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VECTOR-VALUED MEASURE AND BOUNDED VARIATION IN HILBERT SPACE

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1. Introduction. The object of this paper is to exhibit the relation between two different constructions of vector-valued measure in Hilbert space due respectively to Gelfand [3] and Cramér [1].

Let X be a real Hilbert space with zero element θ and let $\mathcal{B}$ be the ring of bounded Borel subsets of the real line R^1. A function $Z(\cdot)\colon \mathcal{B} \to X$ will be called a *vector-valued measure* if

(i) $$Z(\Phi) = \theta,$$
where Φ is the empty set, and

(ii) $$Z\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} Z(E_n)$$

whenever the E_n are disjoint sets of $\mathcal{B}$ with a bounded union, the series on the right being strongly convergent. It follows at once that $\sum_{n=1}^{\infty} Z(E_n)$ in fact converges unconditionally (see Hildebrandt [6]). If $z(\cdot)\colon R^1 \to X$ is now defined by

(1) $$z(0) = \theta$$

(2) $$z(b)-z(a) = Z(E) \qquad \text{when} \qquad E = (a, b],$$

then:

(α) for each fixed $y \in X$ the real-valued function $\big(z(\cdot), y\big)$ is of bounded variation on every bounded interval of R^1;

(β) $z(\cdot)$ is everywhere strongly continuous-to-the-right.

Conversely, if $z\colon R^1 \to X$ is a given function which has the properties (α) and (β), then there exists a unique vector-valued measure $Z(\cdot)\colon \mathcal{B} \to X$ satisfying (2). These results are due, essentially, to Gelfand [3].

On the other hand, Cramér [1] has considered a function $z\colon R^1 \to X$ together with its covariance function $\varrho\colon R^2 \to R^1$ defined by

$$\varrho(t, s) \equiv \big(z(t), z(s)\big).$$

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Cramér's argument establishes a result which can be stated in the following form. If z satisfies (β) and also

(α') ϱ is of Vitali bounded variation on every bounded square,

then there exists a unique vector-valued measure $Z(\cdot)\colon \mathscr{B} \to X$ satisfying (2). The main result of the present paper is that condition (α) is strictly weaker than (α'). Some related but elementary theorems are also included for the sake of completeness.

2. Bounded variation. z is said to be of *Dunford bounded variation* (BV(D)) on the bounded interval $[a, b]$ when and only when there is a finite real constant $K(a, b)$ such that, if $k \geqq 1$ and $a \leqq t_1 < t_2 < \ldots < t_{2k} \leqq b$, then

$$\left\| \sum_{r=1}^{k} \big(z(t_{2r}) - z(t_{2r-1})\big) \right\| \leqq K(a, b) .$$

We state without proof a theorem of Gelfand [3] and Dunford [2].

THEOREM 1. *z satisfies condition (α) if and only if it is of* BV(D) *on every bounded interval of R^1.*

A function $\varphi\colon R^2 \to R^1$ is said to be of *Fréchet bounded variation* (BV(F)) on the bounded rectangle $H = [a, b] \times [c, d]$ when and only when there is a finite real constant $K(H)$ such that, if $p \geqq 1$ and $q \geqq 1$ and

$$\text{(3)} \qquad \begin{cases} a \leqq t_1 < t_2 < \ldots < t_{p+1} \leqq b , \\ c \leqq s_1 < s_2 < \ldots < s_{q+1} \leqq d , \end{cases}$$

and if

$$\text{(4)} \qquad \begin{cases} \varepsilon_i = \pm 1, \quad i = 1, 2, \ldots, p , \\ \eta_j = \pm 1, \quad j = 1, 2, \ldots, q , \end{cases}$$

then

$$\left| \sum_{i=1}^{p} \sum_{j=1}^{q} \varepsilon_i \eta_j \Delta_{ij} \varphi \right| \leqq K(H) ,$$

where

$$\Delta_{ij} \varphi \equiv \varphi(t_{i+1}, s_{j+1}) - \varphi(t_i, s_{j+1}) - \varphi(t_{i+1}, s_j) + \varphi(t_i, s_j) .$$

If a finite real constant $M(H)$ exists such that

$$\sum_{i=1}^{p} \sum_{j=1}^{q} |\Delta_{ij} \varphi| \leqq M(H)$$

whenever conditions (3) are satisfied ($p \geqq 1, q \geqq 1$) then φ is said to be of *Vitali bounded variation* (BV(V)) on H. An immediate and well known consequence of these two definitions is that BV(V) implies BV(F).

THEOREM 2. *z satisfies condition (α) if and only if its covariance function ϱ is of* BV(F) *on every bounded rectangle.*

If z has the property (α) then for each bounded interval $[a, b]$ and each fixed $y \in X$ there is a constant $A(y;\ a, b)$ such that

$$\left|\left(\sum_{i=1}^{p} \varepsilon_i \left(z(t_{i+1}) - z(t_i)\right),\ y\right)\right| \leqq A(y;\ a, b) < \infty$$

whenever the t_i and ε_i satisfy (3) and (4). Hence, by uniform boundedness, there is a constant $A(a, b)$ such that

$$\left\|\sum_{i=1}^{p} \varepsilon_i \left(z(t_{i+1}) - z(t_i)\right)\right\| \leqq A(a, b) < \infty .$$

Similarly,

$$\left\|\sum_{j=1}^{q} \eta_j \left(z(s_{j+1}) - z(s_j)\right)\right\| \leqq A(c, d) < \infty$$

when the s_j and η_j satisfy (3) and (4). And so, using Schwarz's inequality,

$$\left|\sum_{i=1}^{p} \sum_{j=1}^{q} \varepsilon_i \eta_j \Delta_{ij} \varrho\right| \leqq A(a, b)\, A(c, d) < \infty .$$

Conversely, if ϱ is of BV(F) on every bounded rectangle H, then on putting $H = [a, b]^2$, $p = q$, $\eta_i = \varepsilon_i$ and $s_i = t_i$ (all i) it is clear that

$$\left\|\sum_{i=1}^{p} \varepsilon_i \left(z(t_{i+1}) - z(t_i)\right)\right\|^2 \leqq K(H) < \infty ,$$

and hence that

$$\sum_{i=1}^{p} \left|\left(z(t_{i+1}) - z(t_i),\ y\right)\right| \leqq \|y\| \left(K(H)\right)^{\frac{1}{2}} .$$

This completes the proof.

COROLLARY 1. *If z satisfies condition (α') then it satisfies (α).*

COROLLARY 2. *z is of* BV(D) *on every bounded interval if and only if its covariance ϱ is of* BV(F) *on every bounded rectangle.*

COROLLARY 3. *If $\varrho\colon R^2 \to R^1$ is a covariance function and is of* BV(F) *on $[a, b]^2$ then $\varrho(\cdot, s)$ $\left(\equiv \varrho(s, \cdot)\right)$ is of bounded variation on $[a, b]$ for each fixed $s \in [a, b]$.*

These results all have natural and immediate extensions to the case in which X is a complex Hilbert space.

3. A counter-example. Any function $\varphi\colon R^2 \to R^1$ which is of BV(V) on a given rectangle is also of BV(F) on that rectangle. On the other hand, Littlewood [7] has shown how to construct a function which is of BV(F) but not of BV(V) on a given rectangle. In the light of the theorems of § 2 it is natural to ask whether it is possible to construct a *covariance* function which is of BV(F), without being of BV(V), on some rectangle. A strongly continuous function z whose covariance ϱ has this property is now constructed using an adaptation of the method of Littlewood [7].

THEOREM 3. *There exists a vector-valued function* z *defined on* [0, 1] *and such that*:

(i) *the range of* z *spans a separable real Hilbert space* X_0;
(ii) z *is strongly continuous on* [0, 1];
(iii) z *is of* BV(D) *on* [0, 1] (*and hence* ϱ *is of* BV(F) *on* $[0, 1]^2$);
(iv) ϱ *is not of* BV(V) *on* $[0, 1]^2$.

Let $[a_{mn}]$ be the real symmetric matrix defined by

$$\left.\begin{aligned} a_{nn} &= 0\,, \\ a_{mn} &= \frac{\sin \frac{1}{2}\pi(m-n)}{m-n}, \qquad m \neq n. \end{aligned}\right\} \quad m, n = 1, 2, 3, \ldots .$$

Schur [10] has shown that for any real sequence $\{\xi_n\}$

$$\left| \sum_{m=1}^{N} \sum_{n=1}^{N} a_{mn}\, \xi_m \xi_n \right| \leqq \tfrac{1}{2}\pi \sum_{m=1}^{N} \xi_m^{\,2} \quad \text{for} \quad N = 1, 2, 3, \ldots .$$

Using a well known theorem of Hellinger and Toeplitz [5] it follows that $[a_{mn}]$ is the matrix of a bounded self-adjoint operator A in the space L_2 of real sequences $x = \{\xi_n\}$ which are such that

$$\|x\| \equiv \left(\sum_{n=1}^{\infty} \xi_n^{\,2} \right)^{\frac{1}{2}} < \infty .$$

Moreover, A has a non-empty spectrum which is a subset of the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Consequently the operator $B = \frac{1}{2}\pi I + A$, where I is the identity operator, is bounded ($\|B\| \leqq \pi$), self-adjoint and non-negative definite. Now let

$$u_n = \frac{1}{n^{\frac{1}{2}} \log (n+1)}, \qquad n = 1, 2, 3, \ldots ,$$

so that $\sum_{n=1}^{\infty} u_n^{\,2} < \infty$; and let $c_{mn} = b_{mn}\, u_m u_n$ for $m, n = 1, 2, 3, \ldots$, where $[b_{mn}]$ is the matrix of B. If

$$M = \pi \sum_{n=1}^{\infty} u_n^{\,2}$$

then

$$0 \leqq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}\, \varepsilon_m \varepsilon_n \leqq M ,$$

and

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}\, \varepsilon_m \eta_n \right| \leqq M ,$$

whenever $-1 \leqq \varepsilon_m \leqq 1$, $-1 \leqq \eta_n \leqq 1$ for all m and n, the double sum being taken in the Pringsheim sense. In particular $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}$ converges. However, it can be shown by a straightforward application of the method used on p. 214 of Hardy, Littlewood and Pólya [4] that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| = \infty .$$

Lastly it is easily shown that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}{}^2 < \infty ,$$

and so $[c_{mn}]$ is the matrix of a bounded self-adjoint non-negative definite operator in L_2.

Now let X be a given real separable infinite dimensional Hilbert space and let $\{\varphi_n\}$ be a c.o.n.s. for X. Then there is a unique bounded operator C in X such that

$$(\varphi_m, C\varphi_n) = c_{mn} \qquad \text{for} \quad m, n = 1, 2, 3, \ldots .$$

But C is also self-adjoint and non-negative definite and hence (see e.g. Riesz and Nagy [9]) there is a bounded self-adjoint operator T in X such that $T^2 = C$. Now let $x_n = T\varphi_n$ for $n = 1, 2, 3, \ldots$.
Then

$$\begin{aligned} (x_m, x_n) &= (T\varphi_m, T\varphi_n) \\ &= (\varphi_m, T^2 \varphi_n) \\ &= (\varphi_m, C\varphi_n) \\ &= c_{mn}, \qquad m, n = 1, 2, 3, \ldots . \end{aligned}$$

Now let $t_1 = 0$, $t_{m+1} = t_m + 2^{-m}$ for $m = 1, 2, 3, \ldots$, and define $z(\cdot)$ on $[0, 1)$ as follows. Let $z(t_1) = \theta$, $z(t_{m+1}) \equiv z(t_m) + x_m$ and let $z(\cdot)$ be *linear* in each of the intervals $[t_m, t_{m+1}]$. Then, since

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}$$

converges,

$$\left\| \sum_{i=m}^{m+p} x_i \right\|^2 = \sum_{i=m}^{m+p} \sum_{j=m}^{m+p} c_{ij} \to 0 \qquad \text{as} \quad m, p \to \infty .$$

Hence $\{z(t_n)\}$ is a Cauchy sequence in X and we can, and do, define $z(1) = \lim_{n\to\infty} z(t_n)$. Then $z(\cdot)$ plainly satisfies conditions (i) and (ii) of Theorem 3.

To prove (iii) it is enough, by Theorems 1 and 2 to show that z satisfies (α). Now

$$\left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|^2 = \sum_{m=1}^{N} \sum_{n=1}^{N} c_{mn} \varepsilon_m \varepsilon_n \leqq M \qquad \text{for} \quad N = 1, 2, 3, \ldots ,$$

whenever $\varepsilon_n = \pm 1$ for all n. For any fixed $y \in X$ it now follows, on taking $\varepsilon_n = \operatorname{sgn}(x_n, y)$ for each n, that

$$\sum_{n=1}^{N} |(x_n, y)| = \left| \left(\sum_{n=1}^{N} \varepsilon_n x_n, y \right) \right| \leqq M^{\frac{1}{2}} \|y\|, \qquad N = 1, 2, 3, \ldots .$$

Consequently, for each $y \in X$,

$$\sum_{n=1}^{\infty} |(x_n, y)| < \infty .$$

But if $p \geqq 1$ and $0 \leqq s_1 < s_2 < \ldots < s_{p+1} \leqq 1$ then

$$\sum_{j=1}^{p} \left| \big(z(s_{j+1}) - z(s_j), y\big) \right| \leqq \sum_{n=1}^{\infty} |(x_n, y)| < \infty .$$

z therefore satisfies the condition (α).

Finally,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left| \big(z(t_{i+1}) - z(t_i),\ z(t_{j+1}) - z(t_j)\big) \right| = \sum_{i=1}^{N} \sum_{j=1}^{N} |c_{ij}| \to \infty \qquad \text{as} \quad N \to \infty .$$

Hence the covariance ϱ of z is not of BV(V) on $[0, 1]^2$; and so the proof of Theorem 3 is complete.

The use of covariance functions of BV(V) in the theory of stochastic processes was introduced by Loève [8] and continued by Cramér [1], whose main theorem subsumed some of Loève's work; and it now seems desirable that this theorem in turn should be generalized by using Fréchet instead of Vitali bounded variation. A proof that such a generalization is possible will be given in a forthcoming paper, and it is clear from the theorems of the present paper that this extension will, in a certain sense, be the ultimate form of Cramér's theorem.

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