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NOTE ON A GENERALIZATION OF A THEOREM OF BOGOLIOÙBOFF

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The aim of the present note is to show that some of the theorems in a recent paper by the author [4] can be improved in a simple way. It will be assumed that the reader is familiar with this paper.

In Theorem 1 *of* [4, p. 6] *the assumption that E is relatively dense with respect to k elements can be replaced by the assumption that $\bar{m}E > 0$. In the conclusion of the theorem the statement $q \leqq k^2$ should then be replaced by $q \leqq (\bar{m}E)^{-2}$.*

When E is relatively dense with respect to k elements, it follows that $\bar{m}E \geqq \underline{m}E \geqq k^{-1} > 0$ and hence $k^2 \geqq (\underline{m}E)^{-2} \geqq (\bar{m}E)^{-2}$. Thus the new form of Theorem 1 is actually stronger than the original one.

As mentioned in [4, p. 6] Bogolioùboff [2] proved Theorem 1 (in a somewhat different form) for the case where G is the discrete additive group of all integers. From this case he passed to the case where G is the additive group of all real numbers with the usual topology. In the proof for the group of integers Bogolioùboff uses only that the upper Besicovitch mean measure $\bar{m}_B E$ of E is positive. For this group our $\bar{m}E$ is easily seen to be the upper Weyl mean measure $\bar{m}_W E$ of E, and the condition $\bar{m}_W E > 0$ is weaker than the condition $\bar{m}_B E > 0$ since $\bar{m}_W E \geqq \bar{m}_B E$. But a simple change in Bogolioùboff's proof will make it work with the assumption $\bar{m}_W E > 0$.

Stronger forms of Theorem 2 *and Theorem* 3 *of* [4, p. 7] *and Corollary* 1 (*as regards the sufficient condition*) *and Corollary* 2 (*as regards the necessary condition*) *of* [4, p. 6] *can be obtained by replacing the relatively dense sets E which occur in these theorems by sets E with $\bar{m}E > 0$.*

We shall modify the proof of Theorem 1 so as to obtain a proof of the stronger form of Theorem 1. Instead of the theorem of Banach cited in [4, p. 8] we shall use a stronger form of it which is also due to Banach [1, pp. 27–28].

BANACH'S THEOREM. *Let $\overline{M}f$ be a real functional defined on a real linear space L and satisfying* (4) *and* (5) *in* [4, p. 8]. *Let further Mf be a*

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linear functional on a linear subspace of L and satisfying the inequality $Mf \leqq \overline{M}f$. *Then* Mf *can be extended to a linear functional on L which satisfies the same inequality on the whole of L.*

As a simple consequence of this theorem the functional Mf chosen in [4, p. 8] may be chosen so that $Mj = \overline{M}j$ for the special function $j(x)$ introduced in [4, p. 9]. In particular $Mj \geqq \bar{m}E > 0$.

The proof of the stronger form of Theorem 1 is now obtained from the original proof of Theorem 1 by replacing the relation (1) in [4, p. 10] and the argument leading to it by the relation

$$\mathfrak{M}\mu \geqq (Mj)^2 \tag{1'}$$

and the following proof of (1′).

Let $j_1(x) = j(x) - Mj$. Then

$$\mu(x) = \underset{t}{M}\{j(t)j(t+x)\} = \underset{t}{M}\{(j_1(t) + Mj)(j_1(t+x) + Mj)\}$$
$$= \underset{t}{M}\{j_1(t)j_1(t+x)\} + (Mj)^2 .$$

It was proved in [4, p. 9] that $\mu(x)$ is a positive definite function. It follows in the same way that

$$\mu_1(x) = \underset{t}{M}\{j_1(t)j_1(t+x)\}$$

is a positive definite function. We conclude, applying, for instance, (3), (4) in [4, p. 10] to μ_1 instead of μ, that $\mathfrak{M}\mu_1 \geqq 0$. Thus (1′) is obtained by taking mean values in the above relation for $\mu(x)$.

The stronger forms of the other theorems are now proved in the same way as before. Only for the stronger form of Theorem 2 a minor modification is needed in order to prove that $f(x)$ is bounded.

We shall use the fact that when $\bar{m}E > 0$, the set $E-E$ is relatively dense. This has been proved directly in [3, p. 61], but it is also a simple corollary of the present stronger form of Theorem 3.

Since E consists of ε_0-translation elements of $f(x)$, the set $E-E$ consists of $2\varepsilon_0$-translation elements of $f(x)$. The boundedness of $f(x)$ now follows as in [4, p. 13] if E is replaced by $E-E$ and ε_0 by $2\varepsilon_0$.

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