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SETS OF DIVERGENCE OF TAYLOR SERIES AND OF TRIGONOMETRIC SERIES

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1. Introduction. If a set E on the unit circle C is of type G_δ, there exists a Taylor series $\Sigma a_n z^n$ which diverges on E and converges on $C-E$; this was shown by Herzog and Piranian in [3]. The same authors exhibited in [4] certain sets that are not of type G_δ and that are sets of divergence of Taylor series whose partial sums $s_n(z)$ are uniformly bounded on C. In Section 2 of the present paper we will prove, among other things, that every set of type F_σ and of logarithmic measure zero is the set of divergence of a Taylor series.

Zygmund [6] showed that the partial products of the infinite product $\Pi_{k=1}^{\infty} [1 + (i/k)\cos 3^k\theta]$ coincide with certain partial sums of a Fourier series $\Sigma (a_n \cos n\theta + b_n \sin n\theta)$; this Fourier series has uniformly bounded partial sums, and it diverges on a set which has locally the power of the continuum. Tandori [5] exhibited a continuous function whose Fourier series has the same properties. In Section 3, we prove generalizations of Tandori's result which are analogous to two of the theorems in Section 2.

In Section 4, an analogue to Theorem 3 of Section 2 is proved for trigonometric series.

2. Taylor series. A set E on the unit circle C is of logarithmic measure zero provided it can be covered with a set of arcs of lengths L_j (with $L_j < 1$ for $j = 1, 2, \ldots$) such that $\Sigma 1/|\log L_j|$ is arbitrarily small.

THEOREM 1. *If the set E on C is of logarithmic measure zero, there exists a function $f(z) = \Sigma a_n z^n$ with the following properties:*

i) *$f(z)$ is continuous on $|z| \leqq 1$;*

ii) *$\Sigma a_n z^n$ diverges on E;*

iii) *the sequence $\{s_n(z)\}$ is uniformly bounded on C.*

The proof of this theorem is based on the fact that the polynomials

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \ldots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \frac{z^{n+1}}{2} - \ldots - \frac{z^{2n-1}}{n}$$

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are uniformly bounded on C (see Fejér [2]; a very elementary proof that $|P_n(z)| < 4 + 2\pi$ is given in [1, pp. 42–43]). We choose a sequence of numbers ε_k, with $0 < \varepsilon_k < \frac{1}{3}$ and $\Sigma_{k=1}^{\infty} \varepsilon_k < 1$. For each index k, we can construct a set of open arcs A_{kj}, of lengths $L_{kj} < 1$, such that, for each k, $E \subset \bigcup_{j=1}^{\infty} A_{kj}$ and $\Sigma_{j=1}^{\infty} 1/|\log L_{kj}| < \varepsilon_k$. (Since $\varepsilon_k < \frac{1}{3}$, the L_{kj} will actually be less than $\frac{1}{16}$.) We denote by ω_{kj} the midpoint of A_{kj}, and we choose integers n_{kj} such that $1/(4L_{kj}) \leqq n_{kj} \leqq 1/(2L_{kj})$. Finally, we let

$$f(z) = \sum_{k,j} g_{kj}(z) \,, \tag{1}$$

where

$$g_{kj}(z) = (\log n_{kj})^{-1} z^{m_{kj}} P_{n_{kj}}(z/\omega_{kj}) \,. \tag{2}$$

It is assumed here that the integers m_{kj} are chosen in such a way that no two of the polynomials $g_{kj}(z)$ contain like powers of z.

Since, by the foregoing inequalities,

$$\Sigma_{k,j} (\log n_{kj})^{-1} \leqq \Sigma_{k,j} (|\log L_{kj}| - \log 4)^{-1} < \Sigma_{k,j} 2 |\log L_{kj}|^{-1} < 2 \,,$$

the double series in (1) converges uniformly on C, and therefore $f(z)$ is continuous on $|z| \leqq 1$.

If $z = e^{i\theta}$, the argument of each of the first n terms in $P_n(z)$ lies between $-n\theta$ and $n\theta$. If in particular $-1/4n < \theta < 1/4n$, the real part of the sum of the first n terms in $P_n(z)$ is greater than $\frac{1}{2}\log n$. For each point z in E, there exist infinitely many ω_{kj} such that $|\arg(z/\omega_{kj})| < L_{kj}/2 \leqq 1/4n_{kj}$. It follows that, for each z in E, infinitely many of the polynomials $g_{kj}(z)$ have at least one partial sum that is greater than $\frac{1}{2}$ in modulus, and therefore the Taylor series of $f(z)$ diverges at all points of E.

To show that the partial sums $s_n(z)$ of this Taylor series are uniformly bounded on C, it is sufficient to observe that the partial sums of the series (1) are uniformly bounded and that the partial sums of the polynomials $g_{kj}(z)$ are bounded by a universal constant.

Remark. We actually proved a result slightly stronger than the theorem: there exist two positive universal constants K_1 and K_2, such that at each point of E the Taylor series of our function $f(z)$ diverges with an oscillation greater than K_1, while $|s_n(z)| < K_2$ at all points of C.

Theorem 2. *If the set E on C is of type F_σ and of logarithmic measure zero, there exists a function $f(z) = \Sigma a_n z^n$ with the following properties:*

i) *$f(z)$ is continuous on $|z| \leqq 1$;*

ii) *$\Sigma a_n z^n$ diverges on E and converges on $C - E$;*

iii) *the sequence $\{s_n(z)\}$ is uniformly bounded on C.*

First we prove the theorem for the special case where the set E is closed. We observe that, if $z = e^{i\theta}$, $\theta \neq 0$, the partial sums of the polynomial $P_n(z)$ have moduli less than $c/|\theta|$, where c is a universal constant. (See, for example, [1, p. 43].) In the proof of Theorem 1, we now take the precaution of requiring that each of the arcs A_{kj} contains a point of E. Then each point z on $C-E$ is the midpoint of an arc of length 2δ which contains at most a finite number of points ω_{kj}. The convergence of $\Sigma a_n z^n$ at z now follows from the fact that, for every pair k and j for which ω_{kj} does not lie in the δ-neighborhood of z, the partial sums of $g_{kj}(z)$ have modulus less than $c/(\delta \log n_{kj})$. The theorem is therefore proved for the case where the set E is closed.

To deal with the general case, we suppose that $E = \bigcup E_p$, where each of the sets E_p is closed and has logarithmic measure zero. We construct for each set E_p a function $f_p(z)$ in the manner which has just been described and in accordance with the remark made at the end of the proof of Theorem 1. We then set $f(z) = \Sigma_p K_3{}^p f_p(z)$, where K_3 is a positive constant small enough so that $K_1 > 2K_2 \Sigma_{p=1}^{\infty} K_3{}^p$.

If z is a point in E, let q be the smallest value of p for which z lies in E_p. The Taylor series of $f_p(z)$ converges at z when $p < q$; the Taylor series of $K_3{}^q f_q(z)$ diverges at z with an oscillation greater than $K_1 K_3{}^q$; and the partial sums of the Taylor series of the function $\Sigma_{p=q+1}^{\infty} K_3{}^p f_p(z)$ are bounded on C by $K_2 \Sigma_{p=q+1}^{\infty} K_3{}^p < \frac{1}{2} K_1 K_3{}^q$. It follows that the divergence contributed by the term $K_3{}^q f_q(z)$ is not cancelled by the other terms, and therefore the Taylor series of $f(z)$ diverges at z.

If z is a point of $C-E$ then, for all p, the Taylor series of $K_3{}^p f_p(z)$ converges at z and has partial sums that are bounded on C by $K_2 K_3{}^p$. Therefore the Taylor series of $f(z)$ converges at z. It is obvious that $f(z)$ has also the properties i) and iii) stated in Theorem 2. This completes the proof.

THEOREM 3. *If E on C is the union of two sets E_1 and E_2, where E_1 is of type F_σ and of logarithmic measure zero, while E_2 is of type G_δ, then there exists a Taylor series which diverges on E and converges on $C-E$.*

By Theorem 2, there exists a function $f(z)$ whose Taylor series converges on $C-E_1$, diverges on E_1 and has partial sums that are uniformly bounded on C; let $|s_n(z)| < B$. By the method of [3, p. 532, lines 17–20], a function $g(z)$ exists whose Taylor series converges on $C-E_2$ and diverges at every point z_2 of E_2 with an oscillation greater than some positive constant b, which is independent of z_2. If the constant k is chosen greater than $2B/b$, the Taylor series of $f(z) + kg(z)$ provides the proof of the theorem.

3. Fourier series. In this and the next section, the interval $0 \leqq \theta < 2\pi$ will be denoted by I and the nth partial sum of the series

$$\sum_{n=0}^{\infty} a_n \cos n(\theta - \theta_n) \tag{3}$$

by $s_n(\theta)$.

THEOREM 4. *If the set E^* on I is of logarithmic measure zero, there exists a function $f(\theta)$ with the following properties:*

i) *$f(\theta)$ is continuous on the closure of I and $f(0) = f(2\pi)$;*

ii) *the Fourier series* (3) *of $f(\theta)$ diverges on E^*;*

iii) *the sequence $\{s_n(\theta)\}$ is uniformly bounded on I.*

THEOREM 5. *If the set E^* on I is of type F_σ and of logarithmic measure zero, there exists a function $f(\theta)$ with the following properties:*

i) *$f(\theta)$ is continuous on the closure of I and $f(0) = f(2\pi)$;*

ii) *the Fourier series* (3) *of $f(\theta)$ diverges on E^* and converges on $I - E^*$;*

iii) *the sequence $\{s_n(\theta)\}$ is uniformly bounded on I.*

For the proof of these two theorems, we map the set E^* into a set E on the unit circle C in the obvious manner, and we use essentially the real part of the series (1), constructed in the preceding section. As far as continuity, boundedness of the partial sums, and (in the case of Theorem 5) convergence are concerned, no new problems arise. However, a certain refinement is necessary in order to ensure divergence of the Fourier series on the given set E^*. The difficulty lies in the fact that the exponent m_{kj} in (2) may be very large compared with n_{kj}, and that therefore the real part of the sum of the first n_{kj} terms of the polynomial $g_{kj}(z)$ may be zero even when z lies very near to ω_{kj}. We solve this difficulty by replacing the functions $g_{kj}(z)$ by the polynomials

$$h_{kj}(z) = (\log n_{kj})^{-1} \left[z^{m_{kj}} P_{n_{kj}}(z/\omega_{kj}) + z^{2m_{kj}} P_{n_{kj}}(z/\omega_{kj}) \right].$$

The m_{kj} are here assumed to be chosen so that no two of the polynomials $h_{kj}(z)$ overlap, and also so that in each $h_{kj}(z)$ the two "halves" of the polynomial do not contain like powers of z.

Let $\varphi_{kj}(\theta)$ denote the argument of the sum of the first n_{kj} terms of $P_{n_{kj}}(e^{i\theta})$. Then, for $-1/4n_{kj} < \theta < 1/4n_{kj}$, $|\varphi_{kj}(\theta)| < \pi/12$, and therefore at least one of the quantities

$$|\cos [m_{kj}\theta + \varphi_{kj}(\theta)]| \qquad \text{and} \qquad |\cos [2m_{kj}\theta + \varphi_{kj}(\theta)]|$$

is greater than $\cos(5\pi/12)$. (Note that, for any real β, at least one of the two values β and 2β differs from the nearest multiple of π by at most $\pi/3$.) This ensures that, for $|\arg (z/\omega_{kj})| < 1/4n_{kj}$, the polynomial

$h_{kj}(z)$ contains a block of n_{kj} terms whose sum has a real part of modulus greater than a positive universal constant. From here on, the proof proceeds as before, and we omit the details.

4. Trigonometric series. The following result is a natural analogue of Theorem 3.

THEOREM 6. *If E^* on I is the union of a set of type G_δ and a set of type F_σ and of logarithmic measure zero, then there exists a trigonometric series which diverges on E^* and converges on $I-E^*$.*

To prove this theorem, we use the procedure outlined in the proof of Theorem 3, combined with the "doubling" process described in the proof of Theorems 4 and 5. We omit the details.

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