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SOME PROBLEMS IN THE THEORY OF ALMOST PERIODIC FUNCTIONS

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Introduction. In this paper we study some problems concerning almost periodic functions on an abelian topological group G. These functions form a Banach algebra A under the norm $\|x\| = \sup_t |x(t)|$, pointwise addition and convolution multiplication. The main theorems of the theory of almost periodic functions can be expressed very simply in terms of the ideal structure of A.

In section 2 we give a solution of the multiplier problem for almost periodic Fourier series, which corresponds to determining the operators on A that commute with translations. The answer shows that the multipliers are in general non-measurable, and the Weil-van Kampen compactification is therefore an indispensable tool, even for almost periodic functions on the real line. An interesting special case of the multiplier problem is the following question: What subsets S (called distinguished sets) of the character group G^* have the property that $\Sigma_{\chi \in S}\, a(\chi)\chi(t)$ is an almost periodic Fourier series whenever $\Sigma a(\chi)\chi(t)$ is an almost periodic Fourier series? Under an additional (positivity preserving) restriction the distinguished sets are determined (theorem 3), but we have not been able to give any simple identification of these sets in general, and theorem 4 seems to indicate that rather deep algebraic properties of the character group G^* are involved.

In the last section the possibility of permuting Fourier exponents in an almost periodic Fourier series is investigated. This corresponds to the study of the automorphisms of A. In theorem 5 the isometric automorphisms of A are determined.

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1. Preliminaries. Let G be an abelian topological group with the group operation denoted multiplicatively. A complex valued continuous func-

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tion $x(t)$ on G is called almost periodic (Bochner [2], von Neumann [16]) if the set of translates of $x(t)$, $\{x(ht) \mid h \in G\}$, has a compact closure $\mathfrak{H}\{x(ht)\}$ in the Banach space $\mathfrak{C}^{\infty}(G)$ of complex valued bounded continuous functions on G under the norm $\|x\| = \sup_G |x(t)|$. Special almost periodic functions are the characters of G, namely the continuous functions $\chi(t)$ satisfying:

$$\chi(st) = \chi(s)\,\chi(t) \quad \text{for all } s,t \in G \quad \text{and} \quad |\chi(t)| \equiv 1 .$$

The set of all almost periodic functions on G is a Banach space A under the norm $\|x\| = \sup_t |x(t)|$. On A there exists a form $\mathfrak{M}(x)$, the mean value, which is uniquely determined by the four following properties:

(i) $\mathfrak{M}(1) = 1$;

(ii) $\mathfrak{M}(\alpha x + \beta y) = \alpha\mathfrak{M}(x) + \beta\mathfrak{M}(y)$ for all $x,y \in A$, α,β complex;

(iii) $\mathfrak{M}(x(ct)) = \mathfrak{M}(x(t))$ for each $c \in G$;

(iv) $\mathfrak{M}(x) \geqq 0$ if $x \geqq 0$.

Other important properties of $\mathfrak{M}$ are:

(v) (The ergodic property) $\mathfrak{M}(x)$ is the unique constant in the closed convex hull of the translates of $x(t)$;

(vi) $x \geqq 0$ and $\mathfrak{M}(x) = 0$ imply $x \equiv 0$.

The convolution of two almost periodic functions $x(t)$ and $y(t)$ is defined by

$$x * y(t) = \mathfrak{M}_s\{x(ts^{-1})\,y(s)\}$$

and is again an almost periodic function. The Banach space A is a Banach algebra under convolution-multiplication. (For the terminology of the theory of Banach algebras see Loomis [14]). This algebra does not in general have an identity element (in fact only if the group is finite), but a substitute is given by the following property: To every $x \in A$ and $\varepsilon > 0$ there exists an element $y \in A$ such that

$$\|x * y - x\| < \varepsilon . \tag{1}$$

From this it follows at once that a closed ideal in A is an invariant subspace (under translation), and the converse, that a closed invariant subspace is a closed ideal, is a consequence of the above-mentioned properties of the mean value. (See von Neumann [16, p. 451].)

If $x(t)$ is almost periodic and $\chi(t)$ is a character, then

$$x * \chi(t) = a(\chi)\,\chi(t)$$

where $a(\chi)$ is a complex number, the Fourier coefficient of x at χ. The formal series

$$x(t) \sim \sum a(\chi)\,\chi(t) \tag{2}$$

is called the Fourier series of $x(t)$; formal series that occur as Fourier series for almost periodic functions are called *almost periodic Fourier series*. The characters χ at which $a(\chi) \neq 0$ are called the Fourier exponents of $x(t)$.

Let G^* be the character group of G. The group G^* will usually be given the discrete topology and then denoted G^*_d. For an arbitrary subset $S \subset G^*$ consider the set

$$(3) \qquad I_S = \{x \mid x \in A,\ a(\chi) = \mathfrak{M}_t(x(t)\,\overline{\chi(t)}) = 0 \text{ for each } \chi \notin S\},$$

which is the family of functions in A whose Fourier exponents belong to S. It is easily seen that I_S is a closed ideal. Conversely, by the approximation theorem, every closed ideal is of this form.

There is therefore a one-to-one correspondence between all the closed ideals in the algebra A and all subsets of the character group G^*. Since this correspondence preserves the inclusion, the regular maximal ideals, which are closed, will correspond to complements of points and are of the form

$$(4) \qquad M_\chi = \{x \mid x \in A,\ a(\chi) = 0\}.$$

The uniqueness theorem for the Fourier series representation (2) may therefore be expressed: A is semi-simple. Similarly, the approximation theorem has shown that every closed ideal in A is the intersection of the regular maximal ideals containing it.

An application is the following proposition. (Σ^* means there and in the sequel a finite sum.)

PROPOSITION 1. *The absolute value of the mean value can be represented*

$$|\mathfrak{M}(x)| = \inf_{\alpha_i, a_i} \left\| \sum{}^* \alpha_i\, x(ta_i) \right\|$$

where the a_i run through all elements of G and the α_i run through all complex numbers for which $\Sigma^* \alpha_i = 1$.

PROOF. Evidently

$$\inf_{\Sigma \alpha_i = 1,\ \text{all } a_i} \left\| \sum{}^* \alpha_i\, x(ta_i) \right\|$$

$$= \inf_{\Sigma \beta_i = 0,\ \text{all } b_i} \left\| x(t) - \sum{}^* \beta_i\, x(tb_i) \right\| = \text{distance}\{x, \mathfrak{N}\},$$

where $\mathfrak{N}$ is the closure of the set

$$\left\{ \sum{}^* \beta_i\, x(tb_i) \;\middle|\; \sum{}^* \beta_i = 0,\ b_i \text{ arbitrary} \right\}.$$

$\mathfrak{N}$ is a closed invariant subspace, hence a closed ideal. There exists therefore a set $N \subset G^*$ such that $\mathfrak{N}$ is the set of functions whose Fourier exponents belong to N. If $y(t)=\Sigma^*\beta_i x(tb_i)$ then

$$\mathfrak{M}\{y(t)\,\overline{\chi(t)}\} = \left[\sum{}^*\beta_i\chi(b_i)\right] a(\chi)$$

where

$$a(\chi) = \mathfrak{M}_t\{x(t)\,\overline{\chi(t)}\}\,.$$

Thus the set N is identical with the Fourier exponent set of $x(t)$ with the exception of $\chi=1$, which does not belong to N, but may be a Fourier exponent of $x(t)$. We have then for arbitrary $\varepsilon>0$ a function $y_1\in\mathfrak{N}$ such that

$$\begin{aligned} \text{distance } \{x, \mathfrak{N}\} &= \inf_{y\in\mathfrak{N}} \|x-y\| \\ &\geqq \|x-y_1\| - \varepsilon \\ &\geqq \mathfrak{M}(|x-y_1|) - \varepsilon \\ &\geqq |\mathfrak{M}(x-y_1)| - \varepsilon \\ &= |\mathfrak{M}(x)| - \varepsilon\,. \end{aligned}$$

On the other hand

$$\inf_{y\in\mathfrak{N}} \|x-y\| \leqq \|x-(x-a)(1)\| = |a(1)| = |\mathfrak{M}(x)|\,,$$

which proves the proposition.

An interesting problem that suggests itself is the description of the automorphisms of the algebra A, that is, one-to-one mappings T of A onto itself such that

$$\begin{aligned} T(\alpha x+\beta y) &= \alpha T(x) + \beta T(y) \qquad && \alpha, \beta \text{ complex}, \ x, y\in A \\ T(x*y) &= T(x)*T(y) && x, y \in A\,. \end{aligned}$$

Under the mapping regular maximal ideals will be mapped onto regular maximal ideals, such that the automorphism T induces a permutation of the character group G^*. Conversely, if σ is a permutation of G^*, such that for every Fourier series $\Sigma a(\chi)\chi(t)$ for a function in A, the series

$$\sum a(\chi)\,\sigma(\chi)(t) \qquad \text{and} \qquad \sum a(\chi)\,\sigma^{-1}(\chi)(t)$$

are also almost periodic Fourier series, then there is, in the obvious way, induced an automorphism of A. The above-mentioned problem will be considered in section 5.

To avoid trivial discussions *we shall always assume in the sequel that G is a maximal almost periodic group.*

2. Multipliers. We recall that almost periodicity of $x(t)$ on G means compactness of the closure $\mathfrak{H}\{x(ht)\}$. As pointed out by Stepanoff and Tychonoff, this amounts to an embedding of G in a compact group. Later A. Weil [17] and E. van Kampen [13] introduced the universal compactification which reveals the nature and scope of the theory of almost periodicity. Their construction simplifies in the abelian case due to the duality theory which is then available, and we give the main details in this case since these will be useful later.

Let G be an abelian maximal almost periodic group and let $\mathbf{G}$ be the compact character group of G^*_d and let $(\chi, \boldsymbol{t})$ be the function on $G^*_d \times \mathbf{G}$ that defines the duality between G^*_d and $\mathbf{G}$. For each fixed $t \in G$, $\chi(t)$ is a continuous character of G^*_d, and since the group G is maximal almost periodic, different $t \in G$ give different characters $\chi(t)$, such that G may be considered as a subset of $\mathbf{G}$. The topology of $\mathbf{G}$ is the weakest one for which the functions $\boldsymbol{t} \to (\chi, \boldsymbol{t})$ are continuous and these functions are extensions of the functions $t \to \chi(t)$.

We shall prove that G is dense in $\mathbf{G}$. Suppose to the contrary that the closure of G in $\mathbf{G}$, say $\overline{G}$, is $\neq \mathbf{G}$. Choose 1° p in $\mathbf{G} - \overline{G}$, 2° a neighborhood U_p of p such that U_p and $\overline{G}$ are disjoint, 3° compact neighborhoods C_p and C_e of p and e, respectively, such that $C_p C_e^{-1} \subset U_p$, 4° continuous positive functions $\boldsymbol{g}_p(\boldsymbol{t})$ and $\boldsymbol{g}_e(\boldsymbol{t})$ on $\mathbf{G}$ with support contained in C_p and C_e, respectively, and $\boldsymbol{g}_p(p) \neq 0$, $\boldsymbol{g}_e(e) \neq 0$. Further, denote by $d\boldsymbol{s}$ the normalized Haar measure on $\mathbf{G}$ and consider the convolution

$$\boldsymbol{f}(\boldsymbol{t}) = \int_{\mathbf{G}} \boldsymbol{g}_p(\boldsymbol{t}\boldsymbol{s})\, \boldsymbol{g}_e(\boldsymbol{s})\, d\boldsymbol{s} .$$

$\boldsymbol{f}(\boldsymbol{t})$ is not identically 0, but vanishes on G. The functions $\boldsymbol{g}_e(\boldsymbol{t})$, $\boldsymbol{g}_p(\boldsymbol{t})$ and $\boldsymbol{f}(\boldsymbol{t})$ are almost periodic on $\mathbf{G}$ and $\boldsymbol{f}(\boldsymbol{t})$ has an absolutely convergent Fourier series

$$\boldsymbol{f}(\boldsymbol{t}) = \sum a(\chi)\, (\chi, \boldsymbol{t}) .$$

On restricting this equation to G, we get $\sum a(\chi)\chi(t) \equiv 0$ which implies $a(\chi) \equiv 0$ and thus $\boldsymbol{f}(\boldsymbol{t}) \equiv 0$, which gives a contradiction. Hence G is a dense subgroup of $\mathbf{G}$.

The relative topology of G on $\mathbf{G}$ is, of course, the weakest one for which the characters $\chi(t)$ are continuous. If $z(t)$ is almost periodic on G, it is a limit of linear combinations of characters in the topology of uniform convergence on G. As the characters $\chi(t)$ are uniformly continuous on G in the relative topology of $\mathbf{G}$, the same is true of $z(t)$. Since G is dense in $\mathbf{G}$, $z(t)$ can be extended uniquely to a continuous function on $\mathbf{G}$. Conversely, if $\boldsymbol{z}(\boldsymbol{t})$ is continuous on $\mathbf{G}$, the restriction to G, $z(t)$, is continuous in the relative topology of $\mathbf{G}$, and a fortiori continuous in the original topology of G. Moreover, since $\mathbf{z}(\mathbf{t})$ is almost periodic on $\mathbf{G}$, $z(t)$ is almost periodic on G.

We shall now give a solution of the multiplier problem, raised in Bochner [3]. This problem appears in differentiation and integration of almost periodic functions and more generally in the consideration of linear operations on A which commute with translations.

DEFINITION. A complex valued function $q(\chi)$ on G^* is called a *multiplier* if for every almost periodic Fourier series $\Sigma_\chi a(\chi)\chi(t)$, the series $\Sigma_\chi a(\chi)q(\chi)\chi(t)$ is also an almost periodic Fourier series.

For the case $G=G^*=R$ the continuous multipliers have been determined by R. Doss [7]. However, the typical multipliers, which are the Fourier coefficients themselves, show at once that the problem has nothing to do with continuity (or even measurability) and besides, the continuity restriction excludes the most interesting multipliers as we shall see later.

We recall that a complex valued function $p(t)$ on a group G is called positive-definite if

$$\sum_{i,j=1}^{n} p(t_i t_j^{-1})\, \alpha_i \bar{\alpha}_j \geqq 0$$

for arbitrary $t_1, \ldots, t_n \in G$ and arbitrary set $\alpha_1, \ldots, \alpha_n$ of complex numbers (no continuity assumed).

THEOREM 1. *The multipliers are precisely the functions of the form*

$$q(\chi) = p_1(\chi) - p_2(\chi) + i p_3(\chi) - i p_4(\chi) \tag{7}$$

where the $p_\varkappa(\chi)$ are positive definite functions on G^*.

PROOF. We prove first that every function of the form (7) is a multiplier. $q(\chi)$ is a linear combination of continuous positive definite functions on the discrete group G^*_d and, by the generalized Bochner theorem, gives rise to a bounded measure μ on the compact character group $\mathbf{G}$ such that

$$q(\chi) = \int_{\mathbf{G}} (\chi, \mathbf{t})\, d\mu(\mathbf{t})\,. \tag{8}$$

Now let $z(t) \sim \Sigma a(\chi)\chi(t)$ and let $\mathbf{z}(\mathbf{t})$ be the extension of $z(t)$ to $\mathbf{G}$. Then

$$a(\chi) = \mathfrak{M}_t\{z(t)\,\overline{\chi(t)}\} = \int_{\mathbf{G}} \mathbf{z}(\mathbf{t})\,\overline{(\chi, \mathbf{t})}\, d\mathbf{t}\,. \tag{9}$$

From (8) and (9) we get

$$a(\chi)\, q(\chi) = \int_{\boldsymbol{G}} \boldsymbol{y}(\boldsymbol{t})\, \overline{(\chi, \boldsymbol{t})}\, d\boldsymbol{t} \tag{10}$$

where

$$\boldsymbol{y}(\boldsymbol{t}) = \int_{\boldsymbol{G}} \boldsymbol{z}(\boldsymbol{ts})\, d\mu(\boldsymbol{s}) .$$

Since μ is a bounded measure and $\boldsymbol{z}(\boldsymbol{t})$ is continuous on $\boldsymbol{G}$, $\boldsymbol{y}(\boldsymbol{t})$ is continuous on $\boldsymbol{G}$. The restriction to G, $y(t)$, is almost periodic and by (10) has the Fourier series

$$y(t) \sim \sum a(\chi)\, q(\chi)\, \chi(t) , \tag{11}$$

and $q(\chi)$ is a multiplier.

Conversely, let $q(\chi)$ be a multiplier, such that to any almost periodic function $z(t) \sim \Sigma a(\chi)\chi(t)$ corresponds another almost periodic function

$$Q\, z(t) \sim \sum a(\chi)\, q(\chi)\, \chi(t) . \tag{12}$$

Q is a linear transformation of A into itself. We shall prove that Q is continuous. By the closed graph theorem (Banach [1, p. 41]), we only have to prove that if $\|x_n - x\|$ tends to 0 and $\|Qx_n - y\|$ tends to 0, then $Qx = y$; however, this is obvious.

Consider now the linear form

$$L(z) = (Qz)(e)$$

on the space A. By the one-to-one correspondence between A and the space of continuous functions on $\boldsymbol{G}$, $L(z)$ can be viewed as a linear form on this latter space and is continuous in the topology of uniform convergence on $\boldsymbol{G}$, since Q is continuous. Such a form induces a bounded measure μ on $\boldsymbol{G}$, for which

$$L(z) = \int_{\boldsymbol{G}} \boldsymbol{z}(\boldsymbol{t})\, d\mu(\boldsymbol{t}) \qquad \text{for all } z \in A .$$

We apply this when $z(t)$ is a character $\chi(t)$ of G, and consequently $\boldsymbol{z}(\boldsymbol{t})$ is a character $(\chi, \boldsymbol{t})$ on $\boldsymbol{G}$, and we get

$$q(\chi) = L(\chi) = \int_{\boldsymbol{G}} (\chi, \boldsymbol{t})\, d\mu(\boldsymbol{t}) .$$

If $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ is the Jordan decomposition of the measure μ into positive measures, the corresponding decomposition of $q(\chi)$ is of the form (7), and the theorem is proved.

Remark. From the proof it is clear that the multipliers that transfer positive almost periodic functions into functions of the same kind are precisely the positive definite functions.

It is well known (Godement [9, Chap. III]), that on the space of Fourier-Stieltjes transforms

$$q(\chi) = \int_G (\chi, t)\, d\mu(t)$$

there exists a mean value, that is, a form $\mathfrak{M}$ satisfying (i)–(iv) in section 1, and these properties determine the mean value uniquely. $\mathfrak{M}$ also satisfies (v) in section 1 as we shall use later. For a Fourier-Stieltjes transform which is an almost periodic function, the two mean values coincide.

We split the bounded measure μ into a discrete part and a continuous part,

$$\mu = \mu_c + \mu_d ,$$

where μ_d is of the form $\Sigma_i a_i \varepsilon_{t_i}$ and μ_c vanishes on sets consisting of a single point. Here ε_a denotes the measure $\varepsilon_a(f) = f(a)$ for all $f \in \mathfrak{C}^\infty(G)$. The Fourier-Stieltjes transform $q(\chi)$ splits accordingly,

$$q(\chi) = q_c(\chi) + q_d(\chi) ,$$

where

$$q_d(\chi) = \int_G (\chi, t)\, d\mu_d(t) = \sum a_i(\chi, t_i) .$$

Hence $q_d(\chi)$ is almost periodic on the discrete group $G^*{}_d$ and we have the Parseval-Bohr relation

$$\mathfrak{M}(|q_d|^2) = \sum |a_i|^2 . \tag{13}$$

For the transform $q_c(\chi)$ we have by Bochner [4, Kap. IV] and Godement [9, Chap. III]

$$\mathfrak{M}(|q_c|^2) = 0 . \tag{14}$$

From the relation

$$|q|^2 = |q_c|^2 + q_c\bar{q}_d + \bar{q}_c q_d + |q_d|^2$$

and the Schwarz inequality

$$|\mathfrak{M}(q_c\bar{q}_d)|^2 \leqq \mathfrak{M}(|q_c|^2)\, \mathfrak{M}(|q_d|^2)$$

we see that (13) and (14) can be combined in the formula

$$\mathfrak{M}(|q|^2) = \sum |a_i|^2 . \tag{15}$$

(This formula has been published recently by Eberlein [8].)

Now let Q be the endomorphism of A corresponding to the multiplier $q(\chi)$, that is,

$$Q\, z(t) \sim \sum a(\chi)\, q(\chi)\, \chi(t) \qquad \text{if} \qquad z(t) \sim \sum a(\chi)\, \chi(t) .$$

It is easily seen that these endomorphisms are precisely the ones that commute with the translations of G.

THEOREM 2. *The endomorphism Q is isometric if and only if*

$$(16)\qquad q(\chi\psi)\, q(1) = q(\chi)\, q(\psi) \qquad \textit{for all } \chi,\psi \in G^* \quad \textit{and} \quad |q(\chi)| \equiv 1 .$$

PROOF. If Q is isometric it is clear that $|q(\chi)| \equiv 1$. We put

$$r(\chi) \equiv q(\chi)/q(1) .$$

Let χ and ψ be arbitrary elements of G^* and consider the function

$$y(t) = 1 + \chi(t) + \psi^{-1}(t) .$$

Obviously $\|y\| = y(l) = 3$. Then

$$9 = \|(1+\chi+\psi^{-1})^2\| = \|3+\chi+\psi^{-1}+\chi^{-1}+\psi+\chi\psi+\chi^{-1}\psi^{-1}\|$$

and by assumption this is equal to

$$(17)\qquad \|1+1+1+r(\chi)\,\chi+r(\psi^{-1})\,\psi^{-1}+r(\chi^{-1})\,\chi^{-1}+ \\ +r(\psi)\,\psi+r(\chi\psi)\,\chi\psi+r(\chi^{-1}\psi^{-1})\,\chi^{-1}\psi^{-1}\| .$$

Each term in (17) is $\leqq 1$ in absolute value, and three terms are equal to 1, but since there are only 9 terms, and the supremum is 9, the terms simultaneously approach 1 arbitrarily closely. In other words, to any $\varepsilon > 0$ there exists a t_0 such that

$$\begin{aligned} r(\chi)\,\chi(t_0) &= 1 + \langle\varepsilon\rangle , \\ r(\chi^{-1})\,\chi^{-1}(t_0) &= 1 + \langle\varepsilon\rangle , \\ r(\psi^{-1})\,\psi^{-1}(t_0) &= 1 + \langle\varepsilon\rangle , \\ r(\chi\psi)\,\chi\psi(t_0) &= 1 + \langle\varepsilon\rangle , \end{aligned}$$

where $|\langle\varepsilon\rangle| < \varepsilon$. On multiplying together the two first equations and taking into account that ε is arbitrarily small we get $r(\chi^{-1}) = r(\chi)^{-1}$, which together with the three last equations implies $r(\chi\psi) = r(\chi)\,r(\psi)$, and (16) follows.

Conversely, if (16) holds, $q(\chi)/q(1)$ is a character on G^*_d and can therefore be written (χ, s) for some $s \in \boldsymbol{G}$. If $z(t) \sim \sum a(\chi)\chi(t)$ then

$$Q\, z(t) \sim \sum a(\chi)\, (\chi, s)\, q(1)\, \chi(t)$$

and on the compactification $\boldsymbol{G}$

$$(18)\qquad Q\, z(\boldsymbol{t}) = z(\boldsymbol{ts})\, q(1),$$

which shows that Q is isometric since $\|z\| = \|\boldsymbol{z}\|$. Thus the theorem is proved.

REMARK. Theorem 2 and (18) show that all the isometric endomorphisms of A that commute with the translations on G are induced by translations $\boldsymbol{t} \to \boldsymbol{ts}$ on the compactification $\boldsymbol{G}$, followed by multiplication with a constant of absolute value 1.

3. The distinguished sets.

DEFINITION. A distinguished set is a subset S of G^* with the following property: Whenever $\Sigma_{\chi \in G^*} a(\chi)\chi(t)$ is a Fourier series for an almost periodic function $z(t)$ then $\Sigma_{\chi \in S} a(\chi)\chi(t)$ is also a Fourier series for an almost periodic function $z_S(t)$.

We shall be concerned in this section with the problem of determining these sets.

Theorem 1 gives the answer that the distinguished sets are the subsets S of G^* whose characteristic functions are of the form

$$\varphi_S(\chi) = p_1(\chi) - p_2(\chi) + ip_3(\chi) - ip_4(\chi)$$

where the $p_i(\chi)$ are positive definite functions on G^*. However, a more explicit description is desirable.

A large class of distinguished sets is exhibited by

THEOREM 3. *The distinguished sets which preserve positivity in the sense that $z_S \geqq 0$ whenever $z \geqq 0$, are precisely the subgroups of G^*_d.*

We first prove a lemma.

LEMMA. *Let S be a subgroup of G^*_d, $S^\perp$ the compact annihilator of S in $\boldsymbol{G}$ and ds the normalized Haar measure on $S^\perp$.*

If $z(t)$ is almost periodic on G, $z_S(t)$ exists (that is, S is distinguished) and is given by

$$\int_{S^\perp} \boldsymbol{z}(\boldsymbol{ts})\, d\boldsymbol{s} = \boldsymbol{z}_S(\boldsymbol{t}) . \tag{19}$$

PROOF. We put

$$\boldsymbol{y}(\boldsymbol{t}) = \int_{S^\perp} \boldsymbol{z}(\boldsymbol{ts})\, d\boldsymbol{s} .$$

Then $\boldsymbol{y}(\boldsymbol{t})$ is continuous on $\boldsymbol{G}$ and the restriction $y(t)$ to G almost periodic. Let $y(t)$ and $z(t)$ have the Fourier coefficients $b(\chi)$ and $c(\chi)$, respectively,

$$
\begin{aligned}
(20) \qquad b(\chi) &= \int_G \left[\int_{S^\perp} z(ts)\, ds \right] \overline{(\chi, t)}\, dt \\
&= \int_{S^\perp} ds \int_G z(ts)\, \overline{(\chi, t)}\, dt \\
&= \int_{S^\perp} ds\, (\chi, s) \int_G z(ts)\, \overline{(\chi, ts)}\, dt \\
&= c(\chi) \int_{S^\perp} (\chi, s)\, ds\,.
\end{aligned}
$$

By the duality theory, $S^\perp$ is the character group of the factor group G^*_d/S. If $\chi_0 \notin S$, there exists an $s_0 \in S^\perp$ such that $(\chi_0, s_0) \neq 1$. We then have

$$\int_{S^\perp} (\chi_0, s)\, ds = (\chi_0, s_0) \int_{S^\perp} (\chi_0, s s_0^{-1})\, ds = (\chi_0, s_0) \int_{S^\perp} (\chi_0, s)\, ds$$

which implies

$$\int_{S^\perp} (\chi_0, s)\, ds = 0\,.$$

It follows from this and (20) that the Fourier series of $y(t)$ is $\Sigma_{\chi \in S} c(\chi)\chi(t)$. Hence $z_S(t)$ exists and equals $y(t)$, which proves the lemma.

Proof of Theorem 3. From the remark after theorem 1 we know that if S is distinguished and $z \geqq 0$ implies $z_S \geqq 0$, the characteristic function $\varphi_S(\chi)$ must be positive definite, that is,

$$(21) \qquad \sum_{j,k=1} \varphi_S(\chi_j \chi_k^{-1})\, \alpha_j \bar{\alpha}_k \geqq 0$$

for arbitrary $\chi_1, \chi_2, \ldots, \chi_n \in G^*$ and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. From the relations $|\varphi_S(\chi)| \leqq \varphi_S(1)$ and $\overline{\varphi_S(\chi)} = \varphi_S(\chi^{-1})$ for all $\chi \in G^*$ it follows that if S is not empty it contains 1 and furthermore $\chi \in S$ implies $\chi^{-1} \in S$.

Suppose now $\chi, \psi \in S$. We write out the condition (21) for $\chi_1 = \chi$, $\chi_2 = \psi$, $\chi_3 = 1$ and real $\alpha_1, \alpha_2, \alpha_3$. By suitable rearrangement we get

$$(\alpha_1 + \alpha_2 + \alpha_3)^2 + \alpha_1 \alpha_2 [\varphi_S(\chi \psi^{-1}) + \varphi_S(\chi^{-1} \psi) - 2] \geqq 0$$

which shows that $\chi \psi^{-1} \in S$.

Hence S is a subgroup of G^*_d. The rest of the theorem now follows from the lemma.

REMARK. In particular we see that the characteristic function of a subgroup is positive definite, a result proved by Hewitt [12].

The general distinguished sets are more complicated. All finite sets are distinguished, all translated groups are distinguished, and finite intersections and finite unions of distinguished sets are again distinguished.

Suppose S is a distinguished set. We have the representation

$$\varphi_S(\chi) = \int_G (\chi, t)\, d\mu(t)$$

where μ is a complex bounded measure on G. Since $\varphi_S(\chi)\varphi_S(\chi)=\varphi_S(\chi)$ we get

$$\mu * \mu = \mu \,. \tag{22}$$

Now μ can be uniquely decomposed,

$$\mu = \mu_c + \mu_d \,, \tag{23}$$

where μ_d is a discrete measure and μ_c vanishes on sets consisting of one single point. By (22) and (23)

$$\mu_c + \mu_d = \mu_c * \mu_c + 2\mu_c * \mu_d + \mu_d * \mu_d \,.$$

The two first terms on the right are continuous and the third term is discrete. Due to the uniqueness of decomposition,

$$\mu_d = \mu_d * \mu_d$$

and

$$\mu_c = \mu_c * \mu_c + 2\mu_c * \mu_d \,.$$

The corresponding Fourier-Stieltjes transforms then satisfy

$$q_d(\chi) = q_d(\chi)\, q_d(\chi)\,,$$
$$q_c(\chi) = q_c(\chi)\, q_c(\chi) + 2q_d(\chi)\, q_c(\chi)\,.$$

$q_d(\chi)$ is therefore the characteristic function $\varphi_N(\chi)$ of a set N. We shall prove that if G is the additive group of real numbers (here written multiplicatively), or, more generally, if G is a group whose character group G^* is infinitely divisible, the set N either is the entire G^* or is empty.

DEFINITION. A group H is said to be infinitely divisible if $H^n = H$ for every integer $n \neq 0$.

The function $\varphi_N(\chi)$ is the Fourier-Stieltjes transform of a discrete measure on G and is therefore an almost periodic function on the discrete group G^*_d. By Maak's definition of almost periodicity (see [15]) there exists to each $\varepsilon > 0$ a finite covering of G^*_d,

$$\bigcup_{\nu=1}^{n} A_\nu = G^*_d ,$$

such that

$$|\varphi_N(\chi\gamma) - \varphi_N(\chi\delta)| < \varepsilon \qquad \text{for all } \chi \in G^*_d ,$$

if γ and δ are in the same set A_ν. For $\varepsilon < 1$ this implies that $\varphi_N(\chi\gamma) = \varphi_N(\chi\delta)$, that is, $\varphi_N(\chi)$ has the period $\gamma\delta^{-1}$. We express this as

$$(24) \qquad N_{\gamma\delta^{-1}} = N \quad \text{if } \gamma \text{ and } \delta \text{ are in the same } A_\nu .$$

We now define

$$\alpha \sim \beta \quad \text{if} \quad \alpha\beta^{-1} \text{ is a period for } N.$$

This is clearly an equivalence relation in G^*. Let $\{B_\mu\}$ be the corresponding division of G^* into classes. (24) shows that each A_ν is contained in some B_μ; in particular, there are only a finite number of different classes B_μ. Let B_0 be the class that contains the identity element of G^*. Then B_0 is a subgroup, and the other B_μ are the cosets of B_0. Let m be the number of different B_μ and let χ be an arbitrary element of G^*. Since G^* is infinitely divisible, χ can be written $\chi = \psi^m$, $\psi \in G^*$. But then $\chi \in B_0$. Consequently, every element of G^* is a period of N which therefore is either empty or the entire G^*.

If our distinguished set S is $\neq G^*$, then N is empty and $\varphi_S(\chi)$ is therefore the Fourier-Stieltjes transform of a continuous measure. From (14) follows

$$(25) \qquad \mathfrak{M}(\varphi_S) = 0 .$$

As mentioned earlier, the mean value of Fourier-Stieltjes transforms satisfies condition (v) in section 1. Equation (25) therefore implies: To every $\varepsilon > 0$ there exist positive numbers α_i with $\alpha_1 + \ldots + \alpha_n = 1$ and elements $\gamma_1, \ldots, \gamma_n \in G^*$ such that (writing $S_{\gamma_i} = S_i$)

$$(26) \quad \alpha_1 \varphi_{S_1}(\chi) + \alpha_2 \varphi_{S_2}(\chi) + \ldots + \alpha_n \varphi_{S_n}(\chi) < \varepsilon \qquad \text{for all } \chi \in G^* .$$

This produces a vague picture of the distinguished sets. If G is the group of the reals, (26) shows trivially that a distinguished set cannot contain an infinite interval.

We have proved

THEOREM 4. *If $G = R$, or more generally, if G has an infinitely divisible character group G^*, the distinguished sets S different from G^* have "mean-density" 0, that is, $\mathfrak{M}(\varphi_S) = 0$.*

The theorem does not hold if the assumption of infinite divisibility is dropped, as one sees by considering the case when G^* is the group of integers. In this case the distinguished sets have been determined by H. Helson [11].

4. The case $G=R$. We discuss briefly the classical case, where G is the additive group of real numbers R. According to section 2 the compactification $\mathbf{R}$ can be identified with $R_d{}^*$. R is dense in $\mathbf{R}$ and the relative topology of R in $\mathbf{R}$ is the weakest for which the exponentials $e^{i\lambda t}$ are continuous. The function $x(t)$ is uniformly continuous on R in this topology if and only if $x(t)$ is almost periodic on R.

DEFINITION. A subset of R is called *quasiperiodic* if it is the intersection of a finite number of sets of the form

$$\{t \mid |t| < \delta \pmod{l}\}, \qquad 0 < 2\delta < l\,.$$

From the above remark we arrive at

PROPOSITION 2. *The function $x(t)$ is an almost periodic function on R if and only if for every $\varepsilon > 0$ there exists a quasiperiodic set A_ε such that*

$$|x(t+\tau) - x(t)| < \varepsilon \qquad \textit{for all } t \in R \textit{ and all } \tau \in A_\varepsilon\,. \tag{27}$$

It was shown by Bogolioùboff by elementary methods that this is equivalent to Bohr's definition (see Bohr [6]). However, the proof is not simple (see Maak [15, §§ 25–27]).

DEFINITION. A set of almost periodic functions on R is called a *homogeneous system* if for every $\varepsilon > 0$ the set $\{\tau(\varepsilon)\}$ of numbers which are simultaneously translation numbers belonging to ε for all the functions, is relatively dense and contains an interval around 0.

In the classical theory of almost periodic functions, homogeneous systems play an important role, partly due to the following theorem of Bochner [2]: A uniformly bounded infinite homogeneous system contains a uniformly convergent subsequence.

The next proposition explains the importance of this notion and shows that Bochner's theorem can be deduced from a theorem on equicontinuous families.

PROPOSITION 3. *A set of almost periodic functions on R is a homogeneous system if and only if the extensions to $\mathbf{R}$ form an equicontinuous family.*

PROOF. If the family $\{x\}$ is equicontinuous on $\mathbf{R}$, the restrictions to R also form an equicontinuous family in the relative topology of $\mathbf{R}$.

Thus, to every $\varepsilon > 0$ there exists a quasiperiodic set A_ε such that (27) holds for all functions in the family. Since every quasiperiodic set is relatively dense and contains an interval around 0, the family forms a homogeneous system. The converse is a simple consequence of a theorem of Bogolioùboff (see Maak [15, p. 102, Satz 2]).

Similarly as the continuous functions on $\boldsymbol{R}$ can be identified with the Bohr almost periodic functions on R, the space $L^2(\boldsymbol{R})$ can be identified with the Besicovitch almost periodic functions of exponent 2. In the last case, however, the correspondence is purely formal, namely through the Fourier series, because (see Hewitt [12]) the real line R is a null-set in the Haar measure on $\boldsymbol{R}$.

5. Permutations of Fourier exponents. We now return to the problem mentioned in section 1, namely the description of the automorphisms of the convolution algebra of almost periodic functions.

DEFINITION. A permutation σ of G^* is called *permissible* if for any almost periodic Fourier series $\sum_{\chi \in G^*} a(\chi)\chi(t)$, the series $\sum_{\chi \in G^*} a(\chi)\sigma(\chi)(t)$ is also an almost periodic Fourier series.

We have seen earlier that a permutation σ of G^* induces an automorphism T_σ of A if and only if the permutation and its inverse are permissible.

For every trigonometric polynomial $a_1\chi_1(t) + \ldots + a_n\chi_n(t)$ we consider the quantity

$$m(\chi_\nu, a_\nu) = \frac{\|1 + \sum a_\nu \chi_\nu(t)\|}{1 + \sum |a_\nu|}.$$

From the Kronecker theorem it follows that for fixed χ_ν, $\nu = 1, 2, \ldots, n$, the equation $m(\chi_\nu, a_\nu) = 1$ for all a_ν is a necessary and sufficient condition for the independence of the χ_ν. Therefore the quantity $m(\chi_\nu, a_\nu)$, taken for all coefficients a_ν, is a measure of the indepencence of $\chi_1, \ldots, \chi_n$.

PROPOSITION 4. *The permutation σ with $\sigma(1) = 1$ is permissible if and only if*

$$\sup_{a_\nu, \chi_\nu} \left[m(\sigma(\chi_\nu), a_\nu) / m(\chi_\nu, a_\nu) \right] < \infty . \tag{28}$$

PROOF. If σ is a permissible permutation, then

$$T_\sigma x(t) \sim \sum a(\chi)\sigma(\chi)(t)$$

induces a linear transformation T_σ of A into itself. By the closed graph theorem, T_σ is a continuous operator and (28) holds. Conversely (28) shows the existence of an endomorphism on the space of trigonometric polynomials which has the same effect as σ. Since these polynomials are dense in A, the endomorphism has a unique extension to the whole of A, and σ is permissible.

We also note that

$$\|T_\sigma\| = \sup_{a_\nu,\, \chi_\nu} \left[m(\sigma(\chi_\nu), a_\nu)/m(\chi_\nu, a_\nu)\right],$$

so obviously $\|T_\sigma\| \geqq 1$. The extreme case $\|T_\sigma\| = 1$ is described in the next theorem, which is an analogue of a theorem of Beurling, Helson, and Wendel on the automorphisms of L^1, and the proof is similar to that of Helson [10].

THEOREM 5. *Let σ be a permissible permutation for which $\|T_\sigma\| = 1$. Then σ satisfies the equation*

$$\sigma(\chi\psi)\,\sigma(1) = \sigma(\chi)\,\sigma(\psi) \qquad \textit{for all } \chi, \psi \textit{ in } G^*, \tag{29}$$

and T_σ is an isometric automorphism of A. Conversely, every permutation of the form (29) *is permissible, and T_σ is an isometric automorphism.*

PROOF. Suppose σ is permissible and $\|T_\sigma\| = 1$. Consider for an arbitrary $s \in G$ the linear form

$$L_s(z) = T_\sigma z(s) \qquad \text{for all } z \in A.$$

Then we have

$$|L_s(z)| \leqq \|T_\sigma z\| \leqq \|z\|. \tag{30}$$

L_s can be regarded as a continuous linear form on $\mathfrak{C}^\infty(G)$ and induces a measure μ_s on G such that

$$T_\sigma z(s) = \int_G z(t)\, d\mu_s(t). \tag{31}$$

By (30), $\|\mu_s\| \leqq 1$. For a character $z(t) = \chi(t)$ equation (31) takes the form

$$(\sigma(\chi), s) = \int (\chi, t)\, d\mu_s(t). \tag{32}$$

We put $r(\chi) = (\sigma(\chi)\sigma^{-1}(1), s)$. Then $r(\chi)$ is a Fourier-Stieltjes transform of a measure ν with $\|\nu\| = \|\mu_s\| \leqq 1$. Also $r(1) = \nu(G) = 1$. It follows from the relations $\|\nu\| \leqq 1$ and $\nu(G) = 1$ that ν is positive.

The function $r(\chi)$ is therefore positive definite on G^*_d. We split ν into a discrete and a continuous part and get

$$\nu = \sum_i b_i \varepsilon_{s_i} + \lambda \qquad (b_i \geqq 0),$$

where

$$1 = \nu(G) = \sum b_i + \lambda(G). \tag{33}$$

By (15)

$$\mathfrak{M}(|r|^2) = \sum |b_i|^2 .$$

Since $|r(\chi)| \equiv 1$, $b_i \geqq 0$ and $\lambda \geqq 0$, we have $1 = \Sigma b_i^2 \leqq \Sigma b_i \leqq 1$, which implies that all the b_i are 0 except one, say b_0, which is 1. By (33), $\lambda = 0$. Hence $\nu = \varepsilon_{s_0}$ and $r(\chi) = \big(\sigma(\chi)\sigma^{-1}(1), s\big)$ is a character on G^*_d. Since s is arbitrary in G, (29) follows.

Suppose conversely that (29) holds. Since the translation $\chi \to \chi\sigma(1)$ obviously induces an isometric automorphism of A, we can assume $\sigma(1) = 1$. Thus, σ is an automorphism of G^*_d, and the mapping δ defined by $\big(\delta(t), \chi\big) = \big(t, \sigma(\chi)\big)$ for all χ and t is an automorphism of G (algebraically and topologically). Consider the operator T_σ defined by $T_\sigma z(t) = z\big(\delta(t)\big)$. T_σ is linear and isometric. If $z(t) = \chi(t)$, then

$$T_\sigma z(t) = \sigma(\chi)(t) ,$$

so σ is a permissible permutation. The same conclusion holds for σ^{-1}, and T_σ is therefore an automorphism of A. Thus, theorem 5 is proved.

A counterexample. Since an almost periodic Fourier series with independent exponents is absolutely convergent, it is natural to ask whether a permutation that maps an independent set of characters onto itself, leaving the others fixed, is permissible. We show by an example that this is not in general the case. Let

$$x(t) \sim \sum a(\lambda)\, e^{i\lambda t} \tag{34}$$

be an almost periodic function with (β): $\beta_1, \beta_2, \ldots$ as a basis for the set of Fourier exponents such that $\Sigma a(\beta_\varkappa)\, e^{i\beta_\varkappa t}$ is not an almost periodic Fourier series. Such a function exists by a theorem of Doss ([7, § 3]). Suppose that every permutation σ of R which maps (β) onto itself and leaves all other points fixed is permissible. For every such σ

$$\sum a\big((\sigma(\lambda)\big)\, e^{i\lambda t} \tag{35}$$

is an almost periodic Fourier series. The same is true for the formal difference of (34) and (35) so by a theorem of Bohr [5]

$$\sum_{i=1}^{\infty} \big|a(\beta_i) - a\big(\sigma(\beta_i)\big)\big| < \infty ; \tag{36}$$

on the other hand

$$\sum |a(\beta_i)| = \infty \qquad \text{and} \qquad \sum |a(\beta_i)|^2 < \infty .$$

We put $a(\beta_i) = a_i$, $\sigma(\beta_i) = \sigma(i)$ and assume that the a_i are ordered after decreasing magnitude of their absolute values. We shall obtain the desired contradiction by constructing a permutation σ of (β) such that (36) fails to hold.

Let $K_1, K_2, \dots$ be an increasing sequence of positive numbers tending to ∞. We can determine m_1 and m_2 such that

$$|a_1| + |a_2| + \dots + |a_{m_1}| > 2K_1 ,$$
$$|a_{m_2}| < K_1/m_1 .$$

Put

$$\begin{array}{ll} \sigma(1) = m_2 , & \sigma(m_1+1) = 1 , \\ \sigma(2) = m_2+1 , & \sigma(m_1+2) = 2 , \\ \dots & \dots \\ \sigma(m_1) = m_2+m_1-1 , & \sigma(m_1+m_2-1) = m_2-1 , \end{array}$$

and let $n_1 = m_1 + m_2 - 1$. We then have

$$\begin{aligned} \sum_{i=1}^{n_1} |a_i - a_{\sigma(i)}| &\geqq \sum_{i=1}^{n_1} \big||a_i| - |a_{\sigma(i)}|\big| \\ &\geqq \sum_{i=1}^{m_1} \big||a_i| - a_{\sigma(i)}|\big| \\ &= \sum_{i=1}^{m_1} |a_i| - |a_{\sigma(i)}| \\ &\geqq 2K_1 - m_1 K_1/m_1 \\ &= K_1 . \end{aligned}$$

We have defined a permutation σ of the numbers $1, 2, \dots, n_1$ such that

$$\sum_{i=1}^{n_1} |a_i| - |a_{\sigma(i)}| \geqq K_1 .$$

An obvious induction process gives the desired contradiction.

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