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ON THE FOURIER SERIES OF STEPANOV ALMOST PERIODIC FUNCTIONS

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The class of S^2-almost periodic functions was introduced independently by N. Wiener [6] and V. Stepanov [3]. Among the results from Wiener's paper we are particularly interested in Theorem 22, p. 583, which can be formulated in the following way.

WIENER'S THEOREM. *Every series* $\sum_1^\infty a_n e^{i\lambda_n t}$, *where*

$$\sum_{m=-\infty}^{\infty} \Big\{ \sum_{m \leq \lambda_n < m+1} |a_n| \Big\}^2$$

converges, is the Fourier series of an S^2*-almost periodic function.*

The special case of this theorem, where the number of Fourier exponents in an interval of length one remains below a fixed level, was proved by V. Stepanov [4] [5] by a different method. In 1950 the author of the present paper told a group of mathematicians about Stepanov's proof and pointed out that a converse theorem could be proved by the same method. During the ensuing discussion E. Følner, B. Jessen and A. Zygmund suggested that it might be possible to prove Wiener's theorem and an inverse theorem by Stepanov's method. The author's attempt to do this was successfull, and the proof was discussed at another meeting. In the present paper, we shall give a detailed account of the proof, which has not been published before.

Let $f(t)$ denote an S^2-almost periodic function with the Fourier series

$$(1) \qquad f(t) \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n t} .$$

The norm of the function $f(t)$ is the positive quantity

$$\left(D_{S_L^2}[f(t)]\right)^2 = \sup_{-\infty < u < \infty} L^{-1} \int_0^L |f(t+u)|^2 dt .$$

The index L is omitted if $L = 1$. We have

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$$(2) \qquad \varphi_1(L)\left(D_{S^2}[f(t)]\right)^2 \leqq \left(D_{S_L^2}[f(t)]\right)^2 \leqq \varphi_2(L)\left(D_{S^2}[f(t)]\right)^2,$$

where $\varphi_1(L)$ and $\varphi_2(L)$ are positive functions independent of f.

The series (1) is summable in the following sense: there exists a double sequence of numbers $k_{N,n}$, $N=1, 2, \ldots$, $n=1, 2, \ldots$, satisfying the conditions

$$0 \leqq k_{N,n} \leqq 1; \qquad k_{N,n}=0, \quad \text{when} \quad n>N; \qquad \lim_{N=\infty} k_{N,n}=1 ,$$

such that the sequence of finite exponential sums

$$(3) \qquad s_N(t) = \sum_{n=1}^{\infty} k_{N,n} a_n e^{i\lambda_n t}$$

converges to $f(t)$ in the S^2-sense, that is

$$\lim_{N=\infty} D_{S^2}[f(t)-s_N(t)]=0 .$$

The W^2-norm is defined as

$$D_{W^2}[f(t)] = \lim_{L=\infty} D_{S_L^2}[f(t)] .$$

For a W^2-almost periodic function the numbers $k_{N,n}$ introduced above can be chosen such that the following condition is satisfied: To $\varepsilon>0$ correspond L and N_0 such that

$$(4) \qquad D_{S_L^2}[f(t)-s_N(t)] \leqq \varepsilon, \qquad \text{when} \quad N \geqq N_0 .$$

Proofs of these results are given in [1].

In the following proofs $K_1, \ldots, K_7$ denote positive absolute constants and we shall use the notation

$$p(t)=e^{-\frac{1}{2}t^2}; \qquad e_{n+1}=p(n),\ n=0, 1, 2, \ldots; \qquad e_0=1$$

and the well-known relation

$$(5) \qquad \int_{-\infty}^{\infty} p(t)e^{i\lambda t}dt = (2\pi)^{\frac{1}{2}} p(\lambda) .$$

Further, we need a new norm defined by

$$(6) \qquad \left(D_E[f(t)]\right)^2 = \sup_{-\infty<u<\infty} \int_{-\infty}^{\infty} p(t)\,|f(t+u)|^2 dt ,$$

which obviously satisfies the condition

$$(7) \qquad K_1\left(D_{S^2}[f(t)]\right)^2 \leqq \left(D_E[f(t)]\right)^2 \leqq K_2\left(D_{S^2}[f(t)]\right)^2.$$

Let

$$P(t) = \sum_{n=1}^{\infty} c_n e^{i\lambda_n t}$$

be an absolutely convergent exponential sum. According to (5)

$$\int_{-\infty}^{\infty} p(t)\,|P(t+u)|^2 dt = \sum_{m,n=1}^{\infty} c_m \overline{c_n} e^{i(\lambda_m-\lambda_n)u} \int_{-\infty}^{\infty} p(t) e^{i(\lambda_m-\lambda_n)t} dt$$

$$= (2\pi)^{\frac{1}{2}} \sum_{m,n=1}^{\infty} p(\lambda_m-\lambda_n) c_m \overline{c_n} e^{i(\lambda_m-\lambda_n)u}\,.$$

Hence, by (6) and (7),

(8) $$\left(D_{S^2}[P(t)]\right)^2 \leqq K_3 \sum_{m,n=1}^{\infty} p(\lambda_m-\lambda_n)\,|c_m|\,|c_n|$$

and, on the other hand, if every $c_n \geqq 0$,

(9) $$\left(D_{S^2}[P(t)]\right)^2 \geqq K_4 \sum_{m,n=1}^{\infty} p(\lambda_m-\lambda_n) c_m c_n\,.$$

We introduce the notations

$$C_m = \sum_{m \leqq \lambda_\mu < m+1} |c_\mu|$$

and

$$S_{mn} = \sum_{\substack{m \leqq \lambda_\mu < m+1 \\ n \leqq \lambda_\nu < n+1}} p(\lambda_\mu-\lambda_\nu)\,|c_\mu|\,|c_\nu|\,.$$

Obviously

(10) $$S_{mn} \leqq e_{|m-n|} \sum_{\substack{m \leqq \lambda_\mu < m+1 \\ n \leqq \lambda_\nu < n+1}} |c_\mu|\,|c_\nu| = e_{|m-n|} C_m C_n$$

and similarly

(11) $$S_{mn} \geqq p(|m-n|+1) C_m C_n\,.$$

By (8) and (10) we have

$$\left(D_{S^2}[P(t)]\right)^2 \leqq K_3 \sum_{m,n=-\infty}^{\infty} e_{|m-n|} C_m C_n = K_3 \sum_{q=-\infty}^{\infty} e_{|q|} \sum_{m=-\infty}^{\infty} C_m C_{m+q}$$

and Cauchy's inequality yields

$$\left(D_{S^2}[P(t)]\right)^2 \leqq K_3 \sum_{q=-\infty}^{\infty} e_{|q|} \left\{ \sum_{m=-\infty}^{\infty} C_m^2 \sum_{m=-\infty}^{\infty} C_{m+q}^2 \right\}^{\frac{1}{2}} = K_3 \sum_{q=-\infty}^{\infty} e_{|q|} \sum_{m=-\infty}^{\infty} C_m^2.$$

Since the sum $\Sigma_{-\infty}^{\infty} e_{|q|}$ is an absolute constant, it follows that

(12) $$\left(D_{S^2}[P(t)]\right)^2 \leqq K_5 \sum_{m=-\infty}^{\infty} C_m^2\,.$$

On the other hand, if $c_n \geqq 0$ for every n, we have according to (9) and (11)

$$\left(D_{S^2}[P(t)]\right)^2 \geqq K_4 \sum_{m,n=-\infty}^{\infty} p(|m-n|+1) C_m C_n ,$$

hence, omitting the terms with $n \neq m$,

$$(13) \qquad \left(D_{S^2}[P(t)]\right)^2 \geqq K_4 \sum_{m=-\infty}^{\infty} p(1) C_m{}^2 = K_6 \sum_{m=-\infty}^{\infty} C_m{}^2 .$$

Let us now assume that the series $\sum_1^\infty a_n e^{i\lambda_n t}$ satisfies the condition of Wiener's theorem. We introduce the notation

$$(14) \qquad A_m = \sum_{m \leqq \lambda_n < m+1} |a_n| .$$

This sum is finite, and $\sum_{-\infty}^{\infty} A_m{}^2$ is convergent according to the condition. For the exponential sums

$$P_m(t) = \sum_{-m \leqq \lambda_n < m} a_n e^{i\lambda_n t},$$

which are convergent according to the condition, we have for $q > 0$ according to (12)

$$\left(D_{S^2}[P_{m+q}(t) - P_m(t)]\right)^2 \leqq K_5 \sum_{|\mu| \geqq m} A_\mu{}^2 .$$

It follows that the sequence $P_m(t)$ of ordinary almost periodic functions is fundamental with respect to the S^2-metric. According to a well-known theorem ([2, p. 51–53]) this implies that the sequence $P_m(t)$ converges in the S^2-sense to an S^2-almost periodic function $f(t)$. The Fourier series of $f(t)$ is the formal limit of the Fourier series of $f_m(t)$, i.e. the given series. This completes the proof of Wiener's theorem. We observe that the Fourier series is not only summable but even convergent in the S^2-sense.

We shall now prove the following converse of Wiener's theorem.

If the Fourier series (1) *satisfies the condition* $a_n > 0$ *for every* n, *then the condition of Wiener's theorem is satisfied.*

Applying the inequality (13) to the finite exponential sums (3) we have

$$K_6 \sum_{m=-\infty}^{\infty} \left\{ \sum_{m \leqq \lambda_n < m+1} k_{N,n} a_n \right\}^2 \leqq \left(D_{S^2}[s_N(t)]\right)^2,$$

hence, for every positive integer P,

$$(15) \qquad \sum_{m=-\infty}^{\infty} \left\{ \sum_{\substack{m \leqq \lambda_n < m+1 \\ n \leqq P}} k_{N,n} a_n \right\}^2 \leqq K_7 \left(D_{S^2}[s_N(t)]\right)^2 .$$

For $N \to \infty$ we get

$$\sum_{m=-\infty}^{\infty} \left\{ \sum_{\substack{m \leqq \lambda_n < m+1 \\ n \leqq P}} a_n \right\}^2 \leqq K_7 (D_{S^2}[f(t)])^2 .$$

For $P \to \infty$ we get for every positive integer M

$$\sum_{m=-M}^{M} \left\{ \sum_{m \leqq \lambda_n < m+1} a_n \right\}^2 \leqq K_7 (D_{S^2}[f(t)])^2 ,$$

which proves the theorem.

As E. Følner has pointed out, the last theorem holds also, if the function $f(t)$ is not S^2-almost periodic but W^2-almost periodic. In fact, it follows from the definition of $D_{W^2}[f(t)]$ that $D_{S^2_L}[f(t)]$ is finite for some values of L, hence, according to (2) for all values of L. If we choose L and N_0 such that (4) holds for $\varepsilon = 1$ and $N \geqq N_0$, we have by Minkowski's inequality

$$D_{S^2_L}[s_N(t)] \leqq D_{S^2_L}[f(t)] + 1 ,$$

and, according to (2)

$$(D_{S^2}[s_N(t)])^2 \leqq (\varphi_1(L))^{-1} (D_{S^2_L}[f(t)] + 1)^2 .$$

This, in connection with (15), yields for $N \to \infty$

$$\sum_{m=-\infty}^{\infty} \left\{ \sum_{\substack{m \leqq \lambda_n < m+1 \\ n \leqq P}} a_n \right\}^2 \leqq K_7 (\varphi_1(L))^{-1} (D_{S^2_L}[f(t)] + 1)^2 ,$$

and the proof is completed in the same way as in the preceding case.

We may consider the general problem of finding sufficient conditions that a given series $\sum_1^\infty a_n e^{i\lambda_n t}$ be the Fourier series of an S^2- or W^2-almost periodic function. We have proved that Wiener's condition is the weakest possible sufficient condition involving only λ_n and $|a_n|$, $n = 1, 2, \dots$. Any weaker sufficient condition must involve the arguments of the coefficients.

It also follows that the Fourier series of an S^2- or W^2-almost periodic function is S^2- or W^2-convergent if every Fourier coefficient is positive. To a W^2-almost periodic function with positive Fourier coefficients corresponds an S^2-almost periodic function with the same Fourier series.

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