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ON A TYPE OF EIGENVALUE PROBLEMS FOR CERTAIN ELLIPTIC DIFFERENTIAL OPERATORS

GUNNAR EHRLING

This paper treats a problem that can be regarded as a generalization of the eigenvalue problem for the elastic plate. The main term of the potential energy of an n-dimensional "plate" represented by an open bounded domain D is supposed to be given by a certain generalized Dirichlet integral over the domain. The integrand is a form of the type

$$\sum k_{\mu\nu} D^{\mu}f \, D^{\nu}\bar{f} ,$$

where $D^{\mu}f$ and $D^{\nu}f$ are derivatives of the order m of a deformation function f and $k_{\mu\nu}$ are variable and sufficiently regular. The form is supposed uniformly positive definite in all the derivatives of order m (hence hermitian in the same variables). We shall suppose that the boundary of D satisfies certain regularity conditions. Subjecting the "plate" to boundary conditions involving derivatives of order $\leqq m-1$, an eigenvalue problem is defined in the same way as for the ordinary plate (see Friedrichs [5]).

An eigenfunction g of the elastic plate satisfies the differential equation

$$\Delta \Delta g - \lambda g = 0 .$$

In the case treated here there is in general no differential equation associated with the problem, but it comprises cases in which the eigenfunctions satisfy

$$ug - \lambda g = 0 ,$$

where u is an elliptic differential operator of order $2m$.

In the case when the integrand of the Dirichlet integral is hermitian in all the derivatives of f (f itself included) and $2m > n$ we shall by a method due to Carleman [1] deduce asymptotic formulas for the eigenvalues and eigenfunctions that generalize results in the theory of the elastic plate obtained by R. Courant [2] and Å. Pleijel [16] [17]. Similar eigenvalue problems for more general elliptic operators (Dirichlet inte-

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grals) in an arbitrary open domain but with more special boundary conditions have been treated by Gårding [7] [8] [9]. The technique used here is essentially the same as in these papers. The main results of this paper follow from integral inequalities ((3), (4) and (5), pp. 270–271) which are variants of certain estimates by Friedrichs [5]. The problem of proving the inequality (4) was put to me by L. Gårding. I want to express my gratitude to him for his interest and valuable advice.

Notations and preliminaries. In this paragraph we shall introduce a domain D supposed to represent the "plate" and impose a number of regularity conditions upon it. These conditions are chosen with the purpose of being reasonably general and at the same time giving a direct point of departure for the proof of certain integral estimates. They may be dependent on each other, and it is probably possible to reduce them in number and to give them a simpler form.

In n-dimensional euclidean space E we denote the points by $x = (x_1, \ldots, x_n)$, y, etc., distance by $|x-y| = \left((x_1-y_1)^2 + \ldots + (x_n-y_n)^2\right)^{\frac{1}{2}}$ and the volume element by dx. For derivatives we use the notation $D^\nu f = \partial^{|\nu|} f / \partial x_1^{\nu_1} \ldots \partial x_n^{\nu_n}$, where $|\nu| = \nu_1 + \ldots + \nu_n$ is the order of the derivative.

Let D be an open bounded region in E. We introduce an auxiliary concept. By a (Lipschitz) manifold S_j with respect to D of dimension j we shall mean the set-theoretical union of a finite number of pieces T each of which has the following properties:

1°. *In a suitably chosen system of orthogonal coordinates* $y_1, \ldots, y_n$, *the part* T *can be represented by a set of equations*:

$$(1) \qquad \begin{aligned} y_{j+1} &= \varrho_{j+1}(y_1, \ldots, y_j) , \\ &\ldots \\ y_n &= \varrho_n(y_1, \ldots, y_j) , \end{aligned}$$

where $y^\star = (y_1, \ldots, y_j)$ *varies over a closed domain* Ω *and all the functions* ϱ_k *satisfy a Lipschitz condition*

$$(2) \qquad |\varrho_k(y^\star) - \varrho_k(z^\star)| \leqq C\,|y^\star - z^\star| ,$$

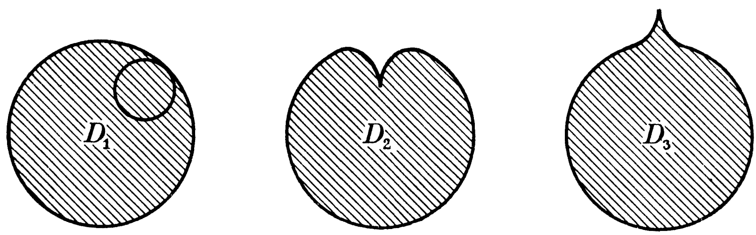
C being independent of $y^\star$ *and* $z^\star$.

2°. *There exists in a plane* $y^\star =$ const. *an* $(n-j)$*-dimensional spherical sector* Σ *with a positive radius and a positive spherical angle so that each point* y *of* T *is the vertex of a sector* Σ_y *obtained from* Σ *by a translation and contained in* D *except perhaps for the vertex. If* $j = n-1$, T *is represented* *by*

$$y_n = \varrho_n(y_1, \ldots, y_{n-1}) ,$$

and the condition 2° *means that to each point* $y = (y_1, \ldots, y_{n-1}, \varrho_n)$ *of* T *the segment* $y^\star = (y_1, \ldots, y_{n-1}) =$ const., $\varrho_n > y_n > \varrho_n - h$, *belongs to* D, h *being a positive number independent of* y.

In the figure the periphery of the circle D_1 and of the tangent circle inside D_1 form together a Lipschitz manifold of dimension 1 with respect to D_1. D_2 and D_3 are two regions where the boundary is, respectively is not, a Lipschitz manifold with respect to the region.



The Lipschitz condition (2) implies that the Jacobians

$$D_{k_1 \ldots k_j} = D(y_{k_1, \ldots, k_j}) / D(y_1, \ldots, y_j)$$

exist almost everywhere on T with respect to $dy^\star = dy_1 \ldots dy_j$ and that S_j has a surface measure defined by

$$dS_j = \Big(\sum_{k_1 < \ldots < k_j} (D_{k_1 \ldots k_j})^2 \Big)^{\frac{1}{2}} dy^\star$$

on T (see Kolmogoroff [12] and Nöbeling [15]). With respect to the measure dS_j the surface has a j-dimensional tangent plane almost everywhere.

For D we make the following assumptions:

I. *The boundary* S *of* D *is a manifold* S_{n-1} *with respect to* D.

II. S *is the boundary of* $S+D$.

III. *There exists an n-dimensional spherical sector* Σ *with a positive radius and a positive spherical angle, so that each point* x *in* $D+S$ *is the vertex of a sector* Σ_x *contained in* $D+S$ *and congruent with* Σ.

IV. D *is normal in the sense of Courant-Hilbert* [3, p. 516], *that is,* D *is the union of a finite number of regions each of which is defined in a suitable system of orthogonal coordinates by the inequalities*

$$0 \leqq y_i \leqq d_i , \qquad i < n ,$$
$$0 \leqq y_n \leqq Y ,$$

where Y *is a continuous function of* $y_1, \ldots, y_{n-1}$, $\inf Y > 0$, *and* d_i *are constants.*

The statements I–IV may be dependent on each other. Physically, in the case of the vibrating plate the condition II means that we restrict ourselves to examining plates without incisures. The requirement is not essential for the theory and could be replaced by a weaker one.

The conditions I and II imply (see Lorentz [13]) that the exterior normal exists almost everywhere on S and that if $f(x)$ is a continuous function in $D+S$ and if $\partial f/\partial x_i$ exists in D and is integrable over D we have

$$\int_D \partial f/\partial x_i \, dx = \int_S f \cos(\nu, x_i) \, dS ,$$

where dS is the surface element on S and $\cos(\nu, x_i)$ is the cosine of the angle between the exterior normal on S and the positive x_i-axis. It follows that if $g(x)$ has the same properties as $f(x)$ we have

$$\int_D g \, \partial f/\partial x_i \, dx = - \int_D f \, \partial g/\partial x_i \, dx + \int_S f \, g \cos(\nu, x_i) \, dS .$$

A set of integral estimates. Let H be the set of all infinitely differentiable complex functions defined on D which, together with all their derivatives, have continuous extensions to $D+S$. Put

$$N^k(f,g) = \int_D \Big(\sum_{|\nu|=k} D^\nu f \, D^\nu \bar{g} \Big) \, dx$$

and

$$N_t^k(f,g) = N^k(f,g) + tN^\circ(f,g)$$

for f and g in H.

For $N^0(f,g)$, which is the ordinary scalar product $\int_D f\bar{g}\,dx$, we shall also use the notation (f,g).

For large positive values of t we shall prove the following estimates (compare Friedrichs [5]),

$$N^k(f,f) = O(t^{-(1-k/m)}) \, N_t^m(f,f) \tag{3}$$

when $0 \leqq k \leqq m$,

$$|D^\nu f(x)|^2 = O(t^{-[1-\frac{1}{2}(n+2|\nu|)/m]}) \, N_t^m(f,f) \tag{4}$$

whenever the exponent of t is negative, uniformly when $x \in D+S$. Further, if S_j is a Lipschitz manifold with respect to D, then

$$\int_{S_j} |D^\nu f(x)|^2 \, dS_j = O(t^{-[1-\frac{1}{2}(n-j+2|\nu|)/m]}) \, N_t^m(f,f) \tag{5}$$

whenever the exponent of t is negative. All the estimates O are independent of f.

Proof: Let us represent one piece T of S in the form

$$y_n = \varrho(y_1, \ldots, y_{n-1}),$$

where $y^\star = (y_1, \ldots, y_{n-1})$ belongs to a closed domain Ω. According to I we can assume that the region T_h defined by $y^\star \in \Omega$, $\varrho > y_n > \varrho - h$ belongs to D for a positive number h. We represent the value on S of a function f in H by

$$f(y^\star, \varrho(y^\star)) = f(y) + \int_{y_n}^{\varrho(y^\star)} \partial f/\partial y_n \, dy_n$$

for $y \in T_h$. Then

$$|f(y^\star, \varrho(y^\star))|^2 \leqq 2|f(y)|^2 + 2h \int_{y_n}^{\varrho(y^\star)} |\partial f/\partial y_n|^2 dy_n .$$

Integrating over T_h, using the Lipschitz condition (2), we get

$$\begin{aligned}(1+C^2)^{-\frac{1}{2}} h \int_{T_h} |f|^2 dT &\leqq 2 \int_{T_h} |f|^2 dy + 2h^2 \int_{T_h} |\partial f/\partial y_n|^2 dy \\ &\leqq 2 \int_{T_h} |f|^2 \, dy + 2h^2 \int_{T_h} \left(\sum_{i=1}^n |\partial f/\partial y_i|^2 \right) dy ,\end{aligned}$$

where we can give h any value that is small enough. Adding the estimates for the various parts T_h we get

$$\int_S |f|^2 dS \leqq A \, [h^{-1}(f,f) + hN^1(f,f)] \tag{6}$$

for small values of h. Here and in the sequel A will denote a constant, not always the same but independent of the function f and the parameter h. Applying (6) to the derivatives of f and adding the resulting estimates we get

$$\int_S (\sum |\partial f/\partial x_i|^2) \, dS \leqq A \, [h^{-1}N^1(f,f) + hN^2(f,f)] . \tag{7}$$

From Green's formula

$$N^1(f,f) + \int_D f \, \Delta \bar{f} \, dx = \int_S f \, \partial \bar{f}/\partial \nu \, dS ,$$

where ν is the exterior normal of S, it follows that

$$N^1(f,f) \leqq [(f,f)nN^2(f,f)]^{\frac{1}{2}} + \left(\int_S |f|^2 dS \int_S \sum |\partial f/\partial x_i|^2 dS\right)^{\frac{1}{2}}$$

$$\leqq h^{-1}(f,f) + nhN^2(f,f) + h^{-\frac{1}{2}}\int_S |f|^2 dS + h^{\frac{1}{2}}\int_S \sum |\partial f/\partial x_i|^2 dS\,,$$

where $h > 0$ is arbitrary. Estimating the surface integrals by means of (6) and (7), putting the number $h^{\frac{1}{2}}/(4A)$ in (6) and the number $4Ah^{\frac{1}{2}}$ in (7) instead of h, we get

$$N^1(f,f) \leqq A\,[h^{-1}(f,f) + hN^2(f,f)]\,.$$

Applying this formula to the derivatives of f of order $k-1$ and adding the resulting estimates we get

$$N^k(f,f) \leqq A\,[h^{-1}N^{k-1}(f,f) + hN^{k+1}(f,f)]\,.$$

Using this, one proves by induction the formula

$$N^k(f,f) \leqq A\,[h^{-k}(f,f) + h^p N^{k+p}(f,f)]\,,$$

where $k \geqq 1$. Putting $k+p = m$ and $h = t^{-1/m}$ we get (3) when $0 < k < m$. When $k = 0$ or $k = m$ the relation (3) is of course trivial.

The value of f at a point is estimated by expressing it in terms of the values $\varphi(t)$ of f on a half-ray $x+t\xi$, $|\xi| = 1$, $t \geqq 0$, through x. Putting

$$\varphi^{(l)}(t) = \frac{(-1)^l}{l!}\left(\frac{d}{dt}\right)^l \varphi(t)$$

we have

$$f(x) = \varphi(0) = \sum_0^{k-1} \varphi^{(l)}(t)t^l + k\int_0^t \tau^{k-1}\varphi^{(k)}(\tau)\,d\tau \tag{8}$$

so that, supposing $2k-n > 0$, we get

$$|f(x)|^2 \leqq (k+1)\left(\sum_0^{k-1} |\varphi^{(l)}(t)|^2 t^{2l} + k^2\frac{t^{2k-n}}{2k-n}\int_0^t \tau^{n-1}|\varphi^{(k)}(\tau)|^2 d\tau\right). \tag{9}$$

By virtue of the condition III we introduce an n-dimensional spherical sector $\Sigma_h \subset D+S$ with center in x and radius h. We integrate (9) in the variable $y = x+t\xi$ over Σ_h. If the volume of Σ_h is αh^n we get

$$\alpha h^n |f(x)|^2 \leqq (k+1)\left(\sum_0^{k-1} h^{2l}\int_{\Sigma_h} |\varphi^{(l)}|^2 dx + k\frac{h^{2k}}{4k-2n}\int_{\Sigma_h} |\varphi^{(k)}|^2 dx\right),$$

which gives

$$|f(x)|^2 \leqq A \sum_{l=0}^{k} h^{2l-n} N^l(f,f)$$

and for the derivatives of f

$$(10) \qquad |D^\nu f(x)|^2 \leqq A \sum_{l=0}^{k} h^{2l-n} N^{l+|\nu|}(f,f)$$

for $h > h_0$ uniformly in $D+S$.

In order to prove (5) let us denote by T_j a part of S_j represented by the equations (1). We calculate the value of a function f at the point $y = (y_1, \ldots, y_j, \varrho_{j+1}, \ldots, \varrho_n)$ on T_j from its values $\varphi(t)$ on a half-ray $y+t\xi$, $t \geqq 0$, $\xi_1 = \ldots = \xi_j = 0$, $|\xi| = 1$, through the $(n-j)$-dimensional sector Σ_y belonging to y. Supposing that $k > \frac{1}{2}(n-j)$ we get from (8) that

$$(11) \qquad |f(y)|^2 \leqq (k+1)\left(\sum_{0}^{k-1} |\varphi^{(l)}(t)|^2 t^{2l} + k^2 \frac{t^{2k-n+j}}{2k-n+j} \int_0^t \tau^{n-j-1} |\varphi^{(k)}(\tau)|^2 d\tau\right).$$

We take the volume integral of both sides of (11) with respect to the variable $x = y+t\xi$ over the region $x \in \Sigma_{y,h}$, $y \in T_j$, denoting by $\Sigma_{y,h}$ the part of Σ_y that lies inside the sphere with radius h and center in y. Adding the estimates that result in this way for the various parts T_j we get

$$\int_{S_j} |f|^2 dS_j \leqq A \sum_{l=0}^{k} N^l(f,f) h^{2l-n+j}$$

for all sufficiently small h. For the derivatives of f this means

$$(12) \qquad \int_{S_j} |D^\nu f|^2 dS_j \leqq A \sum_{l=0}^{k} h^{2l-n+j} N^{l+|\nu|}(f,f)\,.$$

Putting $h = t^{-\frac{1}{2}/m}$ and using (3), the result (5) follows. The formula (4) is deduced from (10) in the same way.

The vibration problem. Green's transformation. Corresponding to the potential energy of the vibrating plate we assume that we have a hermitian form in H

$$V(f,f) = \int_D \sum_{\substack{|\mu|=m \\ |\nu|=m}} a_{\mu\nu}(x)\, D^\mu f\, D^\nu \bar{f}\, dx\,,$$

where the matrix $\big(a_{\mu\nu}(x)\big)$ is hermitian, bounded and uniformly positive definite in D, so that there exists a positive constant d for which

(13) $$d^{-1}\sum|\xi^{\mu}|^2 \geqq \sum a_{\mu\nu}(x)\xi^{\mu}\bar{\xi}^{\nu} \geqq d\sum|\xi^{\mu}|^2$$

for all x in D and all complex numbers ξ^{μ}. Further we shall assume that for all y in D the coefficients $a_{\mu\nu}$ satisfy the Lipschitz condition

(14) $$|a_{\mu\nu}(y)-a_{\mu\nu}(x)| \leqq C_y|y-x|^{\alpha_y},$$

valid for x in D, C_y and α_y being positive and dependent only on y.

Let us put
$$V_t(f,f) = V(f,f)+t(f,f), \qquad t \geqq 0 .$$

It follows from (13) that $N_t(f,f)$ and $V_t(f,f)$ are equivalent square norms in H, uniformly for all $t \geqq 0$.

In addition to V we shall assume that we have another form $R(f,g)$ defined on $H\times H$ which is linear in f, antilinear in g and is small compared to V in the sense that

(15) $$|R(f,g)|^2 = o(1)\ V_t(f,f)\ V_t(g,g),$$

where $o(1)\to 0$ as $t\to+\infty$ and is independent of f and g. It follows from the estimates proved in the preceding section that, for example,

(16) $$R(f,g) = \int_D M(f,g)dx + \sum_{j=1}^{n-1}\int_{S_j} K_j(f,g)dS_j + \int_{D+S} L(f,g)d\alpha(x)$$

has these properties provided that $M(f,g)=\Sigma b_{\mu\nu}(x)D^{\mu}fD^{\nu}\bar{g}$ with $|\mu|\leqq m$, $|\nu|\leqq m$, $|\nu|+|\mu|<2m$ and with bounded coefficients $b_{\mu\nu}$ in D, that S_j is a Lipschitz manifold of dimension j with respect to D and $K_j(f,g)=\Sigma c_{\mu\nu}(x)D^{\mu}fD^{\nu}\bar{g}$ with $|\mu|,|\nu|<m-\frac{1}{2}(n-j)$ and bounded coefficients $c_{\mu\nu}$ on S_j, and that $\alpha(x)$ is of bounded variation in $D+S$ and $L(f,g)$ has the properties of $K_0(f,g)$. It should be observed that the occurrence, for example, of the last term of (16) is impossible unless $2m>n$.

Closing H in the norm-square $N_1{}^m(f,f)$ we get a Hilbert space $\mathfrak{H}_m$. If $\{f_i\}$ is a Cauchy sequence in H, then according to (3) the functions f_i and their derivatives of orders $\leqq m$ converge in square mean on D. If F_{ν} and F are the limit functions of $\{D^{\nu}f_i\}$ and $\{f_i\}$, respectively, and $|\nu|\leqq m$, we have
$$\int_D gF_{\nu}dx = (-1)^{|\nu|}\int_D D^{\nu}g\,F\,dx$$

for any function $g\in H$ which vanishes outside a compact set in D, so that F_{ν} is the generalized (weak) derivative of F. Hence $\mathfrak{H}_m$ consists of functions F having generalized square integrable derivatives of orders

$\leqq m$ in D. If $2m > n$ it follows from (4) that the limit functions F are continuous, the convergence being uniform on $D+S$.

Let $D^\star$ be an open subset of D with the same measure as D and let $L = L(D^\star)$ be the set of all infinitely differentiable functions vanishing outside compact subsets of $D^\star$. Closing L with respect to the square norm $N_1{}^m(f, f)$ we get a subspace $\mathfrak{L}_m$ of $\mathfrak{H}_m$. Let $\mathfrak{H}$ be any closed subspace of $\mathfrak{H}_m$ containing $\mathfrak{L}_m$. The cases $\mathfrak{H} = \mathfrak{H}_m$ and $\mathfrak{H} = \mathfrak{L}_m$ are included. The elements of $\mathfrak{H}$ correspond to the set of admissible functions in the theory of the plate. (This general type of boundary conditions was proposed to me by L. Gårding.) Imposing on the functions in H a number of linear conditions on the set $D+S-D^\star$, for example

$$\sum_{|\nu| \leqq m-1} c_\nu(x) D^\nu f(x) = 0 \,, \tag{17}$$

and closing the resulting space with respect to $N_1{}^m(f, f)$, we obtain a space $\mathfrak{H}$. We do not enter upon the question of which way the conditions (17) are satisfied by the functions in $\mathfrak{H}$. The functions in $\mathfrak{H}_m$ belong to a class which has been studied by Nikodym [14] and Deny [4].

In $\mathfrak{H}_m$ the forms V and R are defined by continuity. Let us put

$$U(f, g) = V(f, g) + R(f, g)$$

and assume that $m > 0$.

If R is hermitian, that is, if $R(f, f)$ is real for all f, then we may consider $U(f, f)$ as a modified potential energy of the "plate". For given U and $\mathfrak{H}$ we shall consider a vibration problem $\{U, \mathfrak{H}\}$ which, loosely speaking, consists in finding the eigenfunctions and eigenvalues of the form U, defined in $\mathfrak{H}$, with respect to the unit form (f, f).

Definition. An eigenfunction of the vibration problem $\{U, \mathfrak{H}\}$ with the eigenvalue λ is a function $\varphi \in \mathfrak{H}$ that satisfies

$$U(\varphi, h) = \lambda\, (\varphi, h) \tag{18}$$

for all $h \in \mathfrak{H}$.

If, for example, in (16) $L = 0$ and the coefficients of M and of V are sufficiently differentiable this means that φ, after correction upon a null set, is $2m$ times continuously differentiable and satisfies the differential equation

$$u\varphi = (-1)^m \sum D^\mu(a_{\mu\nu} D^\nu \varphi) + \sum (-1)^{|\mu|} D^\mu(b_{\mu\nu} D^\nu \varphi) = \lambda\, \varphi$$

in $D^\star - \Sigma S_j$ where the S_j are the Lipschitz manifolds occurring in (16) (theorem of Schwartz-John, see Gårding [6]).

Let $\mathfrak{L} = \mathfrak{L}_0$ denote the set of all square integrable functions in D, where we introduce $(f, f)^{\frac{1}{2}}$ as norm. We put

$$U_t(f, g) = U(f, g) + t(f, g), \qquad t \geqq 0 . \tag{19}$$

The following theorem (cf. Gårding [7]) shows that the eigenfunctions and essentially also the eigenvalues of the vibration problem can be obtained as the eigenfunctions and eigenvalues of anyone of a set of completely continuous operators.

THEOREM 1. *If t is large enough, the equation*

$$(f, h) = U_t(G_t f, h) , \tag{20}$$

where f belongs to $\mathfrak{L}$ and h and $G_t f$ belong to $\mathfrak{H}$ defines a bounded linear operator G_t from $\mathfrak{L}$ to $\mathfrak{H} \subset \mathfrak{L}$. Considered as an operator from $\mathfrak{L}$ to $\mathfrak{L}$, G_t is completely continuous and if R is hermitian it is selfadjoint and positive. The vibration problem and G_t have the same eigenfunctions. An eigenfunction φ of the vibration problem with the eigenvalue λ has the eigenvalue $(\lambda+t)^{-1}$ with respect to G_t.

PROOF. Let us first put $R = 0$ so that $U_t = V_t$ using in this case the notation $A_t = G_t$. Obviously (f, h) is an antilinear bounded function of $h \in \mathfrak{H}$ so that introducing into $\mathfrak{H}$ instead of $N_1{}^m(f, g)$ the equivalent scalar product $V_t(f, g)$, $t > 0$, it follows that $A_t f$ exists and that A_t is a bounded linear transformation from $\mathfrak{L}$ to $\mathfrak{H}$. Let us show that it is also completely continuous considered as an operator from $\mathfrak{L}$ to $\mathfrak{L}$. We have

$$|V_t(A_t f, A_t f)|^2 = |(A_t f, f)|^2 \leqq (A_t f, A_t f)(f, f) \leqq t^{-1} V_t(A_t f, A_t f)(f, f)$$

so that

$$V_t(A_t f, A_t f) \leqq t^{-1}(f, f) .$$

Consider a sequence $\{f_i\}$ which is bounded in $\mathfrak{L}$. Then the sequence $\{A_t f_i\}$ is bounded with respect to the norm-square $N_t{}^m$ and hence by virtue of (3) also with respect to $N_1{}^1$. It follows from the condition IV on D and a theorem by Rellich (see Courant-Hilbert [3, p. 513]) that there exists a subsequence $\{A_t f_{i'}\}$ which converges in square mean in D. (The proofs of Courant-Hilbert [3] carry over immediately to n dimensions.) Hence A_t is completely continuous.

When $R \neq 0$ we proceed as follows (compare Gårding [8, p. 64]). The equation

$$R(f, g) = V_t(L_t f, g), \qquad f, g, L_t f \in \mathfrak{H} , \tag{21}$$

defines a bounded linear operator L_t from $\mathfrak{H}$ to $\mathfrak{H}$ the norm of which,

$$|L_t|_t = \Big(\sup_{f,g} |R(f,g)|^2/[V_t(f,f)\,V_t(g,g)]\Big)^{\frac{1}{2}},$$

by virtue of (15) tends to zero as $t \to \infty$. Now we can write

$$U_t(f,g) = V_t\big((1+L_t)f, g\big),$$

and if t is so large that $|L_t|_t < 1$ then $(1+L_t)^{-1}$ exists and is bounded. Putting $G_t = (1+L_t)^{-1}A_t$ we get a transformation satisfying (20). It follows immediately from this representation of G_t that G_t is completely continuous. If R is symmetric then

$$(G_t f, f) = V_t(G_t f, G_t f) + R(G_t f, G_t f)$$

is always real and positive, hence G_t is selfadjoint and positive. (A minor modification of the arguments shows that G_t, considered as an operator from $\mathfrak{H}$ to $\mathfrak{H}$, is completely continuous and, if R is hermitian, also selfadjoint and positive with respect to the square norm $U_t(f,f)$.)

Let us assume that $\varphi \in \mathfrak{L}$ and that

(22) $$G_t\varphi = (\lambda+t)^{-1}\varphi.$$

Then by (20), $$U_t(\varphi, h) = (\lambda+t)(\varphi, h), \qquad h \in \mathfrak{H},$$

which implies (18). Conversely, it follows from (18) that φ is an eigenfunction of G_t which completes the proof of Theorem 1.

We shall call G_t Green's transformation corresponding to the bilinear form U_t and the subspace $\mathfrak{H}$ of $\mathfrak{H}_m$.

Green's function. If $2m > n$, which we will assume from now on, then Green's function, i.e., the kernel of Green's transformation, can be obtained very simply (cf. Gårding [7]).

In fact, it follows from (5) that $\overline{f(y)}$ for any $y \in D+S$ is an antilinear function of $f \in \mathfrak{H}$. Hence it can be written as a scalar product

(23) $$\overline{f(y)} = V_t\big(a_t(\cdot, y), f\big).$$

This relation implies in particular that a weakly convergent sequence in $\mathfrak{H}$ converges pointwise in $D+S$. We approximate the integral over D of $\bar{f}h$, f and $h \in \mathfrak{H}$, by a sequence of Riemann sums

$$\textstyle\sum_k \overline{f(y)}\,h(y) = \sum_i \overline{f(y_i^k)}\,h(y_i^k)\,|D_i^k|, \qquad y_i^k \in D_i^k,$$

where D_i^k for fixed k are non-overlapping regions with the sum D and each with a diameter $< \delta_k \to 0$. The measure of D_i^k is denoted by $|D_i^k|$. From (23) it follows that

$$\textstyle\sum_k \overline{f(y)}\,h(y) = V_t\big(\sum_k a_t(\cdot, y)\,h(y), f\big),$$

where $\Sigma_k a_t(\cdot, y)h(y)$ is an analogous Riemann sum. If we let k tend to infinity the left side tends to (h, f) so that $\Sigma_k a_t(\cdot, y)h(y)$ is weakly convergent in $\mathfrak{H}$ to $A_t h$. But then the sum converges pointwise and passing to the limit we get

$$\int_D a_t(x, y)h(y)dy = A_t h(x) .$$

Putting (see (21))

$$g_t(\cdot, y) = (1+L_t)^{-1}a_t(\cdot, y) \tag{24}$$

(t large enough), we get

$$U_t(g_t(\cdot, y), f) = V_t(a_t(\cdot, y), f) = \overline{f(y)} . \tag{25}$$

This relation defines Green's function $g_t(x, y)$ uniquely and it follows as above that

$$\int_D g_t(x, y)h(y)dy = G_t h(x) .$$

A fundamental solution. If the coefficients $a_{\mu\nu}$ in the expression $V(f, g)$ are sufficiently differentiable, we get, integrating by parts in $V(f, g)$, supposing that $f, g \in H$ and vanish outside compact sets in D,

$$V(f, g) = (vf, g) = (f, vg) , \tag{26}$$

where v is the differential operator

$$v = (-1)^m \sum_{\substack{|\mu|=m \\ |\nu|=m}} D^\mu(a_{\mu\nu}(x) D^\nu) . \tag{27}$$

In the rest of this paragraph we shall suppose the $a_{\mu\nu}$ constant. We denote by v_t the differential operator $v+t$ and by $v_t(\xi)$ the polynomial

$$v_t(\xi) = \sum_{\substack{|\mu|=m \\ |\nu|=m}} a_{\mu\nu} \xi_1^{\mu_1+\nu_1} \ldots \xi_n^{\mu_n+\nu_n} + t \quad .$$

in the components ξ_i of the n-dimensional vector ξ.

The inverse Fourier transform

$$A_t(x) = (2\pi)^{-n} \int e^{-ix\xi}[v_t(\xi)]^{-1} d\xi$$

of $[v_t(\xi)]^{-1}$, where $x\xi = x_1\xi_1 + \ldots + x_n\xi_n$, is analytic for $x \neq 0$, and $A_t(x-y)$ is a fundamental solution belonging to v_t, that is,

$$\int_E A_t(x-y)\, v_t f(y)\, dy = f(x) \tag{28}$$

for any infinitely differentiable function $f(x)$ that vanishes outside a bounded region (Schwartz [18, pp. 149–151]). Integrating by parts in (28), chosing functions f that vanish in a neighbourhood of $x = 0$, we find that

$$v_t A_t(x) = 0 \tag{29}$$

for $x \neq 0$.

From Gårding [9] we quote the following lemma:

LEMMA. *Let $p(\xi)$ be a polynomial of degree μ whose coefficients are majorized by a number c_1 and suppose that $|p(\xi)| \geqq c_2(1+|\xi|^\mu)$. Then the (generalized) inverse Fourier transform of $[p(x)]^{-1}$ is an infinitely differentiable function $P(x)$ in the region $x \neq 0$ satisfying*

$$|D^\alpha P(x)| \leqq C e_{|\alpha|}(x)(1+|x|)^{-N}, \begin{cases} e_{|\alpha|}(x) = 1 & \text{when } \mu-|\alpha|-n > 0\,, \\ e_{|\alpha|}(x) = |x|^{\mu-|\alpha|-n-\varepsilon} & \text{when } \mu-|\alpha|-n \leqq 0\,. \end{cases}$$

Here $N \geqq 0$ and $1 > \varepsilon > 0$ are arbitrary, and the number C depends on c_1, c_2, $|\alpha|$, N, and ε, but is otherwise independent of the polynomial p.

Putting $t^{\frac{1}{2}/m} = s$ we have

$$A_t(x) = s^{n-2m} A_1(sx)\,. \tag{30}$$

For $A_t(x)$ we thus have the estimate

$$D^\nu A_t(x) \leqq C s^{n+|\nu|-2m} e_{|\nu|}(sx)(1+|sx|)^{-N}. \tag{31}$$

In this formula the functions $e_{|\nu|}$ are defined as in the lemma but with the number μ replaced by $2m$. The formula is valid for all $v_t(\xi)$ satisfying $|a_{\mu\nu}| < c_1$ and $|v_1(\xi)| \geqq c_2(1+|\xi|^{2m})$.

For a later purpose we note that for y arbitrary but fixed, the function $A_t(\cdot - y)$ belongs to $\mathfrak{H}_m$. In fact, writing

$$A_t(N, x-y) = (2\pi)^{-n} \int\limits_{|\xi| \leqq N} e^{-i(x-y)\xi}[v_t(\xi)]^{-1} d\xi$$

the function $A_t(N, x-y)$ is infinitely differentiable in E, and from Plancherel's theorem it follows that $A_t(N, x-y)$ and its derivatives up to at least the order m converge in square mean on E when N tends to infinity. Hence, as $A_t(N, x-y)$ itself tends to $A_t(x-y)$, the statement follows.

Asymptotic formulas. In the following we are going to show that the function

$$A_t(y, x-y) = (2\pi)^{-n} \int e^{-i(x-y)\xi} [v_t(y, \xi)]^{-1} d\xi ,$$

where

$$v_t(y, \xi) = \sum_{\substack{|\mu|=m \\ |\nu|=m}} a_{\mu\nu}(y) \xi_1^{\mu_1+\nu_1} \dots \xi_n^{\mu_n+\nu_n} + t ,$$

behaves asymptotically as $g_t(x, y)$ for large values of t or more precisely, putting $\frac{1}{2}n/m = \sigma$,

$$\lim_{t\to+\infty} t^{1-\sigma}\big(A_t(y, x-y) - g_t(x, y)\big) = 0 ,$$

if at least one of the points x and y belongs to $D^\star$.

When $x \neq y$, we have by Riemann-Lebesgue's lemma

$$\lim_{t\to+\infty} t^{1-\sigma} A_t(y, x-y) = \lim_{s\to+\infty} A_1\big(y, s(x-y)\big) = 0 ,$$

and from (30) it follows that

$$\lim_{t\to+\infty} t^{1-\sigma} A_t(y, 0) = A_1(y, 0) = (2\pi)^{-n} \int [v_1(y, \xi)]^{-1} d\xi .$$

The integral on the right in the last formula can be transformed in the following way (cf. Gårding [7]): we put $\xi = p\eta$, where $\eta = (\eta_1, \dots, \eta_n)$ is restricted to the domain $\Omega = \{\eta \mid v_0(y, \eta) = 1\}$ and $p = v_0(y, \xi)^{\frac{1}{2}/m}$. Putting $d\xi = p^{n-1} dp\, d\omega_\eta$ we have

$$\int_{p<p_0} d\xi = V_y p_0^n = \int_0^{p_0} p^{n-1} dp \int_\Omega d\omega_\eta ,$$

where $V_y = \int_{p<1} d\xi$. Hence we get $\int_\Omega d\omega_\eta = n V_y$, which gives

$$\int [v_1(y, \xi)]^{-1} d\xi = \int_0^\infty \frac{p^{n-1} dp}{1+p^{2m}} \int_\Omega d\omega_\eta = \Gamma(1+\sigma)\, \Gamma(1-\sigma)\, V_y .$$

Hence we can formulate our proposition as follows (compare Gårding [7]).

THEOREM 2. *For the asymptotic behaviour of $g_t(x, y)$ we have the formula*

$$\lim_{t\to+\infty} t^{1-\sigma} g_t(x, y) = \delta_{xy} (2\pi)^{-n}\, \Gamma(1+\sigma)\, \Gamma(1-\sigma)\, V_y , \tag{32}$$

where $\delta_{xy} = 0$ when $x \neq y$, $\delta_{xx} = 1$ and at least one of the points x and y belongs to $D^\star$.

PROOF. Writing

$$\gamma_t(\cdot, y) = g_t(\cdot, y) - a_t(\cdot, y)$$

we have by (24)

$$\gamma_t(\cdot, y) = [(1+L_t)^{-1} - 1]\, a_t(\cdot, y)\,,$$

so that by (4)

$$\gamma_t(x, y) = O\big(t^{-\frac{1}{2}(1-\sigma)}\big)\, V_t\big(\gamma_t(\cdot, y), \gamma_t(\cdot, y)\big)^{\frac{1}{2}}$$

$$= O\big(t^{-\frac{1}{2}(1-\sigma)}\big) \frac{|L_t|_t}{1-|L_t|_t}\, V_t\big(a_t(\cdot, y), a_t(\cdot, y)\big)^{\frac{1}{2}}\,.$$

But applying (23) and (4)

$$V_t\big(a_t(\cdot, y), a_t(\cdot, y)\big) = \overline{a_t(y, y)} = O\big(t^{-\frac{1}{2}(1-\sigma)}\big)\, V_t\big(a_t(\cdot, y), a_t(\cdot, y)\big)^{\frac{1}{2}}\,,$$

so that

(33) $$V_t\big(a_t(\cdot, y), a_t(\cdot, y)\big)^{\frac{1}{2}} = O\big(t^{-\frac{1}{2}(1-\sigma)}\big)\,,$$

and hence

$$\gamma_t(x, y) = \frac{|L_t|_t}{1-|L_t|_t}\, O\big(t^{-(1-\sigma)}\big)\,.$$

But as $|L_t|_t = o(1)$, we get

(34) $$\lim_{t\to+\infty} t^{1-\sigma}[g_t(x, y) - a_t(x, y)] = 0\,,$$

so that it is sufficient to prove the theorem for $a_t(x, y)$.

Noting that the convergence in (34) is uniform for all x and y in $D+S$ and that the uniform estimate $a_t(y, y) = O\big(t^{-(1-\sigma)}\big)$ follows directly from (33) we get

(35) $$g_t(y, y) = O\big(t^{-(1-\sigma)}\big)$$

uniformly, an estimate that will be used later.

Let φ be a function in L. From the fact that $A_t(y, \cdot - y)$ belongs to $\mathfrak{H}_m$ it follows that $\varphi A_t(y, \cdot - y)$ belongs to $\mathfrak{L}_m$, because if $A_t(y, \cdot - y)$ is approximated by a sequence $\{f_i\}$ in H, it is easily proved by means of (3) that $\varphi A_t(y, \cdot - y)$ is approximated by the sequence $\{\varphi f_i\}$ in L. Let y be a point in $D^\star$. Then we can find two functions φ and ψ in L and two neighbourhoods ω and $\omega^\star$ of y, $\omega^\star \subset \omega \subset D^\star$, such that

$$\varphi(x) = 1 \quad \text{when} \quad x \in \omega\,,$$
$$\psi(x) = 1 \quad \text{when} \quad x \in \omega^\star\,,$$
$$\psi(x) = 0 \quad \text{when} \quad x \notin \omega\,,$$
$$0 \leqq \psi(x) \leqq 1 \quad \text{for} \quad x \text{ arbitrary}\,.$$

If f is an arbitrary function in H, we put $f = f_1 + f_2$, where $f_1 = \psi f$ and

$f_2 = (1-\psi)f$. Obviously $f_1 = 0$ outside ω and $f_2 = 0$ in $\omega^\star$. Let us denote by $V_t^y(g, h)$ the form obtained from $V_t(g, h)$ by making its coefficients $a_{\mu\nu}$ constant and equal to their values at the point y, and let v_t^y be the corresponding differential operator. We then have

$$V_t^y(\varphi A_t(y, \cdot - y), f) = V_t^y(\varphi A_t(y, \cdot - y), f_1) + V_t^y(\varphi A_t(y, \cdot - y), f_2) .$$

Integrating by parts in each term on the right side, noting that $A_t(y, x-z)$ is the fundamental solution belonging to v_t^y, we get

$$V_t^y(\varphi A_t(y, \cdot - y), f_1) = \int_D \varphi(x)\, A_t(y, x-y)\, v_t^y \overline{f_1(x)}\, dx = \overline{f(y)}$$

and

$$V_t^y(\varphi A_t(y, \cdot - y), f_2) = \int_{D-\omega^\star} v_t^y[\varphi(x) A_t(y, x-y)]\, \overline{f_2(x)}\, dx .$$

Hence, noting that $|f_2(x)| \leqq |f(x)|$,

$$|V_t^y(\varphi A_t(y, \cdot - y), f_2)| \leqq C \int_{D-\omega^\star} \sum_{|\nu| \leqq 2m-1} |D^\nu A_t(y, x-y)|\, |f(x)|\, dx ,$$

denoting here and in the sequel by C a constant, not always the same. Estimating the right side by means of (4) and (31) we get the result

$$V_t^y(\varphi A_t(y, \cdot - y), f) = \overline{f(y)} + O(t^{-N})\, V_t(f, f)^{\frac{1}{2}} \tag{36}$$

for N arbitrary. It should be noted that the estimate is not uniform in y.

We then estimate the difference

$$\begin{aligned} V_t^y(\varphi A_t(y, \cdot - y), f) &- V_t(\varphi A_t(y, \cdot - y), f) \\ &= \int_D \sum_{\substack{|\mu|=m \\ |\nu|=m}} \big(a_{\mu\nu}(y) - a_{\mu\nu}(x)\big)\, D^\mu[\varphi(x) A_t(y, x-y)]\, D^\nu \overline{f(x)}\, dx \\ &\leqq C\, V_t(f, f)^{\frac{1}{2}} \sum_{\substack{|\mu|=m \\ |\nu|=m}} \Big(\int |a_{\mu\nu}(y) - a_{\mu\nu}(x)|^2\, |D^\mu[\varphi(x) A_t(y, x-y)]|^2\, dx\Big)^{\frac{1}{2}} . \end{aligned}$$

The $a_{\mu\nu}$ satisfying the Lipschitz condition (14) in y, we estimate the derivatives of $A_t(y, \cdot - y)$ of orders $\leqq m$ according to (31) by

$$D^\nu A_t(y, x-y) \leqq C s^{-m+\frac{1}{2}n-\varepsilon} |x-y|^{-\frac{1}{2}n-\varepsilon} ,$$

where $0 < \varepsilon < \alpha_y$, and get

$$\int_D |a_{\mu\nu}(y) - a_{\mu\nu}(x)|^2\, |D^\mu[\varphi(x) A_t(y, x-y)]|^2\, dx = O\big(t^{-(1-\sigma)-\varepsilon/m}\big) = o\big(t^{-(1-\sigma)}\big),$$

so that

$$(37)\qquad V_t^y(\varphi A_t(y, \cdot - y), f) - V_t(\varphi A_t(y, \cdot - y), f) = o(t^{-\frac{1}{2}(1-\sigma)})\, V_t(f, f)^{\frac{1}{2}} .$$

By continuity the estimates (36) and (37) are valid also for an arbitrary f in $\mathfrak{H}_m$. Hence, noting (23), we get for f arbitrary in $\mathfrak{H}$

$$\begin{aligned} V_t(a_t(\cdot, y) - \varphi A_t(y, \cdot - y), f) &= [V_t(a_t(\cdot, y), f) - V_t^y(\varphi A_t(y, \cdot - y), f)] + \\ &\quad + [V_t^y(\varphi A_t(y, \cdot - y), f) - V_t(\varphi A_t(y, \cdot - y), f)] \\ &= o(t^{-\frac{1}{2}(1-\sigma)})\, V_t(f, f)^{\frac{1}{2}} . \end{aligned}$$

Hence, writing

$$\delta_t(x, y) = a_t(x, y) - \varphi(x) A_t(y, x-y) ,$$

we have

$$V_t(\delta_t(\cdot, y), \delta_t(\cdot, y))^{\frac{1}{2}} = o(t^{-\frac{1}{2}(1-\sigma)}) .$$

By (4) we get

$$|\delta_t(x, y)| = O(t^{-\frac{1}{2}(1-\sigma)})\, V_t(\delta_t(\cdot, y), \delta_t(\cdot, y))^{\frac{1}{2}} = o(t^{-(1-\sigma)}) ,$$

which completes the proof.

From Theorem 1 it follows that if R is hermitian there exists in $\mathfrak{L}$ a complete orthonormal system of eigenfunctions of the vibration problem. The set of corresponding eigenvalues is bounded from below and has no finite limit point.

We are now in a position to prove the following theorem (cf. Gårding [7]).

Theorem 3. *If R is hermitian and if $\{\varphi_i\}$ is a complete system of eigenfunctions of the problem that is orthonormalized with respect to the norm $(\varphi, \varphi)^{\frac{1}{2}}$ and if φ_i are labelled so that the corresponding eigenvalues λ_i form a non-decreasing sequence, we have, denoting by $N(t)$ the number of eigenvalues less than a number t, the following asymptotic relations when $t \to +\infty$:*

$$(38)\qquad N(t) = t^{\sigma} (2\pi)^{-n} \int_D V_y dy\, (1 + o(1))$$

and

$$(39)\qquad \lim_{N \to \infty} N^{-1} \sum_1^N \varphi_i(x) \overline{\varphi_i(y)} = \delta_{xy} V_y \Big/ \int_D V_y dy ,$$

where at least one of the points x and y belongs to $D^\star$, the notations being the same as in Theorem 2.

PROOF. Expanding the kernel $g_t(x, y)$ in terms of the functions $\varphi_i(x)/(\lambda_i+t)^{\frac{1}{2}}$ which form a complete orthonormal system in the norm square U_t we get

$$g_t(x, y) = \sum \varphi_i(x)\,\overline{\varphi_i(y)}/(\lambda_i+t)\,,$$

where the series converges pointwise. Combining this formula with (32) we get

$$\text{(40)}\qquad \lim_{t\to+\infty} t^{1-\sigma} \sum \varphi_i(x)\,\overline{\varphi_i(y)}/(\lambda_i+t) = \delta_{xy}(2\pi)^{-n}\,\Gamma(1+\sigma)\,\Gamma(1-\sigma)\,V_y\,.$$

Putting $x = y$ in (40) and integrating over $D^\star$ we get

$$\text{(41)}\qquad \lim_{t\to+\infty} t^{1-\sigma} \sum 1/(\lambda_i+t) = (2\pi)^{-n}\,\Gamma(1+\sigma)\,\Gamma(1-\sigma)\int_D V_y\,dy$$

by the theorem of Lebesgue using (35). From (41) we get (38) by means of a Tauberian theorem due to Hardy and Littlewood (Pleijel [16, pp. 3–5]). By the same theorem applied to (40) with $x = y$ we get

$$\text{(42)}\qquad \sum_{\lambda_i<t} |\varphi_i(x)|^2 = t^\sigma(2\pi)^{-n}\,V_y\big(1+o(1)\big)\,.$$

If θ is a complex number and $|\theta| = 1$ we have by (40) if $x \neq y$

$$\lim_{t\to+\infty} t^{1-\sigma} \sum \frac{|\varphi_i(x)+\theta\varphi_i(y)|^2}{\lambda_i+t} = (2\pi)^{-n}\,\Gamma(1+\sigma)\,\Gamma(1-\sigma)\,(V_x+V_y)\,.$$

Hence, using the Tauberian theorem once more and then the formula (42) we get

$$\sum_{\lambda_i<t} \varphi_i(x)\,\overline{\varphi_i(y)} = t^\sigma(2\pi)^{-n}\,V_y\big(\delta_{xy}+o(1)\big)\,.$$

Combining this with (38) we get (39).

REMARK. It follows from results announced by Keldych [11] that (38) is true also in the non-selfadjoint case provided that we put

$$N(t) = \sum_{\mathrm{Re}\,\lambda_i<t} 1\,,$$

where each eigenvalue is counted with its multiplicity.

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UNIVERSITY OF LUND, SWEDEN