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THE MAXIMUM VALUE OF A FOURIER-STIELTJES TRANSFORM

EDWIN HEWITT and HERMAN RUBIN

1. Introduction. Let G be a locally compact Abelian group with character group G^*. Let (y, x) denote the function on $G \times G^*$ equal to the value of $y \in G^*$ at $x \in G$. Let φ be a bounded Radon measure on G, with Fourier-Stieltjes transform

$$\Phi(y) = \int_G (y, x)\, d\varphi(x).$$

Let $|\varphi|$ be the total variation of the measure φ (see [2, p. 459]). That is,

$$|\varphi|(A) = \sup \sum_{\nu=1}^{n} |\varphi(A_\nu)|,$$

the supremum being taken over all pairwise disjoint families $\{A_\nu\}_{\nu=1}^{n}$ of Borel sets whose union is the Borel set A. For other notation and terminology, see [3].

We are concerned in this note with the sets

$$A(\varphi) = E\left[y;\quad y \in G^*,\ |\Phi(y)| = \int_G d|\varphi|(x)\right]$$

and

$$M(\varphi) = E\left[y;\quad y \in G^*,\ \Phi(y) = \int_G d|\varphi|(x)\right].$$

We shall establish the following results, which characterize the possible sets $A(\varphi)$ and $M(\varphi)$ completely.

1.1 THEOREM. *The following conditions on a subset E of G^* are equivalent:*

1.1.1 *E has the form $A(\varphi)$ for some bounded Radon measure φ;*

1.1.2 *E has the form $M(\varphi)$ for some bounded Radon measure φ;*

1.1.3 *E contains a non-void G_δ and is a closed subgroup of G^* or is a translate of such a subgroup, or $E = 0$.*

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1.2 THEOREM. *The measure φ referred to in Theorem 1.1 can be chosen to be absolutely continuous with respect to Haar measure on G if and only if E is compact.*

These theorems appear to be known for the case $G = G^* =$ the real line R under addition. The only compact subgroup of R being $\{0\}$, Theorem 1.2 implies that integrals

$$\int_{-\infty}^{\infty} e^{ixy}\, d\varphi(x)$$

for non-negative absolutely continuous measures φ are equal to $\varphi(R)$ for $y = 0$ and are less than $\varphi(R)$ in absolute value for $y \neq 0$. The only proper closed subgroups H of R are of the form $\{n\alpha\}_{n=-\infty}^{\infty}$ $(\alpha \in R)$. The measure $\varphi = \frac{1}{2}\varepsilon_{\pi/\alpha} + \frac{1}{2}\varepsilon_{-\pi/\alpha}$ has the property that $A(\varphi) = H$. The measure $\varphi = \frac{1}{2}[\varepsilon_0 + \varepsilon_{2\pi/\alpha}]$ has the property that $M(\varphi) = H$. These observations show that 1.1.3 is sufficient for 1.1.1 and 1.1.2 in the case $G = R$.

2. Measure-theoretic observations. We first prove some simple properties of $M(\varphi)$ and $A(\varphi)$.

2.1 THEOREM. *Let φ be a bounded Radon measure on G. Then there exists a bounded complex-valued Borel-measurable function h on G such that*

$$d\varphi(x) = h(x)\, d|\varphi|(x) .$$

PROOF. Since $\varphi = \varphi_1 + i\varphi_2$, where φ_j is a real-valued measure and since φ_j is absolutely continuous with respect to $|\varphi|$, we can apply the Radon-Nikodym theorem ([1, p. 129 *et. seq.*]) to write

$$d\varphi_j(x) = h_j(x)\, d|\varphi|(x) \qquad (j = 1, 2) .$$

We then take $h = h_1 + ih_2$.

2.2 THEOREM. *Let φ be a bounded Radon measure on G, and let y_0 denote an element of G^*. Then*

$$y_0 \in M(\varphi) \quad \textit{if and only if} \quad d\varphi(x) = (y_0^{-1}, x)\, d\pi(x) ,$$

where π is a non-negative bounded Radon measure on G.

PROOF. It follows from 2.1 that

$$\int_G (y_0, x)\, d\varphi(x) = \int_G (y_0, x)\, h(x)\, d|\varphi|(x) .$$

It is easy to see that the relation

$$\int_G (y_0, x)\, h(x)\, d|\varphi|(x) = \int_G 1 \cdot d|\varphi|(x)$$

holds if and only if $(y_0, x)\, h(x) = \text{ess sup}\, |h(x)|$ almost everywhere with respect to $|\varphi|$, the ess sup being taken with respect to $|\varphi|$. (See for example [4, Theorem 3.1.].) It follows that $h(x) = \beta\, (y_0^{-1}, x)$ with a positive constant β almost everywhere with respect to $|\varphi|$. Hence we take $\pi = \beta\, |\varphi|$.

2.3 THEOREM. *Let φ be a bounded Radon measure on G, and let y_0 be an element of G^*. Then*

$$y_0 \in A(\varphi) \quad \textit{if and only if} \quad d\varphi(x) = \alpha\, (y_0^{-1}, x)\, d\pi(x)\,,$$

where π is a non-negative bounded Radon measure on G and $|\alpha| = 1$.

PROOF. We have

$$\left| \int_G (y_0, x)\, d\varphi(x) \right| = \int_G d|\varphi|(x)$$

if and only if there is a number δ of absolute value 1 such that

$$\int_G (y_0, x)\, d[\delta\varphi](x) = \int_G d|\delta\varphi|(x);$$

this brings us back to Theorem 2.2, and we may take $\alpha = \delta^{-1}$.

2.4 THEOREM. *Suppose that $M(\varphi) \neq 0$. Then the set $M(\varphi)$ is a closed G_δ and is a subgroup or a translate of a subgroup of G^*.*

PROOF. Since $M(\varphi)$ is the set where the continuous function Φ assumes a fixed value, it is clearly a closed G_δ. It remains only to show that it is a subgroup or a translate of a subgroup. Applying Theorem 2.2, we can multiply $d\varphi$ by a character and suppose that the measure φ is non-negative and that accordingly the identity of G^* lies in $M(\varphi)$. This of course amounts simply to translating $M(\varphi)$. Under these hypotheses, $y \in M(\varphi)$ if and only if $(y, x) = 1$ almost everywhere with respect to φ. The set of such y clearly forms a subgroup of G^*.

2.5 THEOREM. *Suppose that $A(\varphi) \neq 0$. Then $A(\varphi)$ is a closed G_δ which is either a subgroup of G^* or a translate of a subgroup of G^*.*

PROOF. Similar to the proof of Theorem 2.4.

Theorems 2.4 and 2.5 show that 1.1.3 is necessary for 1.1.1 and 1.1.2.

3. Group-theoretic observations. Let S be a subset of G. The set $N(S)$, the annihilator of S, is the set of all $y \in G^*$ such that $(y, x) = 1$ for all $x \in S$. It is obvious that $N(S)$ is a closed subgroup of G^* and it is well known that $N(N(S)) = S$ if S is a closed subgroup of G.

3.1 Theorem. *Let S be a subset of G. If S contains a non-void G_δ, then $N(S)$ is σ-compact.*

Proof. Let $\{Q_n\}_{n=1}^{\infty}$ be a sequence of open subsets of G such that

$$0 \neq \bigcap_{n=1}^{\infty} Q_n = T \subset S .$$

Let x be any element of T. Then there exists a sequence of open sets $\{U_n\}_{n=1}^{\infty}$ such that $x \in U_n$, $U_n \subset Q_n$, U_n^- is compact $(n = 1, 2, 3, \ldots)$ and $U_n^- \subset U_{n-1}$ $(n = 2, 3, 4, \ldots)$. Now let δ be a positive real number less than $\frac{1}{10}$, and let

$$V_n = E[y;\quad y \in G^*, |(y, x) - 1| < \delta \text{ for all } x \in U_n^-]$$

$(n = 1, 2, 3, \ldots)$. It is known that V_n^- is compact in G^*. Hence

$$W = \bigcup_{n=1}^{\infty} V_n^-$$

is σ-compact. We now show that $N(S) \subset W$. In fact, if

$$y \in N(S) \cap W',$$

then for every positive integer n, there exists $x_n \in U_n^-$ such that

$$|(y, x_n) - 1| \geqq \delta .$$

Since $x_n \in U_m^-$ for $m \leqq n$, there is a point

$$x_0 \in \bigcap_{n=1}^{\infty} U_n^- \subset T \subset S$$

such that every neighborhood of x_0 contains an infinite number of the points x_n. It follows that $|(y, x_0) - 1| \geqq \delta$, and this is inconsistent with the relation $y \in N(S)$. Thus $N(S)$ is contained in a σ-compact set. Since $N(S)$ is closed, it follows that $N(S)$ is σ-compact.

We note also that the annihilator of a σ-compact subgroup of G is a G_δ.

3.2 Corollary. *A closed subgroup of G contains a non-void G_δ if and only if it is a non-void G_δ.*

3.3 Theorem. *Let H be a closed σ-compact subgroup of G. Then there exists a non-negative bounded Radon measure φ on G such that:*

3.3.1 $\varphi(A) > 0$ *for every non-void relatively open subset* A *of* H;
3.3.2 $\varphi(H) = 1$;
3.3.3 $\varphi(H') = 0$.

PROOF. Let λ be a Haar measure on the group H (H is certainly a locally compact Abelian group). Since H is σ-compact, the measure λ is σ-finite. This implies that

$$H = \bigcup_{n=1}^{\infty} P_n ,$$

where the sets P_n are pairwise disjoint and $0 < \lambda(P_n) < \infty$ $(n = 1, 2, 3, \dots)$. Let the function f on H be defined by the relations

$$f(x) = 2^{-n}[\lambda(P_n)]^{-1} \quad \text{for} \quad x \in P_n \quad (n = 1, 2, 3, \dots) .$$

It is clear that $f \in \mathfrak{L}_1(H)$ and that

$$\int_A f(x)\, d\lambda(x) > 0 \qquad \text{if} \qquad \lambda(A) > 0 .$$

For an arbitrary Borel set $Q \subset G$, let

$$\varphi(Q) = \int_{Q \cap H} f(x)\, d\lambda(x) .$$

It is obvious that this set-function satisfies all requirements of the present theorem.

3.4 THEOREM. *If the subgroup* H *of Theorem* 3.3 *is also open, then the measure* φ *of Theorem* 3.3 *can be taken as absolutely continuous with respect to Haar measure on* G.

PROOF. This follows immediately from the fact that Haar measure on an open subgroup H of G is simply Haar measure on G relativized to H.

3.5 REMARK. Theorem 3.3 is not true for general locally compact σ-compact Hausdorff spaces. Let D denote a countably infinite discrete space and let βD denote the Stone-Čech compactification of D. Then, as Nakamura and Kakutani have shown [5], the compact Hausdorff space $\beta D \cap D'$ contains a continuum of pairwise disjoint non-void open sets. It is clear that no Borel measure on $\beta D \cap D'$ can assign positive measure to every non-void open set.

4. Completion of the proof of Theorem 1.1. We shall now show that given a set $E \subset G^*$ which contains a non-void G_δ and is a closed subgroup or a translate of such a subgroup, there exists a bounded Radon measure φ on G such that $A(\varphi)=M(\varphi)=E$. Upon translating E if necessary, which is equivalent to multiplying $d\varphi(x)$ by a character, we may suppose that E is a subgroup of G^*. Now consider $N(E)\subset G$. By Theorem 3.1, $N(E)$ is a σ-compact subgroup of G. Consider the measure φ described in Theorem 3.3, for $H=N(E)$. Since $N(N(E))=E$, we have $\Phi(y)=1$ for all $y\in E$. Conversely, if $|\Phi(y)|=1$ for an element y of G^*, there exists a complex number β of absolute value 1 such that

$$\int_G \beta(y, x)\, d\varphi(x)=1 ,$$

and $\beta(y, x)=1$ almost everywhere with respect to φ. Accordingly, $(y, x)=\beta^{-1}$ for all $x\in N(E)$, and as $(y, e)=1$ (e the identity of G), we find $\beta=1$ and $y\in N(N(E))=E$. This proves that $|\Phi(y)|<1$ for $y\notin E$, and establishes Theorem 1.1.

To prove Theorem 1.2, we note that if E is compact, then $N(E)$ is open, and then apply Theorem 3.4.

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THE UNIVERSITY OF WASHINGTON, SEATTLE, WASH., U.S.A.
AND
STANFORD UNIVERSITY, STANFORD, CALIF., U.S.A.