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LOGIC-FREE FORMALISATIONS OF RECURSIVE ARITHMETIC

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This note outlines a new version of a logic-free formalisation of recursive arithmetic known as the equation calculus. In this version the only axioms are explicit and recursive function definitions, and the only inference rules are the substitution schemata

Sb_1
$$\frac{F(x) = G(x)}{F(A) = G(A)}\,,$$

Sb_2
$$\frac{A = B}{F(A) = F(B)}\,,$$

T
$$\frac{A = B,\quad A = C}{B = C}\,,$$

where $F(x)$, $G(x)$ are recursive functions and A, B, C are recursive terms, and the primitive recursive uniqueness rule

U
$$\frac{F(Sx) = H\big(x, F(x)\big)}{F(x) = H^x F(0)}\,,$$

where the iterative function $H^x t$ is defined by the primitive recursion $H^0 t = t$, $H^{Sx} t = H(x, H^x t)$; in the schema U, F may contain additional parameters but H is a function of not more than two variables. In Sb_1, the function $G(x)$ may be replaced by a term G independent of x, provided that $G(A)$ is also replaced by G.

The novelty in this version lies in the derivation of the key equation $a + (b \dot{-} a) = b + (a \dot{-} b)$ by means of the primitive recursive uniqueness rule, instead of requiring a doubly recursive uniqueness rule (or a postulated induction schema) as in earlier versions [2] [3].

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In the following account we shall not assume a previous knowledge of any formalisation of recursive arithmetic but shall construct recursive arithmetic, *ab initio*, on the basis of Sb_1, Sb_2, T and U.

We start by proving a few auxiliary schemata. From the defining equations $x+0 = x$, $x+0 = x$ follows $x = x$ by T, and thence by Sb_1 we reach $A = A$. Since $B = A$ follows from $A = B$, $A = A$ by T, we have proved

K
$$\frac{A = B\,,}{B = A\,.}$$

A schema equivalent to U is

U_1
$$\frac{f(0) = g(0)\,,\quad f(Sx) = H(x, f(x))\,,\quad g(Sx) = H(x, g(x))}{f(x) = g(x)\,.}$$

The passage from U_1 to U is obvious; for the converse we derive

$$f(x) = H^x f(0), \qquad g(x) = H^x g(0)$$

from the stated hypotheses, by U, and $H^x f(0) = H^x g(0)$ from $f(0) = g(0)$ by Sb_2, whence $f(x) = g(x)$ follows by T and K.

As an illustration of the use of Sb_2 we derive $F(a,b) = F(A,B)$ from the pair of equations $a = A$, $b = B$. First we derive $F(a,b) = F(a,B)$ from $b = B$ by Sb_2, and similarly $F(a,B) = F(A,B)$ from $a = A$; hence by using K and T we derive $F(a,b) = F(A,B)$ from $a = A$, $b = B$.

Two further schemata of importance are

E_1
$$\frac{F(Sx) = F(x)}{F(x) = F(0)\,,}$$

E_2
$$\frac{F(0) = 0, \quad F(Sx) = 0}{F(x) = 0\,.}$$

To prove E_1, we define $H_1(x,t)$, $C(t)$ explicitly by the axioms

$$H_1(x,t) = t, \qquad C(t) = F(0)\,,$$

whence we readily derive $C(0) = F(0)$, $C(Sx) = H_1(x, C(x))$, $F(Sx) = H_1(x, F(x))$ which, by U_1, yields $F(x) = C(x)$, and from this we reach

$F(x) = F(0)$ by Sb_1, T. For E_2 we define $Z(t) = 0$, so that $Z(F(x)) = 0$ whence from $F(Sx) = 0$ follows $F(Sx) = Z(F(x))$. This equation together with $Z(Sx) = Z(Z(x))$ and $F(0) = Z(0)$ yields $F(x) = Z(x)$, by U_1, and E_2 follows.

We establish next some results for addition, subtraction and multiplication, taking the defining equations for these operations to be:

$$a+0 = a, \quad a+Sb = S(a+b);$$
$$0\dot{-}1 = 0, \quad Sa\dot{-}1 = a, \quad a\dot{-}0 = a, \quad a\dot{-}Sb = (a\dot{-}b)\dot{-}1;$$
$$a\cdot 0 = 0, \quad a\cdot Sb = a\cdot b+a\,.$$

(1) $$(a\dot{-}b)\dot{-}1 = (a\dot{-}1)\dot{-}b\,.$$

For $(a\dot{-}0)\dot{-}1 = (a\dot{-}1)\dot{-}0$, $(a\dot{-}Sb)\dot{-}1 = \{(a\dot{-}b)\dot{-}1\}\dot{-}1$, $(a\dot{-}1)\dot{-}Sb = \{(a\dot{-}1)\dot{-}b\}\dot{-}1$, and the result follows by U_1.

(2) $$Sa\dot{-}Sb = a\dot{-}b\,.$$

For $Sa\dot{-}Sb = (Sa\dot{-}1)\dot{-}b = a\dot{-}b$, using (1).

(3) $$a\dot{-}a = 0\,.$$

For $Sa\dot{-}Sa = a\dot{-}a$ and so $a\dot{-}a = 0\dot{-}0 = 0$.

(4) $$0\dot{-}a = 0\,.$$

Proof by E_1, using $0\dot{-}Sa = (0\dot{-}1)\dot{-}a = 0\dot{-}a$.

(5) $$(a+b)\dot{-}b = a\,.$$

For $(a+Sb)\dot{-}Sb = S(a+b)\dot{-}Sb = (a+b)\dot{-}b$ so that $(a+b)\dot{-}b = a$, by E_1.

(5.1) $$(a+n)\dot{-}(b+n) = a\dot{-}b\,.$$

For $(a+Sn)\dot{-}(b+Sn) = S(a+n)\dot{-}S(b+n) = (a+n)\dot{-}(b+n)$, and $(a+0)\dot{-}(b+0) = a\dot{-}b$.

(5.2) $$n\dot{-}(b+n) = 0\,.$$

By (5.1) and (4).

(6) $$0+a = a\,.$$

For $0+0 = 0$, $0+Sa = S(0+a)$, $Sa = Sa$, and the result follows by U_1.

(7) $$a+Sb = Sa+b\ .$$

We use $a+S0 = Sa = Sa+0$, $a+SSb = S(a+Sb)$, $Sa+Sb = S(Sa+b)$ and U_1.

(8) $$a+b = b+a\ .$$

One has $a+0 = 0+a$, $a+Sb = S(a+b)$, and, using (7), $Sb+a = b+Sa = S(b+a)$. Then (8) follows by U_1.

(9) $$(a+b) \dot{-} a = b\ .$$

By (8) and (5).

(10) $$(a+b)+c = a+(b+c)\ .$$

With c as variable, apply U_1.

(11) $$Sa \cdot b = a \cdot b + b\ .$$

For $Sa \cdot 0 = a \cdot 0 + 0$, $Sa \cdot Sb = Sa \cdot b + Sa$, $a \cdot Sb + Sb = (a \cdot b + a) + Sb = S\{(a \cdot b + a) + b\} = S\{(a \cdot b + b) + a\}$, by (8), (10), and so $a \cdot Sb + Sb = (a \cdot b + b) + Sa$, whence (11) follows by U_1.

(12) $$0 \cdot a = 0\ .$$

For $0 \cdot Sa = 0 \cdot a$ so that $0 \cdot a = 0 \cdot 0 = 0$.

(13) $$a(1 \dot{-} a) = 0\ .$$

For $0(1 \dot{-} 0) = 0$ and $Sa(1 \dot{-} Sa) = Sa(0 \dot{-} a) = Sa \cdot 0 = 0$.

(14) $$a \cdot b = b \cdot a\ .$$

For $a \cdot 0 = 0 \cdot a$ and $a \cdot Sb = a \cdot b + a$, $Sb \cdot a = b \cdot a + a$.

(15) $$a(b+c) = a \cdot b + a \cdot c\ .$$

This is a consequence by U_1 of the provable equations

$$a(b+0) = a \cdot b = a \cdot b + a \cdot 0\ ,$$
$$a(b+Sc) = a \cdot S(b+c) = a(b+c) + a\ ,$$
$$a \cdot b + a \cdot Sc = ab + (ac+a) = (ab+ac) + a\ .$$

(15.1) $$a(bc) = (ab)c\ .$$

For $a(b \cdot 0) = 0 = (a \cdot b) \cdot 0$ and $a(b \cdot Sc) = a(bc+b) = a(bc) + ab$, $(ab) \cdot Sc = (ab)c + ab$.

We prove now an extension of schema E_2.

$$\text{E}_3 \qquad \frac{f(0) = g(0)\,, \quad f(Sx) = g(Sx)}{f(x) = g(x)\,.}$$

Define $H_2(x,t) = 0\cdot t + g(Sx) = 0\cdot t + f(Sx)$, then

$$f(Sx) = 0\cdot f(x) + f(Sx) = H_2(x, f(x)),$$
$$g(Sx) = 0\cdot g(x) + g(Sx) = H_2(x, g(x)),$$

whence E_3 follows by U_1.

(16) $$(1 \dot- a)b = b \dot- ab\,.$$

For $(1 \dot- 0)b = b = b \dot- 0\cdot b$, $(1 \dot- Sa)b = (0 \dot- a)b = 0$, and $b \dot- Sa\cdot b = $ $= b \dot- (b + a\cdot b) = 0$.

Next we prove the key equation

(17) $$a + (b \dot- a) = b + (a \dot- b)\,.$$

Define $f(a,b) = a + (b \dot- a)$, so that $f(a,0) = a$, $f(0,b) = b$, $f(Sa,Sb) = Sf(a,b)$, and define $g(a,b) = b + (a \dot- b)$ so that $g(a,0) = a$, $g(0,b) = b$, $g(Sa,Sb) = Sg(a,b)$. We start by proving

(17.1) $$f(a,b) = f(a \dot- 1, b \dot- 1) + \{1 \dot- (1 \dot- (a+b))\}$$

By E_3, $a = (a \dot- 1) + \{1 \dot- (1 \dot- a)\}$, whence $f(a,0) = f(a \dot- 1, 0) +$ $\{1 \dot- (1 \dot- a)\}$ which establishes (17.1) with 0 for b. With Sb for b, (17.1) becomes

$$f(a, Sb) = Sf(a \dot- 1, b)$$

which is a consequence of the equations $f(0,Sb) = Sb = Sf(0,b)$, $f(Sa,Sb) = Sf(a,b)$, completing the proof of (17.1). Next we define

$$\varphi(0,a,b) = 0, \qquad \varphi(Sn,a,b) = \varphi(n,a,b) + \{1 \dot- [1 \dot- ((a \dot- n) + (b \dot- n))]\}$$

and prove

$$f(a \dot- n, b \dot- n) + \varphi(n,a,b) = f(a \dot- Sn, b \dot- Sn) + \varphi(Sn,a,b);$$

in fact, by (17.1)

$$\begin{aligned} f(a \dot- n, b \dot- n) &+ \varphi(n,a,b) \\ &= f(a \dot- Sn, b \dot- Sn) + \varphi(n,a,b) + \{1 \dot- [1 \dot- ((a \dot- n) + (b \dot- n))]\} \\ &= f(a \dot- Sn, b \dot- Sn) + \varphi(Sn,a,b) \end{aligned}$$

so that $f(a \dot{-} n, b \dot{-} n) + \varphi(n,a,b) = f(a,b) + \varphi(0,a,b) = f(a,b)$, whence $f(a,b) = f(a \dot{-} b, 0) + \varphi(b,a,b) = (a \dot{-} b) + \varphi(b,a,b)$.

Similarly $g(a,b) = (a \dot{-} b) + \varphi(b,a,b)$ whence equation (17) follows.

Defining $|a, b| = (a \dot{-} b) + (b \dot{-} a)$ we derive from (17) the schema

$$\frac{|A, B| = 0}{A = B;}$$

for from $|A, B| = 0$ follows $|A, B| \dot{-} (B \dot{-} A) = 0$ by (4), whence by (5), $A \dot{-} B = 0$, and similarly, $B \dot{-} A = 0$; from these we reach

$$A + (B \dot{-} A) = A, \ B + (A \dot{-} B) = B$$

and thence $A = B$, by (17). The derivation of $|A, B| = 0$ from $A = B$ is of course trivial.

We come now to some induction schemata. Let $P(x)$ denote the equation $f(x) = g(x)$ and $P(x) \to P(Sx)$ the equation

(i) $$\{1 \dot{-} |f(x), g(x)|\} \cdot |f(Sx), g(Sx)| = 0$$

(the use of implication in this connection being justified on the grounds that if $P(x)$ holds then $|f(x), g(x)| = 0$ and this together with equation (i) yields $|f(Sx, g(Sx)| = 0$ and therefore $P(Sx)$ holds).

The familiar induction schema is

I_1 $$\frac{P(0), \quad P(x) \to P(Sx)}{P(x)}$$

or writing $p(x) = |f(x), g(x)|$

$$\frac{p(0) = 0, \quad \big(1 \dot{-} p(x)\big)p(Sx) = 0}{p(x) = 0\,.} \quad .$$

Define $q(0) = 1$, $q(Sn) = q(n)\big(1 \dot{-} p(n)\big)$; then

$$\begin{aligned} q(SSn) &= q(Sn)\big(1 \dot{-} p(Sn)\big) = q(n)\big(1 \dot{-} p(n)\big)\big(1 \dot{-} p(Sn)\big) \\ &= q(n)\,\big\{\big(1 \dot{-} p(n)\big) \dot{-} \big(1 \dot{-} p(n)\big)p(Sn)\big\} = q(Sn) \end{aligned}$$

where the last equality sign holds according to hypothesis since

$$\big(1 - p(n)\big)p(Sn) = 0;$$

whence $q(Sn) = q(S0) = 1$, that is $q(n)\bigl(1 \dot{-} p(n)\bigr) = 1$, and multiplying by $p(n)$, $p(n) = 0$, by (13).

$$\mathrm{I}_2 \qquad \frac{f(a,0) = 0, \quad f(0,Sb) = 0, \quad \{f(a,b) = 0\} \to \{f(Sa,Sb) = 0\}}{f(a,b) = 0\,.}$$

We observe first that from $f(0,0) = 0, f(0,Sb) = 0$ follows $f(0,b) = 0$. The implication hypothesis stands for the equation

$$\{1 \dot{-} f(a,b)\}f(Sa,Sb) = 0\,.$$

Now $\{1 \dot{-} f(0,b \dot{-} 1)\}f(0,b) = 0$ and from

$$\{1 \dot{-} f(a,0)\}f(Sa,0) = 0, \qquad \{1 \dot{-} f(a,b)\}f(Sa,Sb) = 0$$

follows

$$\{1 \dot{-} f(a,b \dot{-} 1)\}\, f(Sa,b) = 0\,,$$

therefore

$$\{1 \dot{-} f(a \dot{-} 1,b \dot{-} 1)\}\, f(a,b) = 0$$

and so

$$\{1 \dot{-} f(a \dot{-} Sn,b \dot{-} Sn)\}\, f(a \dot{-} n,b \dot{-} n) = 0\,.$$

Next we show that

(j) $$f(a,b)\,\{1 \dot{-} f(a \dot{-} n,b \dot{-} n)\} = 0\,.$$

To this end we prove

$$[1 \dot{-} f(a,b)\,\{1 \dot{-} f(a \dot{-} n,b \dot{-} n)\}]\,f(a,b)\,\{1 \dot{-} f(a \dot{-} Sn,b \dot{-} Sn)\} = 0;$$

with p, q, r standing for $f(a,b)$, $f(a \dot{-} n,b \dot{-} n)$ and $f(a \dot{-} Sn,b \dot{-} Sn)$, respectively, the left hand side of this equation has the form

$$\begin{aligned}\{1 \dot{-} p(1 \dot{-} q)\}p(1 \dot{-} r) &= p\{(1 \dot{-} r) \dot{-} p(1 \dot{-} q)(1 \dot{-} r)\}\\ &= p\{(1 \dot{-} r) \dot{-} p(1 \dot{-} r)\}, \qquad \text{since } q(1 \dot{-} r) = 0\,,\\ &= p(1 \dot{-} r)(1 \dot{-} p) = 0\end{aligned}$$

which completes the proof of (j) by I_1 (the validity of (j) with 0 for n being evident).

By writing $|\varphi(a, b), \psi(a, b)| = f(a, b)$ it follows from I_2 that the schema

$$\mathrm{I}_3 \qquad \frac{\varphi(a,0) = \psi(a,0), \quad \varphi(0,Sb) = \psi(0,Sb), \quad \{\varphi(a,b) = \psi(a,b)\} \to \{\varphi(Sa,Sb) = \psi(Sa,Sb)\}}{\varphi(a,b) = \psi(a,b)}$$

is valid. As particular cases of the I_2, I_3 we note that from $f(a,0) = 0$, $f(0,Sb) = 0$ and $f(Sa,Sb) = 0$ follows $f(a,b) = 0$; from $f(a,0) = 0$, $f(0,Sb) = 0$, $f(a,b) = f(Sa,Sb)$ follows $f(a,b) = 0$; and from $\varphi(a,0) = \psi(a,0)$, $\varphi(0,Sb) = \psi(0,Sb)$, $\varphi(a,b) = \varphi(Sa,Sb)$ and $\psi(a,b) = \psi(Sa,Sb)$ follows $\varphi(a,b) = \psi(a,b)$, for if we denote $|\varphi(a,b), \psi(a,b)|$ by $f(a,b)$ then $f(a,0) = 0, f(0,Sb) = 0$; and from $\varphi(a,b) = \varphi(Sa,Sb)$, $\psi(a,b) = \psi(Sa,Sb)$ follows $f(a,b) = f(Sa,Sb)$; whence $f(a,b) = 0$ and so $\varphi(a,b) = \psi(a,b)$.
As instances of this last schema we mention

(18) $$c(a \dot{-} b) = ca \dot{-} cb, \qquad a \dot{-} (b+c) = (a \dot{-} b) \dot{-} c\,.$$

To complete the construction of recursive arithmetic there remains only to prove the substitution theorem

$$(x = y) \rightarrow \{F(x) = F(y)\}\,.$$

This is readily derived from the equation

$$(1 \dot{-} |x, y|)F(x) = (1 \dot{-} |x, y|)F(y)\,.$$

To prove this last equation we start from the equation

$$(1 \dot{-} z)F(y+z) = (1 \dot{-} z)F(y)\,,$$

which is proved by applying E_2 with z as variable, and derive

$$\{1 \dot{-} (x \dot{-} y)\}F\big(y+(x \dot{-} y)\big) = \{1 \dot{-} (x \dot{-} y)\}F(y)$$

and multiplying by $1 \dot{-} |x, y|$, we reach

$$(1 \dot{-} |x,y|)F\big(y+(x \dot{-} y)\big) = (1 \dot{-} |x,y|)F(y)$$

since

$$\begin{aligned}\{1 \dot{-} [(x \dot{-} y)+(y \dot{-} x)]\}\,\{1 \dot{-} (x \dot{-} y)\} \\ &= (1 \dot{-} |x,y|) \dot{-} (x \dot{-} y)\{[1 \dot{-} (x \dot{-} y)] \dot{-} (y \dot{-} x)\} \\ &= 1 \dot{-} |x,y|\,;\end{aligned}$$

similarly $(1 \dot{-} |x,y|)F\big(x+(y \dot{-} x)\big) = (1 \dot{-} |x,y|)F(x)$ and since

$$x+(y \dot{-} x) = y+(x \dot{-} y)\,,$$

the required result follows by T.

We call the foregoing formalisation of recursive arithmetic system $\mathfrak{R}$. System $\mathfrak{R}$ admits the deduction theorem in the form:

THE DEDUCTION THEOREM. *If the equation $A = B$ is derivable in $\mathfrak{R}$ from an hypothesis $F = G$, (i.e. an unproved equation) and if the derivation does not involve substitution for the variables in the hypothesis, then*

$$(F = G) \rightarrow (A = B)$$

is provable in $\mathfrak{R}$.

We multiply each equation of the derivation by $1 \dot{-} |F, G|$. The hypothesis becomes the *proved* equation

$$\{1 \dot{-} |F, G|\}F = \{1 \dot{-} |F, G|\}G$$

and the final equation becomes

$$\{1 \dot{-} |F, G|\}A = \{1 \dot{-} |F, G|\}B$$

from which we may derive

$$\{1 \dot{-} |F, G|\}|A, B| = 0 ,$$

i.e.

$$(F = G) \rightarrow (A = B) .$$

If $P = Q$ is a proved equation then it follows that, for any function R,

$$RP = RQ$$

is a proved equation, and so multiplication by $1 \dot{-} |F, G|$ turns a proved equation into a proved equation.

We show next that multiplication by a factor does not invalidate an application of any of the schemata Sb_1, Sb_2, T and U. For Sb_1 we have to prove that

$$\frac{R \cdot F(x) = R \cdot G(x)}{R \cdot F(A) = R \cdot G(A)}$$

is valid when the factor R does not contain the variable x, and this of course is a consequence of Sb_1 itself. For Sb_2 we have to prove the validity of the derivation

$$\frac{R \cdot A = R \cdot B}{R \cdot F(A) = R \cdot F(B) .}$$

To this end we remark that, since $R|A, B| = |RA, RB|$, by equations (15) and (18), therefore

$$(RA = RB) \rightarrow (A = B) \vee (R = 0) ,$$
$$\big(F(A) = F(B)\big) \rightarrow \big(R \cdot F(A) = R \cdot F(B)\big) ,$$
$$(R = 0) \rightarrow \big(R \cdot F(A) = R \cdot F(B)\big) ,$$

and by the substitution theorem

$$(A = B) \to \{F(A) = F(B)\}$$

whence, using the schemata

$$\frac{P_1 \to Q\,,\quad P_2 \to Q}{P_1 \vee P_2 \to Q\,,} \qquad \frac{H \to K\,,\quad K \to L}{H \to L}$$

which follow from the provable equations

$$(1 \dot{-} p_1) + (1 \dot{-} p_2) = (1 \dot{-} p_1 p_2) + \{1 \dot{-} (p_1 + p_2)\}\,,$$
$$k + (1 \dot{-} k) = 1 + (k \dot{-} 1)\,,$$

we prove

$$(R \cdot A = R \cdot B) \to \{R \cdot F(A) = R \cdot F(B)\}\,,$$

and so, (taking $0 = 0$ for H in the second of the above schemata) we see that $R \cdot F(A) = R \cdot F(B)$ follows from $R \cdot A = R \cdot B$. For T we have only to prove the schema

$$\frac{RA = RB\,,\quad RA = RC}{RB = RC}$$

and this follows by T itself. It remains to prove that an application of U_1 remains valid under multiplication by R, i.e. that the derivation

$$\frac{R \cdot F(0) = R \cdot G(0)\,,\quad R \cdot F(Sx) = R \cdot H(x, F(x))\,,\quad R \cdot G(Sx) = R \cdot H(x, G(x))}{R \cdot F(x) = R \cdot G(x)}\,.$$

is valid, when R does not contain the variable x. We start by proving the schema

$$\frac{P = Q\,,\quad R = S}{(P = R) \to (Q = S)\,.}$$

By Sb_2,

$$\frac{P = Q}{|P, R| = |Q, R|\,,} \qquad \frac{R = S}{|Q, R| = |Q, S|}$$

whence, by T,

$$\frac{P = Q\,, \quad R = S}{|P, R| = |Q, S|}$$

and the desired derivation follows by the schema

$$\frac{a = b}{(1 \dot{-} a)\, b = 0}$$

which is also proved by Sb_2.

From the formula $(ka = kb) \to \{kJ(a) = kJ(b)\}$, which is proved above, follows

$$\{R \cdot F(x) = R \cdot G(x)\} \to \{R \cdot H(x, F(x)) = R \cdot H(x, G(x))\}$$

whence, by the given hypotheses and the above schemata,

$$\{R \cdot F(x) = R \cdot G(x)\} \to \{R \cdot F(Sx) = R \cdot G(Sx)\}$$

and this, with the first hypothesis, proves $R \cdot F(x) = R \cdot G(x)$, by induction schema I_1, and the deduction theorem is proved.

The deduction theorem holds for any number of hypotheses. For instance given a derivation of $A = B$ from two hypotheses $F_1 = G_1$, $F_2 = G_2$ we obtain a proof of the implication

$$(F_1 = G_1) \to \{(F_2 = G_2) \to (A = B)\}$$

by multiplying each equation in the derivation by the factor

$$(1 \dot{-} |F_1, G_1|)(1 \dot{-} |F_2, G_2|)\,.$$

Similarly we discharge the hypotheses $F_1 = G_1$, $F_2 = G_2$, $F_3 = G_3$ by multiplying each equation in the derivation from these hypotheses by

$$(1 \dot{-} |F_1, G_1|)(1 \dot{-} |F_2, G_2|)(1 \dot{-} |F_3, G_3|)\,,$$

and so on.

Reduction of schema U. In [4] Th. Skolem showed that the induction schema in recursive arithmetic could be replaced by the simple schema

$$\frac{f(0) = 0, \quad f(n) \geqslant f(Sn)}{f(n) = 0}$$

and (in reply to a question raised by Skolem) Bernays showed in [1] that the even simpler schema

E $$\frac{f(0) = 0, \quad f(n) = f(Sn)}{f(n) = 0}$$

suffices, subject to the introduction of the axioms

$$m = n \to \alpha(m,n) = 0\,,$$
$$m \neq n \to \alpha(m,n) = S0$$

for the function $\alpha(m,n)$.

Schema E by itself is insufficient for a logic-free formalisation of recursive arithmetic since it furnishes no means of passing from the equation $|a, b| = 0$ to the equation $a = b$. We shall however consider some alternative logic-free formalisations of recursive arithmetic with schema E in place of schema U, suitably strengthened in other ways.

We consider first a system $\mathfrak{R}_1$ with Sb_1, Sb_2, T and E, the axiom

A $$a + (b \dot{-} a) = b + (a \dot{-} b)$$

and, in place of the familiar introductory equations of the predecessor function, the axiom

P $$Sa \dot{-} Sb = a \dot{-} b\,.$$

The axiom $a + (b \dot{-} a) = b + (a \dot{-} b)$ enables us to deduce $a = b$ from $a \dot{-} b = 0$ and $b \dot{-} a = 0$. For by Sb_2,

$$\frac{b \dot{-} a = 0}{a + (b \dot{-} a) = a + 0 = a\,,} \qquad \frac{a \dot{-} b = 0}{b + (a \dot{-} b) = b + 0 = b}$$

and from $a + (b \dot{-} a) = a$, $b + (a \dot{-} b) = b$ and $a + (b \dot{-} a) = b + (a \dot{-} b)$ follows $a = b$. Derivation of $a = b$ from $a \dot{-} b = 0$, $b \dot{-} a = 0$ we call schema A.

To prove schema E_1, namely

$$\frac{F(Sx) = F(x)}{F(x) = F(0)}$$

we define $\Phi(x) = F(x) \dot{-} F(0)$, then $\Phi(0) = 0$ and

$$\Phi(Sx) = F(Sx) \dot{-} F(0) = F(x) \dot{-} F(0) = \Phi(x)$$

whence $\Phi(x) = 0$ by E, that is $F(x) \dot{-} F(0) = 0$. Similarly $F(0) \dot{-} F(x) = 0$ and so (by schema A)

$$F(x) = F(0)$$

which completes the proof of schema E_1.

We turn now to a reconsideration of the equations and schema we proved in $\mathfrak{R}$.

Equation (1) we leave to the end. Equation (2) is now an axiom. The proofs of (3) and (5) remain unchanged.

Proof in $\mathfrak{R}_1$ of (7).

$$(a+S0) \dot{-} (Sa+0) = Sa \dot{-} Sa = 0 ,$$
$$(a+SSb) \dot{-} (Sa+Sb) = S(a+Sb) \dot{-} S(Sa+b) = (a+Sb) \dot{-} (Sa+b) ,$$

proving $(a+Sb) \dot{-} (Sa+b) = 0$. Similarly $(Sa+b) \dot{-} (a+Sb) = 0$ whence (7) follows by schema A.

Proof of (6).

$$(0+Sa) \dot{-} Sa = S(0+a) \dot{-} Sa = (0+a) \dot{-} a$$

so that $(0+a) \dot{-} a = (0+0) \dot{-} 0 = 0$. Similarly $Sa \dot{-} (0+Sa) = a \dot{-} (0+a)$ whence (6) follows by schema A.

Proof of (8).

$$(a+0) \dot{-} (0+a) = a \dot{-} a = 0 ,$$
$$(a+Sb) \dot{-} (Sb+a) = S(a+b) \dot{-} S(b+a) = (a+b) \dot{-} (b+a) ,$$

so that $(a+b) \dot{-} (b+a) = 0$, whence $(b+a) \dot{-} (a+b) = 0$ and $a+b = b+a$ follows by A.

The proof of equation (9) remains unchanged.

Proof of (4).

$a + (0 \dot{-} a) = 0 + (a \dot{-} 0) = a$, therefore $\{a + (0 \dot{-} a)\} \dot{-} a = a \dot{-} a = 0$, whence, by (9), $0 \dot{-} a = 0$.

The schema

$$\frac{f(0) = g(0) , \quad f(Sa) = f(a) , \quad g(Sa) = g(a)}{f(a) = g(a)}$$

follows by two applications of schema E and two of schema T.

The proofs of (5.1), (5.2) remain unchanged, and from (5.1) it follows that

$$|a+n, b+n| = |a, n| .$$

Proof of (10).
$$|a+(b+0),\ (a+b)+0| = 0\ ,$$
$$|a+(b+Sn),\ (a+b)+Sn| = |a+(b+n),\ (a+b)+n|\ , \qquad \text{etc.}$$

Proof of (11).

Since $Sa \cdot Sb = Sa \cdot b + Sa$ and
$$a \cdot Sb + Sb = (ab+a) + Sb = S\{(a \cdot b + a) + b\}$$
$$= S\{(ab+b)+a\} = (a \cdot b + b) + Sa$$
therefore $|Sa \cdot Sb,\ a \cdot Sb + Sb| = |Sa \cdot b,\ a \cdot b + b|$, etc.

The proof of (12) remains unchanged.

For (14), (15) and (15.1) we note that
$$|a \cdot Sb,\ (Sb)a| = |ab+a,\ ba+a| = |ab,\ ba|\ ,$$
$$|a(b+Sc),\ ab+a \cdot Sc| = |a(b+c)+a,\ (ab+ac)+a| = |a(b+c),\ ab+ac|\ ,$$
$$|a(b \cdot Sc),\ ab \cdot Sc| = |a(bc)+ab,\ (ab)c+ab| = |a(bc),\ (ab)c|\ , \qquad \text{etc.}$$

The schema
$$\frac{F(x)+G(x)=0}{F(x)=0}$$
is proved by means of the equations (9), (4) which yield
$$\{F(x)+G(x)\} \dot{-} G(x) = F(x)$$
and
$$0 \dot{-} G(x) = 0\ .$$

To prove schema E_2
$$\frac{F(0)=0, \quad F(Sx)=0}{F(x)=0}\ .$$
we define $\Phi(0) = 0$, $\Phi(Sx) = \Phi(x) + F(x)$ so that $\Phi(SSx) = \Phi(Sx)$ whence $\Phi(Sx) = \Phi(S0) = 0$ and so $\Phi(x) + F(x) = 0$ whence $F(x) = 0$.

The proof of (13) follows by E_2 in $\mathfrak{R}_1$ as in $\mathfrak{R}$, and the proof of (16) remains unchanged.

So too the proofs of the induction schema (i), I_1, I_2, I_3 and the substitution schema carry straight through from $\mathfrak{R}$ to $\mathfrak{R}_1$.

It remains only to prove schema U_1,

$$\frac{F(0) = G(0)\,, \quad F(Sx) = H(x, F(x))\,, \quad G(Sx) = H(x, G(x))}{F(x) = G(x)\,.}$$

Define $\Phi(x) = |F(x), G(x)|$, then $\Phi(0) = 0$ and by the substitution schema,

$$\{1 \dot{-} \Phi(x)\}\,|H(t, f(x)), H(t, g(x))| = 0$$

whence

$$\{1 \dot{-} \Phi(x)\}\,|H(x, f(x)), H(x, g(x))| = 0$$

and so

$$\{1 \dot{-} \Phi(x)\}\,\Phi(Sx) = 0$$

whence $\Phi(x) = 0$ and therefore $f(x) = g(x)$.

The proof of equation (1) in $\mathfrak{R}$ is therefore valid also in $\mathfrak{R}_1$. (We may obviously take the equation $(a \dot{-} b) \dot{-} 1 = (a \dot{-} 1) \dot{-} b$ as an axiom in $\mathfrak{R}_1$ in place of $Sa \dot{-} Sb = a \dot{-} b$.)

The system $\mathfrak{R}_1$ may be further modified by taking the schema

$$\frac{a \dot{-} b = 0}{a + (b \dot{-} a) = b}$$

in place of axiom A, provided that we add the axiom

$$0 \dot{-} 1 = 0\,.$$

For by S and Sb_2

$$\frac{a \dot{-} b = 0, \quad b \dot{-} a = 0}{a = b\,.}$$

To prove $0 \dot{-} a = 0$ we use $0 \dot{-} Sa = (0 \dot{-} a) \dot{-} 1 = (0 \dot{-} 1) \dot{-} a = 0 \dot{-} a$, and $0 \dot{-} 0 = 0$ (in place of axiom A).

Finally the equation $a + (b \dot{-} a) = b + (a \dot{-} b)$ is proved exactly as in $\mathfrak{R}$.

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