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BERNSTEIN POLYNOMIALS AND SEMIGROUPS OF OPERATORS

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1. Let $\{T_t\colon t \geqq 0\}$ be a semigroup of linear bounded transformations of the real Banach space X into itself which is such that $T_0 = I$ (the identity) and $\|T_t x - x\| \to 0$ as $t \to 0$; also let $\|T_t\| \leqq M < \infty$ when $0 \leqq t \leqq 1$ (such an M necessarily exists). Then if $A_h \equiv (T_h - I)/h$ for $h > 0$, it is known (Hille [2, pp. 189–190]) that for each $x \in X$ and for each $t \geqq 0$,

$$\text{strong}\lim_{h\to 0} \exp(tA_h)x = T_t x\,, \tag{1}$$

the convergence being uniform in any finite t-interval. Dunford and Segal [1] have used this "exponential formula" to obtain a simple (though hardly elementary) proof of the classical theorem of Weierstrass concerning the uniform approximability of continuous functions by polynomials.

Here we give a result of the same general type as (1) which has as a simple consequence the explicit approximation theorem of Bernstein (for this see, for example, Lorentz [3]).

2. THEOREM. *For each fixed* $x \in X$,

$$\text{strong}\lim_{n\to\infty} \{(1-t)I + tT_{1/n}\}^n x = T_t x \tag{2}$$

whenever $0 \leqq t \leqq 1$, *the convergence being uniform in this interval.*

PROOF. Let $U_n \equiv (1-t)I + tT_{1/n}$ and let $V_n \equiv T_{t/n}$, where $n \geqq 1$ and $0 < t \leqq 1$. These operators commute, and $\|U_n{}^r V_n{}^s\| \leqq M$ if $0 \leqq r+s \leqq n$. Thus, if $x \in X$,

$$\|U_n{}^n x - V_n{}^n x\| \leqq nM\|U_n x - V_n x\| \leqq M\|A_{1/n}x - A_{t/n}x\|\,.$$

Now choose x_0 in the (dense) domain of the infinitesimal generator A of the semigroup so that $\|x - x_0\| < \tfrac{1}{4}\varepsilon/M$. Then

$$\|U_n{}^n x - V_n{}^n x\| < \tfrac{1}{2}\varepsilon + M\|A_{1/n}x_0 - Ax_0\| + M\|A_{t/n}x_0 - Ax_0\| < \varepsilon$$

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if $0 < t \leqq 1$ and $n \geqq N(\varepsilon)$. This proves the theorem, for (2) is trivial when $t = 0$.

3. Now let $f(\cdot)$ be any real-valued continuous function defined on $[0, 1]$, and put

$$\begin{aligned} f^\star(u) &\equiv f(u) \qquad (0 \leqq u \leqq 1)\,, \\ &\equiv f(1)/u \qquad (1 < u < \infty)\,, \end{aligned}$$

so that $f^\star \in X$ when X is the space of real-valued continuous functions $x(\cdot)$ defined on $[0, \infty)$ and such that $x(u) \to 0$ as $u \to \infty$, with the customary norm. If we apply the theorem to the semigroup of translations, so that

$$(T_t x)(u) \equiv x(u+t) \qquad (u, t \geqq 0)\,,$$

and then put $x = f^\star$ and $u = 0$ we get Bernstein's result:

$$\lim_{n\to\infty} \sum_{r=0}^{n} \binom{n}{r} (1-t)^{n-r} t^r f(r/n) = f(t) \tag{3}$$

whenever $0 \leqq t \leqq 1$, the convergence being uniform in this interval.

REFERENCES

1. N. Dunford and I. E. Segal, *Semigroups of operators and the Weierstrass theorem*, Bull. Amer. Math. Soc. 52 (1946), 911–914.
2 E. Hille, *Functional analysis and semigroups*, New York, 1948.
3. G. G. Lorentz, *Bernstein polynomials*, Toronto, 1953.

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