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SOME INEQUALITIES FOR FUNCTIONS OF EXPONENTIAL TYPE

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The purpose of this paper is to derive inequalities for functions of exponential type, similar to those for trigonometrical polynomials proved by C. Hyltén-Cavallius in the preceding paper [2]. Our inequalities are in fact limiting cases of his and might be proved as such by means of an approximation method developed below. However, it is preferable not to prove them directly but to establish a result on the zeros of functions of exponential type (Theorem 1), which can replace the properties of trigonometrical polynomials used in the proofs of [2]. Inequalities may then be derived in the same way as there with a complete discussion of the cases of equality, which would not be possible by a direct approach. Theorem 1 may also have other applications. A special case of it has been established by Duffin and Schaeffer [1] in proving Bernstein's theorem on the derivative of a function of exponential type.

An entire function f is here said to be of exponential type σ if for every $\varepsilon > 0$ there is a constant C such that

$$|f(z)| \leqq Ce^{(\sigma+\varepsilon)|z|} . \tag{1}$$

We shall here study *the set* R_σ *of functions* f *of exponential type* $\sigma > 0$ *such that* $f(x)$ *is real and* $-1 \leqq f(x) \leqq 1$ *when* x *is real.* It is easily proved by the Phragmén–Lindelöf principle (cf. [1]) that if $f \in R_\sigma$ we even have

$$|f(x+iy)| \leqq e^{\sigma|y|} \tag{2}$$

which sharpens (1).

By a method due to Lewitan [4] in the form given by Hörmander [3], we first prove that the functions in R_σ can be approximated in a suitable fashion by periodical trigonometrical polynomials. Let φ be the function

$$\varphi(x) = \sin^2 \pi x/(\pi x)^2 . \tag{3}$$

Then we have

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$$\varphi(x) \geqq 0, \qquad \varphi(0) = 1, \qquad \sum_{-\infty}^{+\infty} \varphi(x+n) = 1\,, \tag{4}$$

where the last equality follows from a well-known expansion of $1/\sin^2 \pi x$. Now take any function f in R_σ and put for real x and real $h > 0$,

$$f_h(x) = \sum_{-\infty}^{+\infty} \varphi(hx+n)\, f(x+nh^{-1})\,. \tag{5}$$

Since $-1 \leqq f(t) \leqq 1$ it follows from (3) and (4) that the series converges absolutely and locally uniformly, so that $f_h(x)$ is continuous and

$$-1 \leqq f_h(x) \leqq 1\,.$$

Furthermore, $f_h(x)$ has the period $1/h$ and the Fourier coefficients

$$c_\nu(h) = h \int_0^{h^{-1}} f_h(x)\, e^{-2\pi i\nu hx} dx = h \int_{-\infty}^{+\infty} \varphi(hx)\, f(x)\, e^{-2\pi i \nu hx}\, dx\,. \tag{6}$$

It follows from (6) that $c_\nu = 0$ if $2\pi|\nu h| \geqq \sigma + 2\pi h$. In fact, using the estimate (2) and the explicit form (3) of φ, we find that we may integrate over any line $\operatorname{Im} z = \text{constant}$ instead of the real axis, and get

$$c_\nu(h) = h \int_{-\infty}^{+\infty} \varphi[h(x+iy)]\, f(x+iy)\, e^{-2\pi i \nu h(x+iy)}\, dx\,.$$

Now $|\varphi[h(x+iy)]| \leqq (\pi h)^{-2}(x^2+y^2)^{-1} e^{2\pi h|y|}$ and $|f(x+iy)| \leqq e^{\sigma|y|}$, so that we get for any $y \neq 0$,

$$\begin{aligned} |c_\nu(h)| &\leqq h^{-1}\pi^{-2} e^{2\pi\nu hy + |y|(\sigma+2\pi h)} \int_{-\infty}^{+\infty} dx/(x^2+y^2) \\ &= (\pi h|y|)^{-1}\, e^{2\pi\nu hy + |y|(\sigma+2\pi h)}\,. \end{aligned}$$

Letting $y \to +\infty$ or $y \to -\infty$ it follows that $c_\nu(h) = 0$ if $\sigma + 2\pi h \leqq |2\pi\nu h|$.

Thus f_h is a trigonometrical polynomial, and hence it is defined in the whole complex plane. We shall prove that $f_h(z) \to f(z)$ uniformly in every bounded set when $h \to 0$. First we note that for real x we have the formulas

$$f_h(x) - f(x) = \big(\varphi(hx) - 1\big) f(x) + \sum_{n \neq 0} \varphi(hx+n)\, f(x+nh^{-1})\,,$$

$$|f_h(x) - f(x)| \leqq \big(1 - \varphi(hx)\big) + \sum_{n \neq 0} \varphi(hx+n) = 2\big(1 - \varphi(hx)\big)\,,$$

from which the uniform convergence on every bounded set of the real axis follows. Since f_h is of exponential type $\sigma + 2\pi h$ it follows from (2) that

$$|f_h(x+iy)| \leqq e^{(\sigma+2\pi h)|y|},$$

and the uniform convergence in every bounded set of the complex plane is hence a consequence of Vitali's theorem.

Summing up we have proved the following lemma.

LEMMA. *The functions f_h defined by* (5) *with an $f \in R_\sigma$ are trigonometrical polynomials with period $1/h$ and order less than $1+\sigma/2\pi h$. When x is real we have $-1 \leqq f_h(x) \leqq 1$, and $f_h(z) \to f(z)$ uniformly in every bounded set when $h \to 0$.*

Let $E_n(x, \alpha)$ be the trigonometrical polynomial in x defined by

$$E_n(x, \alpha) = T_{2n}[\cos(\alpha/2n)\cos\tfrac{1}{2}x] \tag{7}$$

where T_{2n} is the Tchebycheff polynomial of degree $2n$ and $0 \leqq \alpha < n\pi$. (Polynomials of this kind were the essential tool in the proofs of [2]). Let k be the least integer $\geqq \alpha/\pi$; we have $0 \leqq k \leqq n$. In the interval $(-\pi, \pi)$ the graph of E_n has $2n-2k$ branches passing between -1 and $+1$, that is, there are $2n-2k$ intervals without common interior points where $E_n(x, \alpha)$ varies between -1 and $+1$. Hence if $P(x)$ is any real trigonometrical polynomial of order $\leqq n$ with period 2π such that $-1 \leqq P(x) \leqq 1$ for real x and $P \neq E_n$, the polynomial

$$P(x) - E_n(x, \alpha)$$

has at most $2k$ zeros in the strip $-\pi \leqq \mathrm{Re}\, x < \pi$ besides one zero in each of the $2n-2k$ intervals just mentioned.

We now prove an analogous statement for functions of exponential type.

THEOREM 1. *Let $f \in R_\sigma$ and let $F_\sigma(x, \alpha)$ $(\alpha \geqq 0)$ be the function in R_σ defined by*

$$F_\sigma(x, \alpha) = \cos(\sigma^2 x^2 + \alpha^2)^{\frac{1}{2}}. \tag{8}$$

Then $f - F_\sigma$ vanishes identically or else it has a zero in every interval of the real axis where $F_\sigma(x, \alpha)$ varies between -1 and $+1$, and besides those at most $2k$ zeros where k is the least integer $\geqq \alpha/\pi$.

We observe that the $2k$ "free" zeros may be complex or real and that zeros shall be counted with their multiplicities. It is evident that we may replace $F_\sigma(x, \alpha)$ by $F_\sigma(x-\beta, \alpha)$ with real constant β in the theorem. For $\alpha = 0$ it then reduces to a theorem of Duffin and Schaeffer [1].

PROOF OF THEOREM 1. We form the approximating trigonometrical polynomials $f_h(x)$ defined by (5). They are of order

$$N = N(h) = 1 + [\sigma/2\pi h]$$

at most, and have the period $1/h$. We now use the properties of the function $E_n(x, \alpha)$ stated above. It follows that the difference

$$f_h(x/2\pi h) - E_N(x, \alpha)$$

is either $\equiv 0$ or else it has at most $2k$ zeros in the strip $-\pi \leqq \mathrm{Re}\, x < \pi$ besides one in every interval where E_N varies between -1 and $+1$. Hence

$$D_h(x) = f_h(x) - E_N(2\pi h x, \alpha)$$

is either $\equiv 0$ or has at most $2k$ zeros in the strip $-1/2h \leqq \mathrm{Re}\, x < 1/2h$ besides one in every interval where $E_N(2\pi hx, \alpha)$ varies between -1 and $+1$. Now we have proved that $f_h(x) \to f(x)$ uniformly in every compact part of the complex plane when $h \to 0$. Using the explicit form of $E_N(2\pi hx, \alpha)$ it is elementary to prove that

$$E_N(2\pi hx, \alpha) \to F_\sigma(x, \alpha)$$

in the same sense when $h \to 0$. Hence $D_h(x) \to f(x) - F_\sigma(x, \alpha) = D(x)$ when $h \to 0$. But if $D(x)$ does not vanish identically, it follows from a classical theorem by Hurwitz that the zeros of $D(x)$ are the limits of the zeros of $D_h(x)$, which proves our proposition.

We now generalize Theorem 2 of [2]. The proof only uses our Theorem 1 for $\alpha = 0$ and hence the theorem is contained in the results of Duffin and Schaeffer [1], though not explicitly stated there.

THEOREM 2. *The values which can be assumed by functions in R_σ at a fixed point it, where t is real and $\neq 0$, may be written in the form* $\cos(a+ib)$ *with real a and b and $|b| \leqq \sigma |t|$. If $|b| = \sigma |t|$ this value is only attained by* $f(x) = \cos(bt^{-1}x + a)$. *If $|b| < \sigma |t|$ it is taken by several f in R_σ.*

PROOF. If $|b| \geqq \sigma |t|$, a function f in R_σ is of exponential type $|bt^{-1}|$. Hence according to Theorem 1, the difference

$$f(x) - F_{|bt^{-1}|}(x + ab^{-1}t, 0) = f(x) - \cos(bt^{-1}x + a)$$

cannot vanish at a complex point unless it vanishes identically. This proves that if $f(it) = \cos(a+ib)$, we must have $|b| < \sigma |t|$ or otherwise $f(x) = \cos(bt^{-1}x + a)$ and $|b| = \sigma |t|$.

Finally, in order to prove the last assertion of the theorem and at the same time to prepare Theorem 3, we shall study the functions f in R_σ such that

$$f(it) = \cos(a+ib) \tag{9}$$

with given fixed a, b and $t \neq 0$. It will be supposed that $|b| < \sigma |t|$ since (9) is otherwise impossible or determines f uniquely.

A function of the form $F_\sigma(x-\beta, \alpha)$ with F_σ defined by (8) satisfies (9) if with some integer ν we have $\sigma^2(it-\beta)^2+\alpha^2=(a+2\nu\pi+ib)^2$, or

$$
\left.\begin{aligned}
\alpha_\nu^2 &= (\sigma^2t^2-b^2)\big(1+(a+2\nu\pi)^2/\sigma^2t^2\big)\,,\\
\beta_\nu &= -(a+2\nu\pi)b/\sigma^2t\,,
\end{aligned}\right\}\ \nu = 0,\ \pm 1,\ \pm 2,\ \ldots\,. \tag{10}
$$

We may also write $F_\sigma(x-\beta_\nu, \alpha_\nu)=\cos A_\nu(x)$ where

$$
\begin{aligned}
A_\nu(x)^2 &= \sigma^2(x-\beta_\nu)^2+\alpha_\nu^2\\
&= \sigma^2x^2+2x(a+2\nu\pi)b/t+\sigma^2t^2-b^2+(a+2\nu\pi)^2\,.
\end{aligned} \tag{11}
$$

It is clear that all the functions $\cos A_\nu(x)$, $\nu=0,\ \pm 1,\ \ldots$, are in R_σ and satisfy (9). This completes the proof of Theorem 2.

THEOREM 3. *For any $f\in R_\sigma$ such that $f(it)=\cos(a+ib)$, where $t\neq 0$ and $b^2<\sigma^2t^2$, we have in the interval I_ν where $A_\nu(x)\leqq\pi$,*

$$f(x) \geqq \cos A_\nu(x)\,, \tag{12}$$

with inequality at every point of I_ν unless $f(x)\equiv\cos A_\nu(x)$. The intervals I_ν are disjoint. If x is not in any I_ν there exist several functions $f\in R_\sigma$ such that $f(it)=\cos(a+ib)$ and $f(x)=-1$.

PROOF. First suppose that I_ν contains more than one point, that is $\alpha_\nu<\pi$, and that

$$f(x) - \cos A_\nu(x) \not\equiv 0\,.$$

Since this difference vanishes at the complex points $\pm it$ it follows from Theorem 1 that it cannot have any other zeros except the trivial ones. Hence there is no zero in the interior of I_ν, nor any at an endpoint of I_ν since this would be a double zero of which only one belongs to the trivial ones.

Now suppose that I_ν contains only the point β_ν, that is $\alpha_\nu=\pi$, and that $f(\beta_\nu)=-1$ but $f(x)\not\equiv\cos A_\nu(x)$. A graph shows that if α is less than and close to π, the difference

$$f(x) - F_\sigma(x-\beta_\nu, \alpha)$$

has its two "free" zeros in the vicinity of β_ν. Hence letting $\alpha\to\pi$ it follows from Hurwitz' theorem, that the limit $f(x)-\cos A_\nu(x)$ has its two "free" zeros at β_ν, and therefore cannot vanish for $x=it$.

We now prove the latter part of the theorem. If we replace σ in (11) by $\sigma^\star$ where $|bt^{-1}|\leqq\sigma^\star\leqq\sigma$ we get a function $A_\nu^\star$. It is obvious that $f^\star(x)=\cos A_\nu^\star(x)$ is in $R_{\sigma^\star}$ and hence in R_σ, and that $f^\star(it)=\cos(a+ib)$. When $\sigma^\star=|bt^{-1}|$ we have $A_\nu^\star(x)=|bxt^{-1}+a+2\nu\pi|$, so that for a given x_0 there is at least one index μ such that $A_\mu^\star(x_0)\leqq\pi$. If x_0 is not in any I_ν

it now follows by using (11) that $A_\mu{}^\star(x_0)$ decreases monotonically from a value $>\pi$ to a value $\leqq \pi$ when $\sigma^\star$ decreases from σ to $|bt^{-1}|$. For a suitable $\sigma^\star$ in this interval we must hence have $A_\mu{}^\star(x_0)=\pi$ so that $f^\star(x_0)=-1$.

Since $\sigma>|bt^{-1}|$ it follows from (11) that $A_\nu(x)>|bxt^{-1}+a+2\nu\pi|$ so we cannot have $A_\nu(x)\leqq\pi$ for two values of ν at the same point x. Hence the intervals I_ν are disjoint.

To prove the last assertion we note that if x_0 is not in any I_ν we have constructed a function f in $R_{\sigma-\varepsilon}$ with some $\varepsilon>0$ such that $f(x_0)=-1$ and $f(it)=\cos(a+ib)$. Now take any function g of exponential type ε such that $0\leqq g(x)\leqq 2$ if x is real and $g(it)=g(x_0)=0$. Then the function

$$f_1 = f - fg$$

is in R_σ and we have $f_1(it)=\cos(a+ib)$, $f_1(x_0)=-1$, which completes the proof.

Since a function f is in R_σ and satisfies $f(it)=\cos(a+ib)$ if and only if $-f$ is in R_σ and $-f(it)=\cos(a+\pi+ib)$, it is easy to derive upper estimates also from Theorem 3.

The special case of Theorem 3 when $b=0$ corresponds to Theorem 1 of [2] and has a simple form which merits explicit formulation.

COROLLARY. *If $f\in R_\sigma$ and $f(it)=\cos a$, where $t\neq 0$ and $0\leqq a<\pi$, we have*

$$f(x) \geqq \cos\,[\sigma^2(x^2+t^2)+a^2]^{\frac{1}{2}}$$

when $\sigma^2(x^2+t^2)+a^2\leqq\pi^2$.

An examination of the proof shows that Theorem 3 and hence the corollary are also true when $t=0$ provided that not only $f(0)=\cos a$ but also $f'(0)=0$. The full proof can be omitted and we end the paper by a simple application of the corollary.

THEOREM 4. *If $f\in R_\sigma$ and $f(0)=\cos c$, $0<c\leqq\pi$, we have $f(u+iv)\neq 1$ when*

$$\sigma^2(u^2+v^2) < c^2; \tag{12}$$

but if $u+iv$ is not in this circle we have $f(u+iv)=1$ for some $f\in R_\sigma$ with $f(0)=\sin c$.

PROOF. Suppose that $f(u+iv)=1$. If $v=0$ it follows that also $f'(u)=0$, for we have $f(x)\leqq 1$ for every real x. The function $g(x)=f(x+u)$ is evidently in R_σ and since $g(iv)=1=\cos 0$, $g(-u)=\cos c$, we can apply the corollary. It follows that

$$g(-u) = \cos c \geqq \cos [\sigma^2(u^2+v^2)]^{\frac{1}{2}} \quad \text{if} \quad \sigma^2(u^2+v^2) \leqq \pi^2 .$$

Hence the inequality

$$\sigma^2(u^2+v^2) \geqq c^2 \tag{13}$$

follows if the left hand side is $\leqq \pi^2$. Since $c^2 \leqq \pi^2$ it is also true if the left hand side is $> \pi^2$. Hence (13) is valid without restriction, which proves one part of the theorem.

On the other hand, the function

$$f(x) = \cos \left(\frac{c^2[(x-u)^2+v^2]}{u^2+v^2} \right)^{\frac{1}{2}}$$

is in R_σ if $u+iv$ is not in the circle (12) and we have evidently $f(0) = \cos c$, $f(u+iv) = 1$. This completes the proof.

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