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SOME INTEGRAL FORMULAS FOR CLOSED HYPERSURFACES

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Dedicated to the author's mother for her 70th birthday

Introduction. Let V^n be an orientable hypersurface twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1 \geqq 3$ dimensions, and let $\varkappa_1, \ldots, \varkappa_n$ be the n principal curvatures at a point P of V^n. The r-th mean curvature M_r of V^n at the point P is defined to be the r-th elementary symmetric function of $\varkappa_1, \ldots, \varkappa_n$ divided by the number of terms, that is,

$$\binom{n}{r} M_r = \sum \varkappa_1 \ldots \varkappa_r, \qquad r = 1, \ldots, n \,. \tag{0.1}$$

It is convenient to define $M_0 = 1$. Let $p = p(P)$ denote the oriented distance from a fixed point O in E^{n+1} to the tangent hyperplane of V^n at P, and let dA be the area element of V^n at P. The purpose of this paper is first to show that for an orientable hypersurface V^n with a closed boundary V^{n-1} of dimension $n-1$ the integrals

$$\int_{V^n} (M_{r+1} p + M_r) dA, \qquad r = 0, \ldots, n-1 \,,$$

can be expressed as integrals over the boundary V^{n-1}. These relations, which have been obtained by W. Scherrer [5] for $n = 2$, are then used to prove the following three theorems concerning closed hypersurfaces.

THEOREM 1. *Let V^n be a closed orientable hypersurface twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1 \geqq 3$ dimensions, then*

$$\int_{V^n} M_{r+1} p \, dA + \int_{V^n} M_r \, dA = 0, \qquad r = 0, \ldots, n-1 \,. \tag{0.2}$$

For convex hypersurfaces, these formulas have been obtained by H. Minkowski for $n = 2$ and by T. Kubota for a general n (for references see [1, p. 64]).

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THEOREM 2. *Let V^n satisfy the same conditions as in Theorem* 1. *Suppose that there exist a point O in E^{n+1} and an integer s, $1 \leqq s \leqq n-1$, such that $M_s > 0$ and either $p \leqq -M_{s-1}/M_s$ or $p \geqq -M_{s-1}/M_s$ at all points of V^n. Then V^n is a hypersphere.*

In the case where the hypersurface V^n is convex, $s = 1$, and the equality holds in the last condition, this theorem has been obtained by K.-P. Grotemeyer [2] for $n = 2$ and by W. Süss [6] for a general n. Grotemeyer and Süss have also shown that a convex hypersurface satisfying a condition of the form $(-p)^s = 1/M_s$ is a hypersphere. It may be mentioned that this result can also be obtained for a more general class of hypersurfaces by using Theorem 1 and the method of Süss.

THEOREM 3. *Let V^n satisfy the same conditions as in Theorem* 1. *Suppose that there exist a point O in E^{n+1} and an integer s, $1 \leqq s \leqq n$, such that at all points of V^n the function p is of the same sign, $M_i > 0$ for $i = 1, \ldots, s$, and M_s is constant. Then V^n is a hypersphere.*

In the case $n = 2$, Theorem 3 reduces to the known results that a closed surface with constant Gaussian curvature is a sphere, and that a closed surface with constant mean curvature is necessarily a sphere if there exists a point which is on the same side of all tangent planes of the surface. For convex hypersurfaces of arbitrary dimensions the theorem is due to W. Süss (for references see [1, p. 118]). The proof of Theorem 3 is similar to that of Süss.

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1. Preliminaries. In a Euclidean space E^{n+1} of dimension $n+1 \geqq 3$ let us consider a fixed orthogonal frame $O\mathfrak{Y}_1 \ldots \mathfrak{Y}_{n+1}$ with a point O as the origin. With respect to this orthogonal frame we define the vector product of n vectors $A_1, \ldots, A_n$ in E^{n+1} to be the vector A_{n+1}, denoted by $A_1 \times \ldots \times A_n$, satisfying the following conditions:

(a) the vector A_{n+1} is normal to the n-dimensional space determined by the vectors $A_1, \ldots, A_n$,

(b) the magnitude of the vector A_{n+1} is equal to the volume of the parallelepiped whose edges are the vectors $A_1, \ldots, A_n$,

(c) the two frames $OA_1 \ldots A_n A_{n+1}$ and $O\mathfrak{Y}_1 \ldots \mathfrak{Y}_{n+1}$ have the same orientation.

Let σ be a permutation on the n numbers $1, \ldots, n$, then

$$A_{\sigma(1)} \times \ldots \times A_{\sigma(n)} = (\operatorname{sgn} \sigma)\, A_1 \times \ldots \times A_n\,, \tag{1.1}$$

where $\operatorname{sgn}\sigma$ is $+1$ or -1 according as the permutation σ is even or odd. Let $i_1, \ldots, i_{n+1}$ be the unit vectors from the origin O in the directions of the vectors $\mathfrak{Y}_1, \ldots, \mathfrak{Y}_{n+1}$ and let A^j_α, $j = 1, \ldots, n+1$, be the components of the vector A_α, $\alpha = 1, \ldots, n$, with respect to the frame $O\mathfrak{Y}_1 \ldots \mathfrak{Y}_{n+1}$, then the scalar product of any two vectors A_α and A_β and the vector product of n vectors $A_1, \ldots, A_n$ are, respectively,

$$A_\alpha \cdot A_\beta = \sum_i A^i_\alpha A^i_\beta , \tag{1.2}$$

$$A_1 \times A_2 \times \ldots \times A_n = (-1)^n \begin{vmatrix} i_1 & i_2 & \ldots & i_{n+1} \\ A^1_1 & A^2_1 & \ldots & A^{n+1}_1 \\ \cdot & \cdot & \cdot & \cdot \\ A^1_n & A^2_n & \ldots & A^{n+1}_n \end{vmatrix} . \tag{1.3}$$

If A^j_α are differentiable functions of n variables $x^1, \ldots, x^n$, then by equation (1.3) and the differentiation of determinants

$$\frac{\partial}{\partial x^\alpha}(A_1 \times \ldots \times A_n) = \sum_{\beta=1}^{n} \left(A_1 \times \ldots \times A_{\beta-1} \times \frac{\partial A_\beta}{\partial x^\alpha} \times A_{\beta+1} \times \ldots \times A_n \right). \tag{1.4}$$

Now we consider a hypersurface V^n twice differentiably imbedded in E^{n+1}. Let $(y^1, \ldots, y^{n+1})$ be the coordinates of a point P in E^{n+1} with respect to the orthogonal frame $O\mathfrak{Y}_1 \ldots \mathfrak{Y}_{n+1}$. Then V^n can be given by the parametric equations

$$y^i = f^i(x^1, \ldots, x^n), \qquad i = 1, \ldots, n+1 , \tag{1.5}$$

or the vector equation

$$Y = F(x^1, \ldots, x^n) , \tag{1.6}$$

where y^i and f^i are respectively the components of the two vectors Y and F, the parameters $x^1, \ldots, x^n$ take values in a simply connected domain D of the n-dimensional real number space, $f^i(x^1, \ldots, x^n)$ are of the second class and the Jacobian matrix $\|\partial y^i / \partial x^\alpha\|$ is of rank n at all points of D. (See, for instance, also for the remainder of this section, [7, Chap. IX].) For quantities of the V^n, tensor notation with Greek indices will be used. In particular, the summation convention is adopted for these indices. If we denote the vector $\partial Y / \partial x^\alpha$ by Y_α for $\alpha = 1, \ldots, n$, then the first fundamental form of V^n at a point P is

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad g_{\alpha\beta} = Y_\alpha \cdot Y_\beta , \tag{1.7}$$

where the matrix $\|g_{\alpha\beta}\|$ is positive definite and thus, the determinant

(1.8) $$g = |g_{\alpha\beta}| > 0 .$$

Let N be the unit normal vector at a point P of V^n and N_α the vector $\partial N/\partial x^\alpha$, then

(1.9) $$N_\alpha = - b_{\alpha\beta} g^{\beta\gamma} Y_\gamma ,$$

where

(1.10) $$b_{\alpha\beta} = b_{\beta\alpha} = - N_\alpha \cdot Y_\beta$$

are the coefficients of the second fundamental form of V^n and $g^{\beta\gamma}$ denotes the cofactor of $g_{\beta\gamma}$ in g divided by g so that

(1.11) $$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma ,$$

δ^α_γ being the Kronecker deltas. The n principal curvatures $\varkappa_1, \ldots, \varkappa_n$ of V^n at P are the roots of the determinant equation

(1.12) $$|b_{\alpha\beta} - \varkappa g_{\alpha\beta}| = 0 .$$

From equations (0.1) and (1.12) follow immediately

(1.13) $$M_n = b/g, \qquad n M_1 = b_{\alpha\beta} g^{\alpha\beta}, \qquad n M_{n-1} = g_{\alpha\beta} B^{\alpha\beta}/g ,$$

where

(1.14) $$b = |b_{\alpha\beta}| ,$$

and $B^{\alpha\beta}$ is the cofactor of $b_{\alpha\beta}$ in b.

The area element of V^n at P is given by

(1.15) $$dA = g^{1/2} dx^1 \ldots dx^n .$$

Now we choose the direction of the unit normal vector N in such a way that the two frames $P Y_1 \ldots Y_n N$ and $O \mathfrak{Y}_1 \ldots \mathfrak{Y}_{n+1}$ have the same orientation. Then from equations (1.3) and (1.15) it follows that

(1.16) $$g^{1/2} N = Y_1 \times \ldots \times Y_n ,$$

(1.17) $$|Y_1, \ldots, Y_n, N| = g^{1/2} .$$

2. Proof of the formula (0.2) for $r = 0$. At first, we observe that the vector $Y_1 \times \ldots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \ldots \times Y_n$ is perpendicular to the normal vector N and can therefore be written in the form

(2.1) $$Y_1 \times \ldots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \ldots \times Y_n = a^{\alpha\beta} Y_\beta .$$

Taking the scalar products of the both sides of equations (2.1) with the vector Y_γ and making use of equations (1.1), (1.3), (1.7), (1.16), we obtain

(2.2) $$a^{\alpha\beta} g_{\beta\gamma} = -g^{1/2} \delta^\alpha_\gamma , \qquad \alpha, \gamma = 1, \ldots, n .$$

Solving equations (2.2) for $a^{\alpha\beta}$ for each fixed α and substituting the results in equations (2.1), we are led to

$$(2.3)\qquad Y_1\times\ldots\times Y_{\alpha-1}\times N\times Y_{\alpha+1}\times\ldots\times Y_n=-g^{1/2}\,g^{\alpha\beta}\,Y_\beta\,.$$

Making use of equations (1.4), (1.9), (1.13) and (1.16), it is easily seen that

$$(2.4)\qquad \sum_{\alpha=1}^{n}\frac{\partial}{\partial x^\alpha}(Y_1\times\ldots\times Y_{\alpha-1}\times N\times Y_{\alpha+1}\times\ldots\times Y_n)$$
$$=\sum_{\alpha=1}^{n}Y_1\times\ldots\times Y_{\alpha-1}\times N_\alpha\times Y_{\alpha+1}\times\ldots\times Y_n=-ng^{1/2}\,M_1N\,.$$

Thus, from equations (2.3) and (2.4),

$$(2.5)\qquad ng^{1/2}\,M_1N=\frac{\partial}{\partial x^\alpha}(g^{1/2}\,g^{\alpha\beta}\,Y_\beta)\,.$$

Taking the scalar products of the both sides of equation (2.5) with the vector Y, we obtain in consequence of the relations (1.7) and (1.11)

$$(2.6)\qquad nM_1\,p\,g^{1/2}=\frac{\partial}{\partial x^\alpha}(g^{1/2}\,g^{\alpha\beta}\,\eta_\beta)-ng^{1/2}\,,$$

where we have put

$$(2.7)\qquad p=Y\cdot N,\qquad \eta_\alpha=Y\cdot Y_\alpha\,.$$

Now let us consider a hypersurface V^n having a closed boundary V^{n-1} and twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1\geqq 3$ dimensions. Integrating equation (2.6) with respect to $x^1,\ldots,x^n$ over this hypersurface V^n and applying the general Green's theorem (cf., for instance, [4, pp. 75–76]) to the first term on the right side of equation (2.6), we then obtain

$$(2.8)\qquad \int_{V^n}M_1p\,dA+A=n^{-1}\int_{V^{n-1}}\sum_{\alpha=1}^{n}(-1)^{\alpha-1}\,g^{1/2}\,g^{\alpha\beta}\,\eta_\beta\,dx^1\ldots dx^{\alpha-1}\,dx^{\alpha+1}\ldots dx^n\,.$$

In particular, when V^n is closed and orientable the integral on the right side of equation (2.8) drops out and hence the formula (0.2) for $r=0$ follows.

3. Proof of the formula (0.2) for a general r. In this section we shall use the formula (2.8) to derive an analogous formula for a general r. To this end, in E^{n+1} we first consider a hypersurface $\bar{V}^n$ parallel to a hypersurface V^n with a closed boundary V^{n-1} so that $\bar{V}^n$ and V^n have the same normals. It is evident that the vector equation of $\overline{V}^n$ can be written in the form

$$\overline{Y} = Y - tN\,, \tag{3.1}$$

where t is a real parameter. From equations (3.1), $N \cdot N = 1$ and $N \cdot \overline{Y}_\alpha = 0$, it follows immediately that $\partial t / \partial x^\alpha = 0$ and therefore that t is constant. Making use of equations (1.7), (1.9), (1.10) and their analogous ones for $\overline{V}^n$ we obtain the coefficients of the first and the second fundamental forms of $\overline{V}^n$:

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + 2b_{\alpha\beta}t + b_{\alpha\varrho}b_{\beta\sigma}g^{\varrho\sigma}t^2 = (g_{\alpha\varrho} + b_{\alpha\varrho}t)(\delta^\varrho_\beta + b_{\beta\sigma}g^{\varrho\sigma}t)\,, \tag{3.2}$$

$$\bar{b}_{\alpha\beta} = b_{\alpha\beta} + b_{\alpha\varrho}b_{\beta\sigma}g^{\varrho\sigma}t = b_{\alpha\varrho}(\delta^\varrho_\beta + b_{\beta\sigma}g^{\varrho\sigma}t)\,, \tag{3.3}$$

from which it follows easily by an elementary calculation that

$$\bar{b} = b\Delta\,, \tag{3.4}$$

$$\bar{g} = g\Delta^2\,, \tag{3.5}$$

$$|\overline{R}\bar{b}_{\alpha\beta} - \bar{g}_{\alpha\beta}| = |(\overline{R} - t)b_{\alpha\beta} - g_{\alpha\beta}|\,\Delta\,, \tag{3.6}$$

where $\bar{g}$ and b are defined by equations similar to (1.8), (1.14), and

$$\Delta = |\delta^\beta_\alpha + b_{\alpha\varrho}g^{\varrho\beta}t|\,, \tag{3.7}$$

$$\overline{R}_i = 1/\bar{\varkappa}_i, \qquad i = 1, \ldots, n\,, \tag{3.8}$$

$\bar{\varkappa}_i$ being the principal curvatures of $\overline{V}^n$. In consequence of equations (3.4), (3.5), (3.6) and (1.12), (1.13), (1.15) together with their analogues for $\overline{V}^n$, we have

$$\overline{M}_n\,d\overline{A} = M_n\,dA\,, \tag{3.9}$$

$$\overline{R}_i = R_i + t\,, \tag{3.10}$$

where $d\overline{A}$ is the area element of $\overline{V}^n$ and $R_i = 1/\varkappa_i$. Moreover, let $\bar{g}^{\alpha\beta}$ be the cofactor of $\bar{g}_{\alpha\beta}$ in $\bar{g}$ divided by $\bar{g}$, then from equations (2.7), (3.1), (3.2) and (3.7) we obtain

$$\bar{\eta}_\beta = \eta_\beta + tb_{\beta\delta}g^{\delta\gamma}\eta_\gamma = \eta_\gamma(\delta^\gamma_\beta + tb_{\beta\delta}g^{\delta\gamma})\,, \tag{3.11}$$

$$\bar{g}\,\bar{g}^{\alpha\beta}\,\bar{\eta}_\beta = \Phi^\alpha\Delta\,, \tag{3.12}$$

where $\bar{\eta}_\beta = \overline{Y} \cdot \overline{Y}_\beta$ and

$$\Phi^{\alpha} = \begin{vmatrix} g_{11}+tb_{11} & g_{12}+tb_{12} & \cdots & g_{1n}+tb_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{\alpha-1,1}+tb_{\alpha-1,1} & g_{\alpha-1,2}+tb_{\alpha-1,2} & \cdots & g_{\alpha-1,n}+tb_{\alpha-1,n} \\ \eta_1 & \eta_2 & \cdots & \eta_n \\ g_{\alpha+1,1}+tb_{\alpha+1,1} & g_{\alpha+1,2}+tb_{\alpha+1,2} & \cdots & g_{\alpha+1,n}+tb_{\alpha+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}+tb_{n1} & g_{n2}+tb_{n2} & \cdots & g_{nn}+tb_{nn} \end{vmatrix} . \tag{3.13}$$

Now let

$$\Phi^{\alpha} = \sum_{r=0}^{n-1} \binom{n-1}{r} \Theta_r^{\alpha} t^r , \tag{3.14}$$

then it is obvious that

$$\Theta_0^{\alpha} = g\, g^{\alpha\beta}\, \eta_\beta, \qquad \Theta_{n-1}^{\alpha} = B^{\alpha\beta}\, \eta_\beta . \tag{3.15}$$

By means of equations (0.1) and (3.8), equation (2.8) for $\overline{V}^n$ can be written as

$$\int_{\overline{V}^n} \overline{p}(\Sigma \overline{R}_1 \overline{R}_2 \ldots \overline{R}_{n-1}) \overline{M}_n\, d\overline{A} + n \int_{\overline{V}^n} \overline{R}_1 \overline{R}_2 \ldots \overline{R}_n \overline{M}_n\, d\overline{A} = \int_{\overline{V}^{n-1}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} \overline{g}^{1/2} \overline{g}^{\alpha\beta} \overline{\eta}_\beta\, dx^1 \ldots dx^{\alpha-1} dx^{\alpha+1} \ldots dx^n , \tag{3.16}$$

where $\overline{p} = \overline{Y}\cdot N = p-t$ and $\overline{V}^{n-1}$ is the boundary of $\overline{V}^n$. Substitution of equations (3.5), (3.9), (3.10), (3.12) and (3.14) in equation (3.16) yields immediately

$$\int_{V^n} (p-t) \sum_{i=0}^{n-1} (n-i)(\Sigma R_1 \ldots R_i)\, t^{n-i-1} M_n\, dA + n \int_{V^n} \sum_{i=0}^{n} (\Sigma R_1 \ldots R_i)\, t^{n-i} M_n\, dA = \int_{V^{n-1}} \sum_{\alpha=1}^{n} \sum_{r=0}^{n-1} (-1)^{\alpha-1} \binom{n-1}{r} g^{-1/2}\, \Theta_r^{\alpha} t^r\, dx^1 \ldots dx^{\alpha-1} dx^{\alpha+1} \ldots dx^n , \tag{3.17}$$

which is an identity in t. Hence, by equating the coefficients of t^r on the both sides of equation (3.17) and using (0.1), we arrive at the generalization of the formula (2.8) mentioned in the introduction:

$$(3.18)\quad \int_{V^n} M_{r+1} p\, dA + \int_{V^n} M_r\, dA$$
$$= n^{-1} \int_{V^{n-1}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} g^{-1/2}\, \Theta_r^\alpha\, dx^1 \dots dx^{\alpha-1}\, dx^{\alpha+1} \dots dx^n\,,$$
$$r = 0, \dots, n-1\,,$$

from which follow immediately the formulas (0.2) when V^n is closed and orientable.

4. Proofs of Theorems 2 and 3. In order to prove Theorem 2 we first observe that because of $M_s > 0$ the assumptions $p \leqq -M_{s-1}/M_s$ and $p \geqq -M_{s-1}/M_s$ are respectively equivalent to $M_s p + M_{s-1} \leqq 0$ and $M_s p + M_{s-1} \geqq 0$. From (0.2) for $r = s-1$ we have

$$\int_{V^n} (M_s p + M_{s-1}) dA = 0\,.$$

Hence, either assumption implies $p = -M_{s-1}/M_s$. Substituting this in (0.2) for $r = s$, we obtain

$$(4.1)\qquad \int_{V^n} (M_s^2 - M_{s-1} M_{s+1})/M_s\, dA = 0\,.$$

Since $\binom{n}{i} M_i$ is the i-th elementary symmetric function of the real numbers $\varkappa_1, \dots, \varkappa_n$, we have the inequalities

$$(4.2)\qquad M_i^2 - M_{i-1} M_{i+1} \geqq 0, \qquad i = 1, \dots, n-1\,,$$

and equality in (4.2) for any value of i implies $\varkappa_1 = \dots = \varkappa_n$ (cf. [3, pp. 52, 104]). From (4.1) it follows therefore that $\varkappa_1 = \dots = \varkappa_n$ at all points of V^n. It is well known that this implies that V^n is a hypersphere, and hence Theorem 2 is proved.

If $M_{i-1} > 0$ and $M_i > 0$, the inequality (4.2) may be written as

$$(4.3)\qquad M_i/M_{i-1} \geqq M_{i+1}/M_i\,.$$

Let the assumptions of Theorem 3 be satisfied for some $s < n$. Then the inequality (4.3) holds for $i = 1, \dots, s$. In particular, we have $M_1/M_0 \geqq M_{s+1}/M_s$ or

$$(4.4)\qquad M_1 M_s \geqq M_{s+1}\,,$$

and the equality implies $\varkappa_1 = \dots = \varkappa_n$. Since $M_1 > 0$ and it is assumed that p has the same sign at all points of V^n, we must have $p < 0$

because of the formula (0.2) for $r = 0$. Multiplying the both sides of the inequality (4.4) by p, integrating over V^n, and applying the formula (0.2) for $r = 0$ and $r = s$, we obtain

$$-M_s \int_{V^n} dA = M_s \int_{V^n} M_1 p \, dA \leqq \int_{V^n} M_{s+1} p \, dA = -M_s \int_{V^n} dA,$$

since M_s is constant. Consequently, equality must hold in (4.4) at all points of V^n, and hence Theorem 3 for $s < n$ follows.

In the remaining case of Theorem 3, where $s = n$, the assumptions imply

$$M_i > 0, \quad i = 1, \ldots, n. \tag{4.5}$$

It is known that from the inequalities (4.2) and (4.5) it follows that

$$M_1 \geqq M_2^{1/2} \geqq \ldots \geqq M_{n-1}^{1/(n-1)} \geqq M_n^{1/n}, \tag{4.6}$$

and equality at any stage in (4.6) implies $\varkappa_1 = \ldots = \varkappa_n$ (cf. [3, p. 52]). Now put $M_n = c^n$, where c is a positive constant. Then we obtain on one hand, by means of the formula (0.2) for $r = n-1$ and the inequalities (4.6),

$$\int_{V^n} M_n p \, dA = -\int_{V^n} M_{n-1} \, dA \leqq -c^{n-1} \int_{V^n} dA,$$

and on the other hand, by means of $p < 0$, the inequalities (4.6) and the formula (0.2) for $r = 0$,

$$\int_{V^n} M_n p \, dA = c^{n-1} \int_{V^n} M_n^{1/n} p \, dA \geqq c^{n-1} \int_{V^n} M_1 p \, dA = -c^{n-1} \int_{V^n} dA.$$

Thus $M_n^{1/n} = M_1$ and again we have $\varkappa_1 = \ldots = \varkappa_n = c$ at all points of V^n.

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