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AN APPLICATION OF LOGIC TO ALGEBRA

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1. Dilworth [6] has announced a result to the effect that every *finite* lattice is isomorphic with a sublattice of a semi-modular lattice. His result can be generalized to arbitrary lattices. In this paper we shall prove that *every lattice is isomorphic with a sublattice of a semi-modular lattice*, under a wide variety of definitions of the latter. We shall rely on the following facts:

(A) Dilworth's result quoted above;

(B) every finite subset S of a lattice L can be imbedded in a finite lattice L' with the preservation of all g.l.b. and l.u.b. existing in S;

(C) the notions of a lattice and of a semi-modular lattice can be defined by means of the so-called first-order functional calculus (restricted predicate calculus);

(D) in this calculus, the simultaneous satisfiability of a set of formulae is a property of finite character.

We shall use small Latin letters to denote lattice elements, small Greek letters to denote sets of formulae, and capital Greek letters to denote properties of sets of elements. The signs $\subseteq$, $+$, and $\{\,\}$ have their usual set-theoretic meaning. Our logical formalism is that of A. Robinson [8], except for denoting identity by $=$ and shortening $[[X \supset Y] \wedge [Y \supset X]]$ to read $[X \equiv Y]$.

Since in (B) we do not require that L' is a sublattice of L, the result is rather obvious. S is partly ordered by defining $x \geq y$ (in S) to mean $x \frown y = y$ (in L), and L' may be taken to be the completion of S by cuts. (See [1, p. 58]).

(D) has been proved by Henkin [7] in a form applying also to the first-order functional calculus with identity (of individuals). A similar result has been proved by A. Robinson [8, pp. 24–36].

2. The concept of a lattice is easily defined by means of the first-order functional calculus by introducing two relative symbols $U(\ldots, \ldots, \ldots)$

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and $\Omega(\ldots, \ldots, \ldots)$ of order 3. $U(x, y, z)$ is taken to mean that $x \smile y = z$ and $\Omega(x, y, z)$ to mean that $x \frown y = z$. The axioms of the lattice theory are perfectly obvious:

(1) $(x)(y)(\exists z)U(x, y, z)$;
(2) $(x)(y)(z)(u)[U(x, y, z) \wedge U(x, y, u) \supset z = u]$;
(3) $(x)U(x, x, x)$;
(4) $(x)(y)(z)[U(x, y, z) \supset U(y, x, z)]$;
(5) $(x)(y)(z)(u)(v)(w)[U(x, y, u) \wedge U(y, z, w) \supset U(u, z, v) \equiv U(x, w, v)]$;
(6) $(x)(y)(z)[\Omega(x, y, z) \supset U(x, z, x)]$;
(7)–(12) like (1)–(6) except for the interchange of U and Ω.

(1)–(2) provide for the existence and uniqueness of l.u.b. The formulae (3)–(6) are reformulations of the second halves of Birkhoff's identities $L1-L4$ [1, p. 18]. (7)–(12) are the dual postulates. (Duality in the sense of lattice theory, not of logic.)

As to the notion of semi-modularity, we have a choice of several definitions which all agree in the case of a lattice of finite length. Most of them can be formulated by means of the first-order functional calculus. A case in point is the general condition of semi-modularity adopted by R. Croisot in [5]. The condition is as follows: if $x \frown z < y < x < x \smile z$, then there is t such that $x \frown z < t \leq z$ and $(t \smile y) \frown x = y$. The reformulation in terms of U and Ω is a perfectly straightforward matter, for everything else in this condition is already in terms of the lattice operations and logic except the relations $>$ and $\geq$ which are easily defined by means of U, Ω, and quantifiers.

Also the three strongest conditions (R), (I), and (F) of semi-modularity discussed by Croisot in [4, pp. 204–208] (excepting those which are in terms of ideals) are amenable to the first-order functional calculus. (Cf. also [2] and [3].) In order to show this, it suffices to show that the corresponding conditions (r), (i), and (f) can be formulated by means of this calculus. (R), (I), and (F), respectively, are obtained from them by two universal quantifications.

Regarding (r), we note that it may be paraphrased in the following way: if $z_1 = x_1 \frown y_1$ covers $z_2 = x_2 \frown y_2$ where z_1 and z_2 are between $x \frown y$ and $x \smile y$, x_1 and x_2 between x and $x \smile y$, as well as y_1 and y_2 between y and $x \smile y$, then x_1 at most covers x_2 and $y_1 = y_2$, or *vice versa* with x_i, y_i $(i = 1, 2)$ interchanged. For otherwise we could insert pairs between (x_2, y_2) and (x_1, y_1) in $[x, x \smile y] \times [y, x \smile y]$. Here the notions of covering and at-most-covering are easily defined in the vocabulary of the basic lattice relations U, Ω plus that of the first-order functional calculus; whereafter a required formulation of the whole condition is forthcoming.

Similarly, the condition (i) may be paraphrased as follows: suppose that $x_1 = z_1 \frown x$ covers $x_2 = z_2 \frown x$ and $y_1 = z_1 \frown y = y_2 = z_2 \frown y$, or *vice versa* with x_i, y_i ($i = 1, 2$) interchanged, where x_1 and x_2 are between $x \frown y$ and x, y_1 and y_2 between $x \frown y$ and y, as well as z_1 and z_2 between $x \frown y$ and $x \smile y$. Then z_1 at most covers z_2. The details of the formulation are straightforward.

(f) gives rise to the condition (F) which is identical with Birkhoff's condition (γ) [1, p. 101]. It may be dealt with in an analogous fashion.

We may note that if we can prove our main result under some strong definition of a semi-modular lattice, then it automatically holds under all the weaker definitions.

Let X be the formalization of a suitable definition of semi-modularity by means of the first-order functional calculus. We shall denote the (finite) set of formulae $\{(1)\text{–}(12), X\}$ by μ.

3. Granting all the preliminary results (A)–(D), we may argue as follows:

Let L be an arbitrary lattice, finite or infinite, and let the elements of L be denoted by $x, y, z, \ldots$. We construct a set $\lambda(L)$ of formulae in the following way: for every triple x, y, z of elements of L, $\lambda(L)$ contains $U(x, y, z)$ or $\sim U(x, y, z)$ according to whether $x \smile y = z$ or $x \smile y \neq z$. In the same way, $\Omega(x, y, z)$ or $\sim \Omega(x, y, z)$ belongs to (L) according to whether $x \frown y = z$ or $x \frown y \neq z$. The formulae obtained in this way are the only formulae of $\lambda(L)$.

The question whether L can be represented as a sublattice of a semi-modular lattice now amounts to the question whether the set $\mu + \lambda(L)$ of formulae is satisfiable.

Since satisfiability is a property of finite character, it suffices to consider an arbitrary finite subset of $\mu + \lambda(L)$. Furthermore, it suffices to consider finite subsets of the form $\mu + \nu$ where ν is a finite subset of $\lambda(L)$, for all the other finite subsets of $\mu + \lambda(L)$ may be obtained from them by omitting one or more axioms of lattice theory (1)–(12) or the formula X.

Let $S = \{a, b, c, \ldots\}$ be the subset of all those elements of L whose names occur in the formulae of $\mu + \nu$. According to (B), S can be imbedded in a finite lattice L' with the preservation of all g.l.b. and l.u.b. existing in S. This means that $\nu \subseteq \lambda(L')$, where $\lambda(L')$ is obtained from L' exactly in the same way in which $\lambda(L)$ was obtained from L. The set $\mu + \nu$ is satisfiable if $\mu + \lambda(L')$ is satisfiable. But the satisfiability of the latter is tantamount to the feasibility of an imbedding of L' in a semi-modular lattice with the preservation of all g.l.b. and l.u.b., and hence follows from Dilworth's result.

Consequently, $\mu+\nu$ is satisfiable, entailing the satisfiability of all finite subsets of $\mu+\lambda(L)$ and hence the satisfiability of $\mu+\lambda(L)$ itself. But the latter is tantamount to the possibility of imbedding L in a semi-modular lattice with the preservation of all g.l.b. and l.u.b.

4. By reviewing the above argument, we see that it depends only upon the lemmata (A)–(D). We may consider two properties Φ, Ω of sets, and ask whether every set α with the property Φ can be imbedded in a set with the properties Φ *and* Ω. (By an imbedding, we mean in this section an imbedding in which all the properties and relations of the members of α which were used to define Φ are preserved.) *Mutatis mutandis*, the above proof serves to establish this, provided that the following conditions are fulfilled:

(a) every *finite* set with the property Φ can be imbedded in a set having the properties Φ and Ω;

(b) every finite subset of a set with the property Φ can be imbedded in a finite set with the property Φ;

(c) the properties Φ and Ω are formalizable by means of the first-order functional calculus (in the same sense as the notions of a lattice and a semi-modular lattice above).

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