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SOME EXTREMAL PROBLEMS FOR TRIGONOMETRICAL AND COMPLEX POLYNOMIALS

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1. Introduction

In his paper "On rational polynomials" ([7]), P. Turán[1] raised the following problem: Let $P_n(z)$ be a polynomial of a complex variable z with complex coefficients and of degree $\leqq n$. Suppose that on the circle $|z| = 1$, the absolute value of $P_n(z)$ attains its maximum at the point $z = 1$. How near to this point can there be a zero z_0 of $P_n(z)$ if either

A: z_0 is prescribed to lie on the circle $|z| = 1$,

B: no restriction is made about the position of z_0?

Turán pointed out that necessarily $|z_0 - 1| \geqq 1/n$ and proved that in case A, the nearest positions of a zero are $z_0 = e^{\pm i\pi/n}$ and that if $P_n(z)$ has a zero at one of these points it follows that $P_n(z) = c(1+z^n)$. Turán and Erdös [2] found applications of this theorem, namely to derive from a common source certain theorems by Jentzsch-Szegö and E. Schmidt.

As for case B, Turán showed that to every z_0 on the lines $\arg z = \pm\pi/n$ corresponds a polynomial $P_n(z)$ with the maximum-property mentioned and with $P_n(z_0) = 0$, but the rest of the problem was left as an open question.

While investigating this problem, I was led to study some extremal properties of a class of trigonometrical polynomials (see Theorem I), from which the answer to Turán's problem follows (see Theorem V). Theorem I is, however, interesting in itself. At the suggestion of L. Hörmander, I made a generalization of Theorem I (see Theorem III). Using a method for approximating bounded functions by periodic ones developed in [4], Hörmander proved (see the following paper [5]) certain inequalities, corresponding to those of Theorem III, for functions of exponential type.

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[1] I should like to take the opportunity to express my very deep gratitude to Prof. P. Turán, who brought my attention to this and to many other problems during the last few years. I also wish to thank Mr. L. Hörmander and Prof. Å. Pleijel for valuable suggestions and criticism of my paper.

2. Inequalities for trigonometrical polynomials with prescribed value at one complex point

2.1. We start by making the following

Definition. By $\Pi_n = \Pi_n(it, \cos\alpha)$, where $n \geqq 2$ is an integer, $t \neq 0$ is real and $0 \leqq \alpha \leqq \pi$, we denote the class of trigonometrical polynomials Φ_n with real coefficients and of order $\leqq n$, such that

(2.1.1) $$|\Phi_n(x)| \leqq 1 \qquad \text{for all real } x$$

and

(2.1.2) $$\Phi_n(it) = \cos\alpha \,.$$

Observe that x is real throughout the whole paper.

We first note that $\Phi_n = \cos\alpha$ belongs to Π_n, so that Π_n is not empty. Further, since the coefficients of Φ_n are real, the classes $\Pi_n(it, \cos\alpha)$ and $\Pi_n(-it, \cos\alpha)$ coincide.

We are going to solve the problem of determining the functions

$$m(x) = \inf_{\Phi_n \in \Pi_n} \Phi_n(x) \qquad \text{and} \qquad M(x) = \sup_{\Phi_n \in \Pi_n} \Phi_n(x)$$

for every value of x. It will turn out that for instance $m(x)$ has the value -1 except in the interior of a certain interval $|x| \leqq \delta < \pi$ around the origin and its translations by $2\nu\pi$, where ν is an arbitrary integer. In these intervals $m(x)$ is equal to

(2.1.3) $$T_{2n}(a \cos \tfrac{1}{2}x) \qquad \text{where} \qquad a = \cos(\alpha/2n) \cosh^{-1} \tfrac{1}{2}t \,.$$

Here T_r denotes the r^{th} Tchebycheff polynomial defined by

$$T_r(\cos u) = \cos ru$$

and $\cosh^{-1}$ means $1/\cosh$. Note that $0 < a < 1$.

We shall use the notation

$$\Psi_n(x) = \Psi_n(a, x) = T_{2n}(a \cos \tfrac{1}{2}x) \,.$$

From the identity $T_{2n}(q) = T_n(2q^2 - 1)$ it follows that $\Psi_n(x)$ is a polynomial with real coefficients and of degree n in $\cos^2 \frac{1}{2}x$ and hence also in $\cos x$. Since $|a \cos \frac{1}{2}x| \leqq a < 1$, it follows from the definition of T_{2n} that $|\Psi_n(x)| \leqq 1$ for all real x. Finally,

$$\Psi_n(it) = T_{2n}(a \cosh \tfrac{1}{2}t) = T_{2n}\big(\cos(\alpha/2n)\big) = \cos\alpha \,,$$

so that Ψ_n belongs to the class Π_n. Hence, in a certain interval $|x| \leqq \delta < \pi$ and its translations by $2\nu\pi$, the function $m(x)$ equals a polynomial in the class Π_n and a corresponding fact is true for $M(x)$.

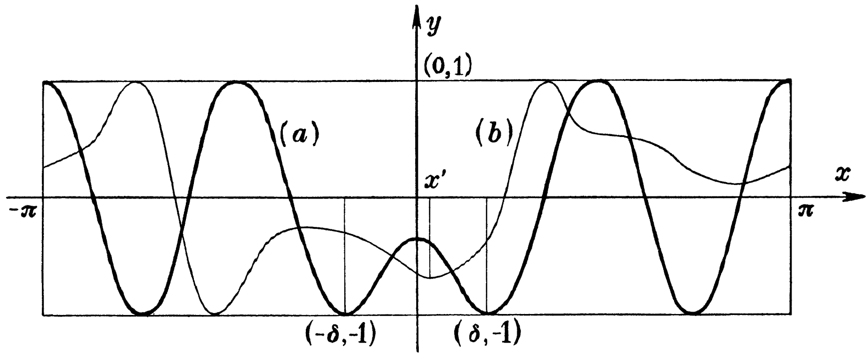

Fig. 1. (a) $y=\Psi_n(x)$; (b) $y=\Phi_n(x)$. ($n=4$.)

The curve $y=\Psi_n(x)$ is drawn in fig. 1. For a detailed discussion of its shape see below.

After these preliminary remarks we state

THEOREM I. *Let $\Pi_n=\Pi_n(it, \cos\alpha)$ be the class and a the number defined above.*

a) *For $\Phi_n \in \Pi_n$ and all x for which $a|\cos\frac{1}{2}x| \geqq \cos(\pi/2n)$ it follows that*

$$\Phi_n(x) \geqq T_{2n}(a\cos\tfrac{1}{2}x) = \Psi_n(x).$$

Equality for one such x implies equality for all x.

b) *To every x for which $a|\cos\frac{1}{2}x| < \cos(\pi/2n)$, there exist infinitely many polynomials $\Phi_n \in \Pi_n$ such that $\Phi_n(x)=-1$.*

Note that, when it is not empty, the set of points x satisfying the condition of a) consists of an interval $|x| \leqq \delta < \pi$ and its translations by $2\nu\pi$.

According to a), $m(x)$ coincides with the polynomial (2.1.3) when x belongs to the interval $|x| \leqq \delta$ or its translations by $2\nu\pi$; according to b), $m(x)=-1$ when x is not in this set. Thus Theorem I implies that $m(x)$ is known for all values of x.

We observe that $\Phi_n \in \Pi_n(it, \cos\alpha)$ is equivalent to

$$-\Phi_n \in \Pi_n\bigl(it, \cos(\pi-\alpha)\bigr),$$

and hence a theorem analogous to Theorem I is valid for the function $M(x)$.

2.2. Proof of Theorem I. Since all functions involved are periodic in the variable x with the period 2π, we can assume that $-\pi \leqq x < \pi$.

PROOF OF I a): If there is an x satisfying the condition of I a), we must have $a \geqq \cos(\pi/2n)$. In order to study $\Psi_n(x)$ in the interval $-\pi \leqq x < \pi$ (see fig. 1), we introduce the number δ defined by

$$a \cos \tfrac{1}{2}\delta = \cos(\pi/2n), \qquad 0 \leqq \delta < \pi .$$

It is easily seen that the function $\Psi_n(x)$ is monotonically decreasing to -1 in the interval $0 \leqq x \leqq \delta$. When x increases from δ to π, then $a \cos \frac{1}{2}x$ decreases from $\cos(\pi/2n)$ to 0. From this it follows that *in the interval* $\delta \leqq x \leqq \pi$ *the curve* $y = \Psi_n(x)$ *has* $n-1$ *branches passing between* $y=-1$ *and* $y=1$. Since Ψ_n is even, we know the curve in the whole interval $(-\pi, \pi)$.

Case 1, $\delta > 0$: Suppose that $\Phi_n(x') \leqq \Psi_n(x')$ for a number x' such that $-\pi \leqq x' < \pi$ and $a \cos \frac{1}{2}x' \geqq \cos(\pi/2n)$. The assumptions imply that $|x'| \leqq \delta$. Let us compare the two trigonometrical polynomials Ψ_n and Φ_n of order $\leqq n$ and count the zeros of the polynomial

$$\Delta_n = \Psi_n - \Phi_n$$

by considering the intersections of the corresponding curves (see fig. 1). If $\Delta_n \not\equiv 0$ it follows that Δ_n has $2n$ zeros in the interval $-\pi \leqq x < \pi$. (More precisely: at least $2n$ zeros, counted with their multiplicities. In the following we use the shorter expression.) Further,

$$\Delta_n(\pm it) = \Psi_n(\pm it) - \Phi_n(\pm it) = 0$$

whence $\Delta_n(z)$ has $2n+2$ zeros in the strip $-\pi \leqq \operatorname{Re} z < \pi$, which is impossible. Thus $\Delta_n \equiv 0$ and I a) is proved if $\delta > 0$.

Case 2, $\delta = 0$: In this case the only value of x we have to consider is 0. Further, $a = \cos(\pi/2n)$ so that

$$\Psi_n(x) = T_{2n}[\cos(\pi/2n) \cos \tfrac{1}{2}x] ,$$

and hence $\Psi_n(0) = -1$. Since $\Phi_n(0) \geqq -1$, we only have to investigate the case $\Phi_n(0) = -1$. If $\Phi_n(0) = -1$, it follows from (2.1.1) that $\Phi_n'(0) = 0$ and $\Phi_n''(0) \geqq 0$. A calculation shows that $\Psi_n'(0) = 0$ and $\Psi_n''(0) = 0$ so that $\Delta_n'(0) = 0$ and $\Delta_n''(0) \leqq 0$.

Suppose now $\Delta_n \not\equiv 0$. If $\Delta_n''(0) < 0$, then $\Delta_n(\pm \varepsilon) < 0$ for a sufficiently small $\varepsilon > 0$ and $\Delta_n(x)$ has altogether $2n-2$ zeros in the intervals $-\pi \leqq x \leqq -\varepsilon$ and $\varepsilon \leqq x < \pi$ and 2 zeros at $x=0$. If $\Delta_n''(0) = 0$ holds, then $\Delta_n(x)$ has altogether $2n-4$ zeros in the intervals $-\pi \leqq x < 0$ and $0 < x < \pi$ and 3 zeros at $x=0$. In both cases we have, in addition, 2 zeros at $x = \pm it$ which is impossible. Hence $\Delta_n \equiv 0$ and I a) is proved.

Proof of I b): Suppose that $a \cos \frac{1}{2}x' < \cos(\pi/2n)$. In order to construct *one* polynomial satisfying the conditions of I b), we write Ψ_n in the form

$$\Psi_n(x) = T_n(2a^2 \cos^2 \tfrac{1}{2}x - 1) .$$

Here we perform a linear transformation on the argument and define with $0 \leqq \varkappa \leqq 1$

$$\Phi_n(x) = T_n[\varphi_\varkappa(x)],$$

where

$$\varphi_\varkappa(x) = \varkappa(2a^2\cos^2\tfrac{1}{2}x - 1) + (1-\varkappa)\cos(\alpha/n).$$

Then $\Phi_n \in \Pi_n(it, \cos\alpha)$, and it is possible to choose $\varkappa$ so that $\Phi_n(x') = -1$.

In fact, we first observe that $\Phi_n(x)$ is a real trigonometrical polynomial of order $\leqq n$. Further, $\varphi_\varkappa(x)$ is a mean value of two terms of modulus less than or equal to 1 and hence $|\varphi_\varkappa(x)| \leqq 1$ so that $|\Phi_n(x)| \leqq 1$ for all real x. Since $a = \cos(\alpha/2n)\cosh^{-1}\frac{1}{2}t$, it follows that $\varphi_\varkappa(it) = \cos(\alpha/n)$ so that $\Phi_n(it) = T_n[\cos(\alpha/n)] = \cos\alpha$. This means that $\Phi_n \in \Pi_n(it, \cos\alpha)$. Further,

$$\varphi_0(x') = \cos(\alpha/n) \geqq \cos(\pi/n)$$

and

$$\varphi_1(x') = 2a^2\cos^2\tfrac{1}{2}x' - 1 < 2\cos^2(\pi/2n) - 1 = \cos(\pi/n)$$

by the assumption of Ib). Thus, there exists a number $\varkappa$ such that $0 \leqq \varkappa < 1$ and $\varphi_\varkappa(x') = \cos(\pi/n)$. This implies that $\Phi_n(x') = -1$. The only case in which $\varkappa = 0$ is $\alpha = \pi$.

For $\varkappa > 0$ it follows from $a < 1$ that $\varphi_\varkappa(0) < 1$ and this inequality also holds for $\varkappa = 0$, $\alpha = \pi$. Further, we have

$$\varphi_\varkappa(\pi) = (1-\varkappa)\cos(\alpha/n) - \varkappa > -1.$$

Thus, in the interval $0 \leqq x \leqq \pi$, the curve $y = \Phi_n(x)$ has at most $n-2$ branches passing between $y = -1$ and $y = 1$. Using this fact it is possible to show that we can submit Φ_n to infinitely many variations so that Φ_n still belongs to $\Pi_n(it, \cos\alpha)$ and $\Phi_n(x') = -1$. However, we do not write out the details of this part of the proof.

By Theorem I we know the functions $m(x)$ and $M(x)$ for all x. Now take x fixed $= x_0$ and suppose that g is a number such that $m(x_0) < g < M(x_0)$. From the theorem it follows that there exist polynomials $\Phi_n{}^m$ and $\Phi_n{}^M$ in Π_n so that $\Phi_n{}^m(x_0) = m(x_0)$ and $\Phi_n{}^M(x_0) = M(x_0)$. A suitable linear combination of these polynomials evidently gives a polynomial Φ_n in Π_n for which $\Phi_n(x_0) = g$. As a matter of fact, one can show that there are infinitely many such polynomials.

2.3. The case when $t = 0$. Let $\Pi_n(0, \cos\alpha)$, where $n \geqq 2$ is an integer and $0 \leqq \alpha \leqq \pi$, be the class of trigonometrical polynomials Φ_n with real coefficients and of order $\leqq n$, such that $|\Phi_n(x)| \leqq 1$ for all real x and $\Phi_n(x) - \cos\alpha$ has a *double* zero at $x = 0$. Then Theorem I is still valid with the following modification in the last line of a): Equality for one such $x \neq 2\nu\pi$, where ν is an arbitrary integer, implies equality for all x.

On the other hand, if we only suppose that $\Phi_n(x)-\cos\alpha$ has a zero (simple or not) at $x=0$, we obtain results of a different type. A calculation of the number of intersections of the curves then shows that for $|x|\leqq(\pi-\alpha)/n$ the inequality $\Phi_n(x)\geqq\cos(n|x|+\alpha)$ holds. Equality for one x such that $0<\pm x\leqq(\pi-\alpha)/n$ implies that $\Phi_n(x)=\cos(\pm nx+\alpha)$ for all x (cf. M. Riesz [6]). Using polynomials of the type

$$T_n[\varkappa\cos(\pm x+\alpha/n)+(1-\varkappa)\cos(\alpha/n)]\,,\qquad 0\leqq\varkappa\leqq 1\,,$$

one can show that if $(\pi-\alpha)/n<|x'|\leqq\pi$, there are infinitely many polynomials $\Phi_n\in\Pi_n(0,\cos\alpha)$ for which $\Phi_n(x')=-1$.

2.4. Now we generalize Theorem I by replacing $\cos\alpha$ in condition (2.1.2) by a complex number.

DEFINITION. Let $\Pi_n(it,\xi+i\eta)$, where $n\geqq 2$ is an integer, t, ξ, η are real and $t\neq 0$, be the class of trigonometrical polynomials Φ_n with real coefficients and of order $\leqq n$, such that

(2.4.1) $$|\Phi_n(x)|\leqq 1\qquad\text{for all real } x$$

and

(2.4.2) $$\Phi_n(it)=\xi+i\eta\,.$$

THEOREM II. *The class $\Pi_n(it,\xi+i\eta)$ is not empty if and only if*

(2.4.3) $$\xi^2\cosh^{-2}nt+\eta^2\sinh^{-2}nt\leqq 1\,.$$

This means that the possible values of $\Phi_n(x)$ for $x=it$ are situated inside or on an ellipse with the semiaxes $\cosh nt$ and $\sinh nt$. Though the theorem follows from the reasoning used by Duffin and Shaeffer in [1] we write down a short proof.

PROOF OF THEOREM II: If (2.4.3) is satisfied, we can write

(2.4.4) $$\xi=b\cos nx_1\cosh nt\,,\qquad \eta=b\sin nx_1\sinh nt\,,$$

where $0\leqq b\leqq 1$ and x_1 is suitably chosen. Then, since

$$\xi+i\eta=b\cos n(it-x_1)\,,$$

the polynomial $b\cos n(x-x_1)$ belongs to $\Pi_n(it,\xi+i\eta)$ and the first part of the theorem is proved. Suppose on the other hand that $\Phi_n\in\Pi_n(it,\xi+i\eta)$, but that (2.4.3) is not fulfilled. Then ξ and η can be written in the form (2.4.4) with $b>1$. The function $\cos n(x-x_1)-\Phi_n(x)/b$ is $\geqq 1-1/b>0$ for $x=x_1$ and hence $\neq 0$. But it has $2n$ real zeros in $-\pi\leqq x<\pi$ and two complex zeros $x=\pm it$, which is impossible. This proves the theorem.

Observe in particular that if (2.4.2) is written in the form

$$\Phi_n(it) = \xi + i\eta = \cos(\alpha + i\beta)\,, \qquad \alpha, \beta \text{ real}\,,$$

the condition (2.4.3) is equivalent to $|\beta| \leqq n|t|$. If $\beta = \pm nt$, that is, if $\cos(\alpha + i\beta)$ is situated on the ellipse, it follows by considering

$$\cos(nx \pm \alpha) - \Phi_n(x)$$

that in this case the only polynomial belonging to $\Pi_n\big(it, \cos(\alpha \pm int)\big)$ is $\cos(nx \pm \alpha)$.

2.5. We shall now solve the problem analogous to that in Theorem I for the class $\Pi_n\big(it, \cos(\alpha + i\beta)\big)$. Also in this case the functions $m(x)$ and $M(x)$ are -1 and 1, respectively, except in the interior of certain intervals where they are equal to polynomials belonging to the class. These polynomials have the form $\pm \Psi_n(a, x - x')$, where

$$\Psi_n(a, x - x') = T_{2n}[a \cos \tfrac{1}{2}(x - x')]\,,$$

i.e. they are obtained from the polynomials $\Psi_n(x)$ by a translation. Note that, if a polynomial $\Psi_n(a, x - x')$ belongs to $\Pi_n\big(it, \cos(\alpha + i\beta)\big)$, we must have $T_{2n}[a \cos \tfrac{1}{2}(it - x')] = \cos(\alpha + i\beta)$.

It is now convenient to make the following

Definition. Let $\{(a_k, x_k)\}$ be the set of all different pairs of real numbers satisfying the equation

$$T_{2n}[a_k \cos \tfrac{1}{2}(it - x_k)] = \cos(\alpha + i\beta)\,, \tag{2.5.1}$$

for which $a_k \geqq \cos(\pi/2n)$ and $-\pi \leqq x_k < \pi$.

By solving the equation $T_{2n}(z) = \cos(\alpha + i\beta)$ with respect to z, the numbers (a_k, x_k) may be obtained explicitly. This will be done later.

Theorem III. *Take $|\beta| \leqq n|t|$ and let $\Pi_n = \Pi_n\big(it, \cos(\alpha + i\beta)\big)$ be the class and $\{(a_k, x_k)\}$ the set of pairs defined above.*

a) *For $\Phi_n \in \Pi_n$ and all x belonging to the point-set I_k defined by the inequality $a_k |\cos \tfrac{1}{2}(x - x_k)| \geqq \cos(\pi/2n)$ it follows that*

$$\Phi_n(x) \geqq T_{2n}[a_k \cos \tfrac{1}{2}(x - x_k)] = \Psi_n(a_k, x - x_k)\,.$$

Equality for one $x \in I_k$ implies equality for all x.

b) *To every x which is outside all sets I_k, there exist infinitely many polynomials $\Phi_n \in \Pi_n$ such that $\Phi_n(x) = -1$.*

The set I_k consists of the points in the interval $|x - x_k| \leqq \delta < \pi$ and its translations by $2\nu\pi$. It will be shown below that intervals belonging to different sets I_k do not overlap.

Using the fact that $\Phi_n \in \Pi_n(it, \cos(\alpha + i\beta))$ is equivalent to

$$-\Phi_n \in \Pi_n(it, \cos(\alpha + \pi + i\beta)) ,$$

we get an analogous theorem giving the function $M(x)$.

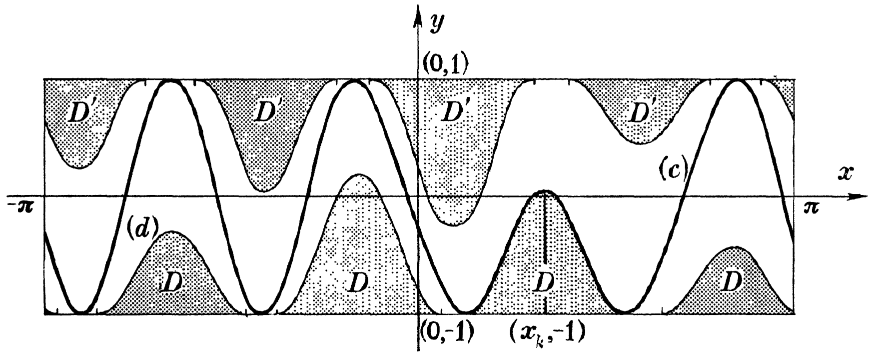

Fig. 2. (c) $y = \Psi_n(a_k, x - x_k)$; (d) $y = \Psi_n(a_{k'}, x - x_{k'})$. $(n = 4.)$

Part a) of Theorem III states that, if $\Phi_n \in \Pi_n$, the curve $y = \Phi_n(x)$ cannot pass through certain domains D of the strip $-1 \leqq y \leqq 1$ (see fig. 2). If q is outside all sets I_k defined in a), it will be shown that there are at most n such domains D in the interval $q \leqq x < q + 2\pi$. In the same way the theorem concerning $M(x)$ gives at most n excluded domains D' in a suitably chosen period.

2.6. Proof of Theorem III. The equation (2.5.1) can be written

$$a_k \cos \tfrac{1}{2}(it - x_k) = \cos[(\alpha + 2k\pi + i\beta)/2n] ,$$

where to begin with $k = 1, 2, \ldots, 2n$. Thus, we get

$$(2.6.1) \quad \begin{aligned} a_k \cosh \tfrac{1}{2}t \cos \tfrac{1}{2}x_k &= \cos[(\alpha + 2k\pi)/2n] \cosh(\beta/2n) , \\ a_k \sinh \tfrac{1}{2}t \sin \tfrac{1}{2}x_k &= -\sin[(\alpha + 2k\pi)/2n] \sinh(\beta/2n) , \end{aligned}$$

and hence

$$(2.6.2) \quad a_k^2 = \left[\frac{\cos[(\alpha + 2k\pi)/2n] \cosh(\beta/2n)}{\cosh \tfrac{1}{2}t}\right]^2 + \left[\frac{\sin[(\alpha + 2k\pi)/2n] \sinh(\beta/2n)}{\sinh \tfrac{1}{2}t}\right]^2 .$$

Since $|\beta| \leqq n|t|$, it follows from (2.6.2) that $a_k \leqq 1$ for all k. This and (2.5.1) imply that $\Psi_n(a_k, x - x_k) \in \Pi_n$ for all k.

According to the definition of a_k given in Section 2.5 we shall, however, only consider such a_k for which $a_k \geqq \cos(\pi/2n)$. On account of this fact we can now accomplish the proof of III a) as that of I a) by considering the curve $y = \Psi_n(a_k, x - x_k)$.

If $\beta \neq 0$, it follows from the formulas (2.6.1) that at most one of the two integers k and $k+n$ gives a pair (a_k, x_k) which satisfies $a_k \geqq \cos(\pi/2n)$ and $-\pi \leqq x_k < \pi$. This shows that in the period $(q, q+2\pi)$ the number of domains D is $\leqq n$. If $\beta = 0$ (cf. Theorem I), this number is 1 or 0.

Since $m(x)$ is unique and $\geqq -1$ in all sets I_k, being -1 only at the endpoints of the intervals, it follows that intervals belonging to different sets I_k cannot overlap.

The midpoints of the intervals constituting the set I_k are $x_k + 2\nu\pi$. If we denote the common length of these intervals by l_k, we have

$$a_k \cos(l_k/4) = \cos(\pi/2n), \qquad 0 \leqq l_k \leqq 2\pi/n .$$

It might be worth noting what happens with the excluded domains D if t varies and α, β and n are fixed. First let $|t| \to \infty$. From (2.6.2) it then follows that $a_k \to 0$ for all k. Hence the condition $a_k \geqq \cos(\pi/2n)$ will not be satisfied if $|t| \geqq t_0$, where t_0 is suitably chosen. Thus for $|t| \geqq t_0$ III a) gives no excluded domains D at all. The same holds for the domains D', introduced at the end of Section 2.5.

On the other hand, by means of Theorem II we conclude that $|t| \geqq |\beta|/n$. If $t = \beta/n \neq 0$ it follows from (2.6.2) and (2.6.1) that all $a_k = 1$ and that $x_k \equiv -\alpha/n \bmod 2\pi/n$. Since the common length of the intervals is now $2\pi/n$, we conclude that the sets I_k together fill up the whole x-axis. The corresponding polynomials $\Psi_n(1, x - x_k) = \cos(nx+\alpha)$ are independent of k. Hence $m(x) = \cos(nx+\alpha)$ for all x.

By studying $M(x)$ we find that $M(x) = -\cos(nx+\alpha+\pi) = \cos(nx+\alpha)$ for all x so that $M(x) = m(x)$. Thus in the case when $t = \beta/n$ the domains D and D' fill up the whole strip $-1 \leqq y \leqq 1$ and as mentioned in Section 2.4, there is only one polynomial in Π_n, namely $\cos(nx+\alpha)$. The corresponding fact is true for $t = -\beta/n$.

To prove IIIb), it is convenient to use a reasoning different from that employed in the proof of Ib). Suppose that $x = x'$ is outside all the sets I_k. From the discussion just concluded, it is clear that necessarily $|\beta| < n|t|$. Thus, if $\cos(\alpha + i\beta) = \xi + i\eta$ it follows that the point $\xi + i\eta$ is inside the ellipse introduced in (2.4.3). Now let $\xi_0 + i\eta_0$ be a point *on* this ellipse. Then to $\xi_0 + i\eta_0$ there correspond certain sets I_k which together cover the x-axis. Thus x' belongs to one of them, and hence to one interval, say to $J(\xi_0 + i\eta_0)$. Let L be the straight line through $\xi_0 + i\eta_0$ and $\xi + i\eta$. If x' is not an endpoint of $J(\xi_0 + i\eta_0)$, we let the point z move on L from $\xi_0 + i\eta_0$ towards $\xi + i\eta$. For z near to $\xi_0 + i\eta_0$, there correspond to $\Pi_n(it, z)$ certain intervals, one of which, $J(z)$, contains x' as an interior point. The interval $J(z)$ varies continuously with z and coincides with $J(\xi_0 + i\eta_0)$ for $z = \xi_0 + i\eta_0$.

Now we observe that when z moves on L from $\xi_0+i\eta_0$ towards $\xi+i\eta$, the interval $J(z)$ cannot cease to exist if we have not first reached a point z for which $J(z)$ has the length 0 (note that $a_k=\cos(\pi/2n)$ implies $l_k=0$). When we arrive at $\xi+i\eta$ we know that x' is outside all sets I_k and from this it follows that there is a point $z_1 \neq \xi+i\eta$ on L, between $\xi_0+i\eta_0$ and $\xi+i\eta$, so that x' is an endpoint of $J(z_1)$. But this means that there is a polynomial $\Phi_n^{(1)} \in \Pi_n(it, z_1)$ for which $\Phi_n^{(1)}(x')=-1$.

The same argument, applied to the other point of intersection of L and the ellipse, shows that there exists a polynomial $\Phi_n^{(2)} \in \Pi_n(it, z_2)$ for which $\Phi_n^{(2)}(x')=-1$. Now, if t_1, t_2 are chosen so that $t_1 \geqq 0, t_2 \geqq 0, t_1+t_2=1$, $t_1 z_1+t_2 z_2=\xi+i\eta$, it follows that

$$\Phi_n = t_1 \Phi_n^{(1)}+t_2 \Phi_n^{(2)} \in \Pi_n(it, \xi+i\eta) \qquad \text{and} \qquad \Phi_n(x') = -1$$

so that Φ_n is one polynomial satisfying the conditions of III b). Using different lines L it is possible to show that there are infinitely many polynomials satisfying these conditions.

2.7. In the applications we shall consider a class of polynomials defined as follows.

Definition. Let $\Omega_n(it, 0)$, where $n \geqq 2$ is an integer and t is real, be the class of trigonometrical polynomials Φ_n with real coefficients and order $\leqq n$, such that $0 \leqq \Phi_n(x) \leqq 1$ for all real x and $\Phi_n(it)=0$.

In this class the extremal polynomials corresponding to Ψ_n are

$$\Theta_n(x) = \tfrac{1}{2}\{1-T_{2n}[\cosh^{-1}\tfrac{1}{2}t \cos\tfrac{1}{2}x]\} .$$

Theorem IV. a) *For $\Phi_n \in \Omega_n$ and all x for which*

$$\cosh^{-1}\tfrac{1}{2}t\,|\cos\tfrac{1}{2}x| \geqq \cos(\pi/2n)$$

it follows that

$$\Phi_n(x) \leqq \tfrac{1}{2}\{1-T_{2n}[\cosh^{-1}\tfrac{1}{2}t \cos\tfrac{1}{2}x]\} = \Theta_n(x) .$$

In the case $t \neq 0$ equality for one such x implies equality for all x. In the case $t=0$ equality for one such $x \neq 2\nu\pi$, ν an arbitrary integer, implies equality for all x.

b) *To every x for which* $\cosh^{-1}\tfrac{1}{2}t\,|\cos\tfrac{1}{2}x| < \cos(\pi/2n)$ *there are infinitely many polynomials $\Phi_n \in \Omega_n$ such that $\Phi_n(x)=1$.*

Proof: We observe that $\Phi_n \in \Omega_n(it, 0)$ is equivalent to $1-2\Phi_n \in \Pi_n(it, 1)$. Thus, for $t \neq 0$ the theorem follows from Theorem I. If $t=0$, the conditions imply that $\Phi_n(x)$ has a double zero at $x=0$ and hence the theorem follows from the remark in Section 2.3.

Let us now consider trigonometrical polynomials $\Phi_n \not\equiv 0$ with real coefficients and of order $\leqq n$, where $n \geqq 2$ is an integer, such that $\Phi_n(x) \geqq 0$ for all real x and such that $\Phi_n(it) = 0$.

For which numbers $x = x_0$ does there exist such a polynomial Φ_n attaining its maximum on the real axis at $x = x_0$? Of course it is no restriction to assume that $\Phi_n(x_0) = 1$, and then it follows from Theorem IV that a necessary and sufficient condition is

$$|\cos \tfrac{1}{2}x_0| \leqq \cosh \tfrac{1}{2}t \cos(\pi/2n) .$$

This result will be used in Section 3.

3. The positions of maxima and complex zeros of trigonometrical polynomials

Let us define Γ_n as the class of trigonometrical polynomials $\Phi_n \not\equiv 0$ of order $\leqq n$, where $n \geqq 2$ is an integer, with complex coefficients and with the property that $|\Phi_n(x)|$ attains its maximum on the real axis at the point $x = 0$. For various subclasses of Γ_n we ask for necessary and sufficient conditions for a point $u + iv$, u and v real, to be a zero of at least one polynomial in the subclass.

a) In the subclass of polynomials having real coefficients and which are non-negative on the real axis the condition is

$$|\cos \tfrac{1}{2}u| \leqq \cosh \tfrac{1}{2}v \cos(\pi/2n) .$$

b) In the subclass of polynomials obtained by the sole restriction that their coefficients are real, the condition is

$$|\cos \tfrac{1}{2}u| \cos(\pi/4n) \leqq \cosh \tfrac{1}{2}v \cos(\pi/2n)$$

if $v \neq 0$. If $v = 0$ and the zero $x = u$ is double, the same is true. If u is not restricted to be a double-zero, the condition is $\cos u \leqq \cos(\pi/2n)$.

c) In the whole class of polynomials with complex coefficients the condition is

$$|\cos \tfrac{1}{2}u| \leqq \cosh \tfrac{1}{2}v \cos(\pi/4n) .$$

The boundaries of the corresponding domains in the (u, v)-plane are called γ_a, γ_b and γ_c and are drawn in the strip $-\pi \leqq u < \pi$ in fig. 3.

The assertion a) is simply proved from the remark at the end of Section 2.7 by a translation.

For $v \neq 0$ the assertion b) is proved by putting $\alpha = \pi/2$ in Theorem I and performing a translation. The same proof holds, if $v = 0$ and the zero is supposed to be double. If $v = 0$ and the zero is not supposed to be double, the assertion follows from Section 2.3.

To prove c), we suppose that Φ_n belongs to Γ_n and $\Phi_n(u+iv)=0$. For complex ζ we write $\Phi_n(\zeta)=K_n(e^{i\zeta})$, where $K_n(z)=\Sigma_{\nu=-n}^{n}\mu_\nu z^\nu$. With this K_n we define $\Lambda_{2n}(z)$ as

$$\Lambda_{2n}(z) = K_n(e^{iz})\overline{K}_n(e^{-iz}) ,$$

where the coefficients of $\overline{K}_n$ are conjugate to those of K_n. Then Λ_{2n} has the following properties: it is a trigonometrical polynomial with real

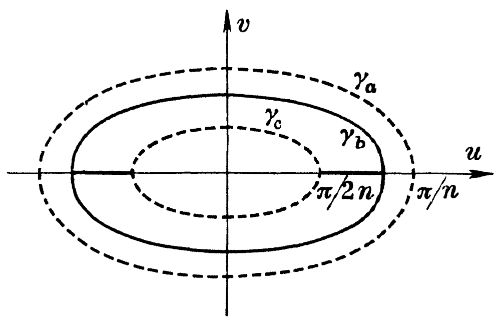

Fig. 3.

coefficients and of order $\leqq 2n$; for real $z=x$ it is $\geqq 0$ and takes its maximum at $x=0$. Hence, Λ_{2n} is in the subclass defined in a) if we replace n by $2n$. Finally, $\Lambda_{2n}(u+iv)=0$.

On the other hand, let Λ_{2n} be a trigonometrical polynomial with these properties. By the theorem of Fejér–F. Riesz [3] about the representation of a non-negative trigonometrical polynomial there exists a function $K_n(z)=\Sigma_{\nu=-n}^{n}\mu_\nu z^\nu$ such that $\Lambda_{2n}(x)=|K_n(e^{ix})|^2$ for real x and $K_n(e^{i(u+iv)})=0$. Then the trigonometrical polynomial Φ_n, defined by $\Phi_n(\zeta)=K_n(e^{i\zeta})$, belongs to Γ_n and $\Phi_n(u+iv)=0$. Thus c) follows from a).

The theorems used here also give the corresponding extremal polynomials explicitly.

4. The positions of maxima and zeros of complex polynomials

4.1. We now turn to the problem mentioned in the introduction of the paper.

Definition. Let $C_n(z_0)$, $z_0 \neq 1$, be the class of polynomials $P_n(z) \not\equiv 0$ of a complex variable z with complex coefficients and of degree $\leqq n$, where $n \geqq 2$ is an integer, which have the following properties: The point $z=z_0$ is a zero of $P_n(z)$ and on the circle $|z|=1$ the absolute value of $P_n(z)$ takes its maximum at $z=1$. Further, let c_n be the curve (see fig. 4) which in polar coordinates ($z=\varrho e^{i\varphi}$) has the equation

$$\cos \tfrac{1}{2}\varphi = \tfrac{1}{2}(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})\cos(\pi/2n), \qquad -\pi/n \leqq \varphi \leqq \pi/n\,. \tag{4.1.1}$$

The curve c_n is closed and contains the point $z=1$ in its interior.

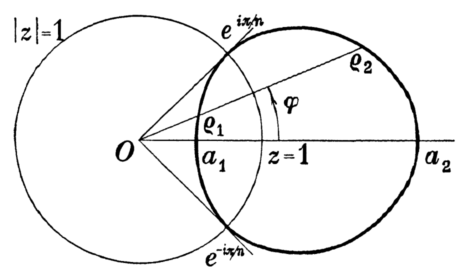

Fig. 4. ($n=4$.)

THEOREM V.

a) *If* z_0 *is a point inside* c_n, *then* $C_n(z_0)$ *is empty.*

b) *If* $z_0=\varrho e^{i\varphi}$ *is a point on* c_n, *then* $C_n(z_0)$ *consists of the polynomials*

$$c \sum_{1\leqq 2\nu+1 \leqq n} \binom{n}{2\nu+1} (ze^{-i\varphi}+1)^{n-2\nu-1}\,(ze^{-i\varphi}-\varrho)^{\nu+1}\,(ze^{-i\varphi}-\varrho^{-1})^{\nu}\,, \tag{4.1.2}$$

where $c \neq 0$ *is an arbitrary complex constant. The polynomials evidently depend on* z_0.

c) *If* z_0 *is a point outside* c_n, *there are infinitely many polynomials* $P_n \in C_n(z_0)$ *which are essentially different* (*not only by a constant factor*).

The cases a), b), and c) correspond to $\cos \frac{1}{2}\varphi \gtreqless \frac{1}{2}(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})\cos(\pi/2n)$, $-\pi \leqq \varphi < \pi$, respectively.

REMARK. Theorems I–V are still valid for $n=1$. However, we have dropped this simple case since it gives exceptions in the proofs. As for Theorem V we note, that for $n=1$ the formula (4.1.1) determines c_1 as the negative real axis.

4.2. Proof of Theorem V. First we consider the case $z_0=0$; this point is always situated outside c_n. Now there are of course infinitely many polynomials $P_{n-1}(z)$ of degree $n-1$ whose absolute values attain their maximum at $z=1$. Then all the polynomials $zP_{n-1}(z)$ belong to $C_n(0)$ and the theorem is proved.

Next assume $z_0 \neq 0$. Let $P_n(z)$ be a polynomial in $C_n(z_0)$ and put $z_0=\varrho e^{i\varphi}$. We define a trigonometrical polynomial Φ_n of order $\leqq n$ by

$$\Phi_n(\zeta) = P_n(e^{i\zeta})\,\overline{P}_n(e^{-i\zeta}),$$

where ζ is a complex variable and the coefficents of $\overline{P}_n$ are conjugate to those of P_n. For real $\zeta=\theta$ we get $\Phi_n(\theta)=|P_n(e^{i\theta})|^2$. The polynomial Φ_n has real coefficients and $\Phi_n(\theta)\geqq 0$ for all real θ. Since $P_n(z_0)=P_n(\varrho e^{i\varphi})=0$, we conclude that $\Phi_n(\varphi-i\log\varrho)=0$.

The polynomial $\Phi_n(\theta)$ attains its maximum on the real axis at $\theta=0$. Hence, according to the result of a) in Section 3,

$$\cos\tfrac{1}{2}\varphi \leqq \cosh(\tfrac{1}{2}\log\varrho)\cos(\pi/2n)\,,$$

and Va) is proved.

To prove Vb), it is convenient to make a rotation through the angle $-\varphi$. After this rotation, $P_n(\varrho)=0$ and $|P_n(z)|$ attains its maximum on $|z|=1$ at $z=e^{-i\varphi}$. Further we assume $|P_n(e^{-i\varphi})|=1$. With this P_n we define

$$\Phi_n(\zeta) = P_n(e^{i\zeta})\,\overline{P}_n(e^{-i\zeta})$$

and conclude that

$$\Phi_n(-i\log\varrho) = 0 \qquad \text{and} \qquad 0 \leqq \Phi_n(\theta) \leqq \Phi_n(-\varphi) = 1$$

for all real θ so that $\Phi_n\in\Omega_n(-i\log\varrho,\,0)$ where Ω_n was defined in Section 2.7.

Theorem IV is now applied with $t=\log\varrho$ and $x=\theta$. Since

$$\cos\tfrac{1}{2}\varphi = \tfrac{1}{2}(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})\cos(\pi/2n)\,,$$

the point $\theta=-\varphi$ belongs to the set considered in IVa). The relation also shows that

$$\Theta_n(-\varphi) = \tfrac{1}{2}\big(1-T_{2n}[\cos(\pi/2n)]\big) = 1\,.$$

Since $\Phi_n(-\varphi)=1$ and the case $\varrho=1,\varphi=0$ is obviously excluded, it follows from IVa) that

(4.2.1) $$\Phi_n(\theta) = |P_n(e^{i\theta})|^2 = \tfrac{1}{2}\{1-T_{2n}[2(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})^{-1}\cos\tfrac{1}{2}\theta]\} = \Theta_n(\theta)$$

for all real θ.

When θ increases from 0 to π, the argument $2(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})^{-1}\cos\frac{1}{2}\theta$ of T_{2n} decreases from

$$2(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})^{-1} = \cos^{-1}\tfrac{1}{2}\varphi\cos(\pi/2n) > \cos(\pi/n)$$

to 0. Hence $\Theta_n(\theta)$ has $n-1$ double zeros in the interval $-\pi\leqq\theta<\pi$. Hence it follows that on the unit circle there are $n-1$ different zeros, not equal to 1, of the polynomial $P_n(z)$. By definition $z=\varrho$ is the n^{th} zero and thus the polynomial $P_n(z)$ is determined up to a constant, non-vanishing, factor. In order to get an explicite expression for $P_n(z)$ we use the identity

(4.2.2) $$\tfrac{1}{2}\left(1-T_{2n}\left(\frac{\zeta+\zeta^{-1}}{2}\right)\right) = \left(\frac{\zeta^n-\zeta^{-n}}{2i}\right)^2.$$

Now, if we solve the equation

$$\tfrac{1}{2}(\zeta+\zeta^{-1}) = 2(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})^{-1}\cos\tfrac{1}{2}\theta$$

for ζ and substitute one of its roots in (4.2.2), we get

(4.2.3) $$\Theta_n(\theta) =$$

$$= -e^{-in\theta}(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})^{-2n}\left(\sum_{1\leq 2\nu+1\leq n}\binom{n}{2\nu+1}(e^{i\theta}+1)^{n-2\nu-1}(e^{i\theta}-\varrho)^{\nu+\frac{1}{2}}(e^{i\theta}-\varrho^{-1})^{\nu+\frac{1}{2}}\right)^2.$$

Let us now consider the polynomial

$$R_n(z) = \varrho^{-\frac{1}{2}}(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})^{-n}\sum_{1\leq 2\nu+1\leq n}\binom{n}{2\nu+1}(z+1)^{n-2\nu-1}(z-\varrho)^{\nu+1}(z-\varrho^{-1})^{\nu}.$$

First we observe that $R_n(\varrho)=0$. Further, (4.2.3) shows that we can write

$$|R_n(e^{i\theta})|^2 = \varrho^{-1}\left|\frac{e^{i\theta}-\varrho}{e^{i\theta}-\varrho^{-1}}\Theta_n(\theta)\right| = \Theta_n(\theta)$$

for all real θ. These two facts together show that $R_n(z)$ is the polynomial $P_n(z)$ we want to determine. A rotation gives V b).

To prove V c), we observe that if $\cos\frac{1}{2}\varphi < \frac{1}{2}(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})\cos(\pi/2n)$, then it follows from IV b) that there exist infinitely many essentially different trigonometrical polynomials $\Phi_n\in\Omega_n(-i\log\varrho,\,0)$ such that $\Phi_n(-\varphi)=+1$. But from the theorem of Fejér–F. Riesz, quoted in Section 3, it then follows that to each Φ_n there exists at least one polynomial $P_n(z)$ such that $|P_n(e^{i\theta})|^2=\Phi_n(\theta)$ for all real θ and for which it is true that $P_n(\varrho)=0$. This proves V c).

4.3. The curve c_n. The curve c_n passes through the points $e^{\pm i\pi/n}$ on the unit circle and through the points

$$a_{1,2} = \frac{1\mp\sin(\pi/2n)}{1\pm\sin(\pi/2n)} = 1\mp\pi/n + O(1/n^2)$$

on the real axis. If we take the point $z=1$ as centre for a new system of polar coordinates (r,τ) with the direction of the positive real axis as principal direction, we get the equation in the form

(4.3.1) $$r = 2\,\mathrm{tg}^2(\pi/2n)\cos\tau + 2\sin(\pi/2n)\cos^{-2}(\pi/2n),$$

which shows that the curve is a "limaçon of Pascal".

From (4.3.1) it follows that

$$r = \pi/n + O(1/n^2)$$

uniformly on c_n so that for large values of n the curve c_n is approximately a circle of radius π/n.

There exists an even better approximation by a circle. Let us consider the circle which passes through $z=e^{\pm i\pi/n}$ and cuts $|z|=1$ orthogonally. Its centre is $\cos^{-1}(\pi/n)=z_1$. If z lies on c_n, we have $|z-z_1|=\pi/n+O(1/n^3)$ uniformly on c_n.

Of course the curve c_n is invariant with respect to an inversion in the circle $|z|=1$ (cf. the equation (4.1.1)). If we write $re^{i\tau}=x+iy$, we find that c_n is of the fourth degree in x, y.

Finally, using (4.3.1) one can prove that c_n is convex for all $n\geqq 3$ (but not for $n=2$).

4.4. The polynomial $P_n(z)$ when z_0 lies on c_n. For $z_0=e^{\pm i\pi/n}$ we get from (4.1.2)

$$P_n(z) = c'(z^n+1)\ ,$$

which is Turán's result mentioned in the introduction.

Generally, if z_0 belongs to c_n it follows from (4.2.1) that the polynomial $P_n(z)$ has on the unit circle the $n-1$ zeros $z = e^{i(\varphi+\delta_\nu)}$, where

$$\cos\tfrac{1}{2}\delta_\nu = \tfrac{1}{2}(\varrho^{\frac{1}{2}}+\varrho^{-\frac{1}{2}})\cos(\nu\pi/n), \qquad 0<\delta_\nu<2\pi, \qquad \nu = 1, 2, \ldots, n-1\ .$$

Besides, we have the zero $z=\varrho e^{i\varphi}$. The zeros of $P_n(z)$ are situated symmetrically with respect to the line $\arg z=\varphi$. Between two zeros of $P_n(z)$ on the unit circle there is always one point in which $|P_n(e^{i\theta})|$ takes the value $|P_n(1)|$ and except for the case $\varphi=0$ there are two such points between $e^{i(\varphi+\delta_1)}$ and $e^{i(\varphi+\delta_{n-1})}$, namely $z=1$ and $z=e^{2i\varphi}$.

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