MATH. SCAND. 3 (1955), 103–106

THE DERIVATIVE OF A SCHLICHT FUNCTION

A. J. LOHWATER, G. PIRANIAN and W. RUDIN

A. Bloch (see Nevanlinna [4, p. 138]) raised the question whether the derivative of a function which is meromorphic and of bounded characteristic in $|z| < 1$ is also of bounded characteristic in $|z| < 1$. Frostman [1, p. 181] answered this question by constructing a Blaschke product $B(z)$ whose derivative $B'(z)$ has the property that for each θ the quantity $\lim_{r \to 1} B'(re^{i\theta})$ fails to exist as a finite number. Since the radial limit of a function meromorphic and of bounded characteristic in $|z| < 1$ exists and is finite, for almost all points on $|z| = 1$ (see [4, p. 134]), Frostman's example answers Bloch's question in the negative. Recently, Rudin [5] has constructed a function $f(z) = \Sigma\, a_n z^n$, with $\Sigma\, |a_n| < \infty$, and with the property that $\lim_{r \to 1} f'(re^{i\theta}) = \infty$ for almost all θ. In [6] he has exhibited another function $f(z)$, regular in $|z| < 1$ and continuous in $|z| \leqq 1$, such that

$$\int_0^1 |f'(re^{i\theta})|\, dr = \infty$$

for almost all θ. It should be noted that the function in [6] cannot possibly be schlicht, because if $f(z)$ is regular and schlicht in $|z| < 1$, then

$$\int_0^1 |f'(re^{i\theta})|\, dr < \infty$$

for almost all θ, by a theorem of Lavrentiev [2].

In the present note, we construct a schlicht function relevant to Bloch's question. The motivation for the construction comes from an example described by Lohwater and Piranian [3] in connection with a geometrical problem.

THEOREM. *For a suitably increasing sequence $\{n_p\}$ of positive integers, the function*

$$(1) \qquad f(z) = \int_0^z \exp\left\{ \tfrac{1}{2} \sum_{p=1}^{\infty} w^{n_p} \right\} dw$$

Received November 28, 1954.

is regular in $|z| < 1$, *continuous and schlicht in* $|z| \leqq 1$, *and has for almost all* θ *the properties*

(2) $$\limsup_{r \to 1} |f'(re^{i\theta})| = \infty ,$$

(3) $$\liminf_{r \to 1} |f'(re^{i\theta})| = 0 ,$$

(4) $$\limsup_{r \to 1} \arg f'(re^{i\theta}) = +\infty ,$$

(5) $$\liminf_{r \to 1} \arg f'(re^{i\theta}) = -\infty .$$

Moreover, the Taylor series of $f(z)$ *converges absolutely on* $|z| = 1$.

Proof. We use the notation

$$h_p(z) = \tfrac{1}{2} \sum_{k=1}^{p} z^{n_k}, \qquad g_p(z) = \exp h_p(z), \qquad f_p(z) = \int_0^z g_p(w)\, dw ,$$

and observe that

(6) $$f_{p+1}(z) - f_p(z) = \int_0^z g_p(w)\,[\exp\{\tfrac{1}{2} w^{n_{p+1}}\} - 1]\, dw .$$

We choose $n_1 = 1$ and define the further elements of the sequence $\{n_p\}$ as follows: having chosen $n_1, \ldots, n_p$, we write

(7) $$a_p = 2/(n_1 + n_2 + \ldots + n_p)$$

and choose n_{p+1} so that n_{p+1}/n_p is an odd integer, and large enough so that

(8) $$|f_{p+1}(z) - f_p(z)| < a_p e^{-p-3} \qquad (|z| \leqq 1);$$

by (6), this construction is always possible. It follows from (8) that the sequence $\{f_p(z)\}$ converges uniformly in $|z| \leqq 1$ to the function $f(z)$ defined by (1), and hence that $f(z)$ is regular in $|z| < 1$ and continuous in $|z| \leqq 1$.

To show that $f(z)$ is schlicht in $|z| \leqq 1$, let $z_1 \neq z_2$, $|z_1| \leqq 1$, $|z_2| \leqq 1$; and let p be that integer for which

$$a_p < |z_1 - z_2| \leqq a_{p-1} .$$

By (8),

$$|f(z) - f_p(z)| < \sum_{k=p}^{\infty} a_k e^{-k-3} < a_p e^{-p-2}$$

for $|z| \leqq 1$, whence

(9) $$|f(z_1) - f(z_2)| > |f_p(z_1) - f_p(z_2)| - 2a_p e^{-p-2} .$$

Also,

$$(10) \qquad f_p(z_1)-f_p(z_2) = \int_{z_2}^{z_1} g_p(w)\,dw = \int_{z_2}^{z_1} \exp\{h_{p-1}(w)+\tfrac{1}{2}w^{n_p}\}\,dw\,,$$

where the path of integration can be taken as the rectilinear segment from z_1 to z_2. Since

$$|h_{p-1}'(z)| \leqq (n_1+n_2+\ldots+n_{p-1})/2 \qquad (|z| \leqq 1)$$

and $|z_1-z_2| \leqq a_{p-1}$, it follows from (7) that, as w passes from z_1 to z_2, the argument of the integrand in (10) varies over an interval whose length is less than $1+1<\frac{2}{3}\pi$. Consequently the modulus of the integral in (10) exceeds

$$\cos\tfrac{1}{3}\pi\cdot|z_1-z_2|\cdot\min_{|z|\leqq 1}|g_p(z)| > \tfrac{1}{2}a_p e^{-p}\,.$$

Applying this estimate to (9), we get

$$|f(z_1) - f(z_2)| > \tfrac{1}{2}a_p e^{-p} - 2a_p e^{-p-2} = \tfrac{1}{2}a_p e^{-p-2}(e^2-4) > 0\,,$$

and therefore $f(z)$ is schlicht in $|z| \leqq 1$.

Next, we observe that (1) implies the relations

$$\log|f'(re^{i\theta})| = \tfrac{1}{2}\sum_{k=1}^{\infty} r^{n_k}\cos n_k\theta\,,$$

$$\arg f'(re^{i\theta}) = \tfrac{1}{2}\sum_{k=1}^{\infty} r^{n_k}\sin n_k\theta\,.$$

We denote by E_2 and E_3, respectively, the sets on $[0, 2\pi]$ on which (2) and (3) hold. Since the coefficients of the lacunary series $\Sigma\cos n_k\theta$ do not tend to zero, the Abel transform of the series is an unbounded function, for almost all θ, so that $m(E_2\cup E_3)=2\pi$ (for the details of this see, for example, [7, pp. 119–122]; and note that there the hypothesis of the convergence of the transform is not used, except inasmuch as it implies the boundedness of the transform).

Since $\{n_k\}$ is a sequence of odd integers, θ belongs to E_2 if and only if $\theta+\pi$ belongs to E_3; hence $m(E_2)=m(E_3)$. Since n_{k+1}/n_k is an integer, both E_2 and E_3 are periodic with arbitrarily small periods $2\pi/n_k$; consequently $m(E_i)$ $(i=2, 3)$ can have only one of the values 0 and 2π. Thus we conclude that $m(E_2)=m(E_3)=2\pi$, and (2) and (3) hold for almost all θ. An analogous discussion of $\Sigma\sin n_k\theta$ establishes (4) and (5).

Finally, we note that $f(z)=\Sigma a_n z^n$, where $a_n \geqq 0$. Since $\lim_{r\to 1}f(r)$ is finite, $\Sigma a_n<\infty$, and the proof is complete.

REFERENCES

1. O. Frostman, *Sur les produits de Blaschke*, Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 12 (1942) no. 15, 169–182 (=Meddelanden från Lunds Universitets Matematiska Seminarium 5 (1943), 1–14).
2. M. Lavrentiev, *Sur la correspondance entre les frontières dans la représentation conforme*, Mat. Sbornik **36** (1929), 112–115.
3. A. J. Lohwater and G. Piranian, *Linear accessibility of boundary points of a Jordan region*, Comment. Math. Helv. 25 (1951), 173–180.
4. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Paris, 1929.
5. W. Rudin, *On a problem of Bloch and Nevanlinna*, to appear in Proc. Amer. Math. Soc.
6. W. Rudin, *On the radial variation of analytic functions*, to appear in Duke Math. J.
7. A. Zygmund, *Trigonometrical series*, Warszawa, 1935.

UNIVERSITY OF MICHIGAN, ANN ARBOR, MICH., U.S.A.,
AND
UNIVERSITY OF ROCHESTER, ROCHESTER, N.Y., U.S.A.