_I f_1(x+iy)\,dx = \int_I f_1{}^*(x)\,dx$ on every finite interval I.

As to the limit of $f_2(x+iy)$ for $y \to -0$ we shall prove the existence of a function $f_2{}^*(x)$ with properties corresponding to 2° and 4° above. This function $f_2{}^*(x)$, however, need not belong to L^2. Let us form

$$f_1(x+iy) + f_2(x-iy) = \pi^{-1} \int_{-\infty}^{\infty} y\,[(\alpha-x)^2+y^2]^{-1} dF(\alpha)\,, \tag{8}$$

where $y > 0$. It follows that

$$\int_{-\infty}^{\infty} |f_1(x+iy) + f_2(x-iy)|\,dx \leqq \int_{-\infty}^{\infty} |dF(\alpha)| = V < \infty\,.$$

For a finite interval $(-a, a)$ we thus obtain

$$\int_{-a}^{a} |f_2(x-iy)|\,dx \leqq \int_{-a}^{a} |f_1(x+iy)|\,dx + V\,,$$

or by (6) and Schwarz' inequality:

$$\int_{-a}^{a} |f_2(x-iy)|\,dx \leqq (ka)^{\frac{1}{2}} + V, \qquad y > 0\,, \tag{9}$$

k being a constant. We now consider the function $\varphi(z) = f_2(z)(z-i)^{-2}$ which is analytic for $\operatorname{Im} z < 0$. By partial integration, assuming $y < 0$, we get

$$\int_{-a}^{a} |\varphi(x+iy)|\,dx = \int_{-a}^{a} |f_2(x+iy)(x+iy-i)^{-2}|\,dx$$

$$\leqq \int_{-a}^{a} |f_2(x+iy)|\,(x^2+1)^{-1}\,dx$$

$$= (a^2+1)^{-1}\int_{-a}^{a} |f_2(\xi+iy)|\,d\xi + 2\int_{-a}^{a}\left\{\int_{0}^{x} |f_2(\xi+iy)|\,d\xi\right\} x\,(x^2+1)^{-2}\,dx\,,$$

or on account of (9):

$$\int_{-a}^{a} |\varphi(x+iy)|\,dx \leqq (a^2+1)^{-1}\big((ka)^{\frac{1}{2}} + V\big) + 2\int_{-a}^{a}\big((k|x|)^{\frac{1}{2}} + V\big)x\,(x^2+1)^{-2}\,dx\,.$$

Hence, letting $a \to \infty$,

$$\int_{-\infty}^{\infty} |\varphi(x+iy)|\,dx \leqq K_1 \qquad \text{for} \qquad y < 0\,, \tag{10}$$

K_1 being a constant independent of y. By (10) it follows that the theorem mentioned above [2, Theorem 2.1] may now be applied to $\varphi(z)$. Thus there exists a function $\varphi^*(x)$ such that

1° $\displaystyle\int_{-\infty}^{\infty} |\varphi^*(x)|\,dx \leqq K_1$;

2° $\displaystyle\lim_{y\to -0} \varphi_2(x+iy) = \varphi^*(x)$ almost everywhere on the real axis;

3° $\displaystyle\lim_{y\to -0} \int_{-\infty}^{\infty} |\varphi_2(x+iy) - \varphi^*(x)|\,dx = 0$;

4° $\displaystyle\lim_{y\to -0} \int_I \varphi_2(x+iy)\,dx = \int_I \varphi^*(x)\,dx$ on every finite interval I.

If $f_2^*(x) = (x-i)^2\,\varphi^*(x)$, it results immediately that $f_2^*(x)$ belongs to L on every *finite* interval I and that

1° $\displaystyle\lim_{y\to -0} f_2(x+iy) = f_2^*(x)$ almost everywhere;

(11) 2° $\displaystyle\lim_{y\to -0} \int_I f_2(x+iy)\,dx = \int_I f_2^*(x)\,dx$.

By partial integration we obtain from (8):

$$f_1(x+iy)+f_2(x-iy) = -\pi^{-1}\int_{-\infty}^{\infty} F(\alpha)\,\frac{d}{d\alpha}\{y[(\alpha-x)^2+y^2]^{-1}\}\,d\alpha$$

$$= \pi^{-1}\int_{-\infty}^{\infty} F(\alpha)\,\frac{d}{dx}\{y[(\alpha-x)^2+y^2]^{-1}\}\,d\alpha\,.$$

Now let ξ and ξ_0 be two arbitrary points where $F(\alpha)$ is continuous. By integration we get

$$\int_{\xi_0}^{\xi}\big(f_1(x+iy)+f_2(x-iy)\big)\,dx \tag{12}$$

$$= \pi^{-1}\int_{-\infty}^{\infty} F(\alpha)\,y[(\alpha-\xi)^2+y^2]^{-1}\,d\alpha \;-\; \pi^{-1}\int_{-\infty}^{\infty} F(\alpha)\,y[(\alpha-\xi_0)^2+y^2]^{-1}\,d\alpha\,.$$

Letting $y \to +0$ in (12) we obtain by (7), (11), and well-known properties of the kernel $\pi^{-1}y[(\alpha-\xi)^2+y^2]^{-1}$ that

$$\int_{\xi_0}^{\xi}\big(f_1^*(x)+f_2^*(x)\big)dx = F(\xi)-F(\xi_0)\,.$$

Hence $F(\xi)$ is absolutely continuous and the theorem is proved. The special choice of ξ and ξ_0 is, of course, unimportant.

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