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ESTIMATES OF THE FRIEDRICHS-LEWY TYPE FOR A HYPERBOLIC EQUATION WITH THREE CHARACTERISTICS

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The aim of this paper is to prove the uniqueness of a solution of a boundary value problem of mixed type for a linear hyperbolic differential equation with three characteristics and to estimate its solution by means of the boundary values. Problems involving more than two characteristics occur in gas dynamics, and uniqueness theorems of the kind considered below are stated without proofs in [1] (see e.g. p. 86). The method used in this paper is due to Friedrichs and Lewy [2], and has recently been used in the theory of linear hyperbolic equations of order greater than two by Leray [5] and Gårding [3]. For the sake of simplicity we have only considered the case when the differential operator has real coefficients and its principal part constant coefficients. Certain restrictions are also made concerning the boundary of the region considered. These specializations, however, do not seem to be imposed by the problem, but are made in order to avoid difficulties.

I wish to express my gratitude to my teacher, professor Lars Gårding, who suggested the subject of this paper.

1. Let V be a closed region in the plane whose boundary S is piecewise smooth and let L be a real hyperbolic differential operator of order three with constant principal part and continuous coefficients. In a suitable coordinate system, a suitable real multiple of such an operator has the form

$$(1)\quad L = (D_2-\alpha_1 D_1)(D_2-\alpha_2 D_1)(D_2-\alpha_3 D_1) + \sum_{i,k=1}^{2} a_{ik} D_i D_k + \sum_{i=1}^{2} b_i D_i + c$$

where $D_i = \partial/\partial x_i$; the constants α_1, α_2 and α_3 are real and $\alpha_1 < \alpha_2 < \alpha_3$ and the coefficients a_{ik}, b_i and c are real-valued continuous functions in V. The characteristic form associated with (1),

$$(2)\qquad \Lambda = \Lambda(\xi) = (\xi_2-\alpha_1\xi_1)(\xi_2-\alpha_2\xi_1)(\xi_2-\alpha_3\xi_1)\,,$$

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divides the ξ-space (the dual space) into four parts Σ_i $(i=0, 1, 2, 3)$, where Σ_i is the set of points $\xi=(\xi_1, \xi_2)$ making exactly i factors of (2) <0 and the others $\geqq 0$. Since every factor $\xi_2-\alpha_k\xi_1$ is negative on the negative ξ_2-axis, the division is seen to be the one represented in fig. 1. For practical reasons we also distinguish between the parts of Σ_1 and Σ_2 which correspond to positive and negative values of ξ_1 and denote these parts by Σ_1^-, Σ_1^+, Σ_2^- and Σ_2^+, respectively.

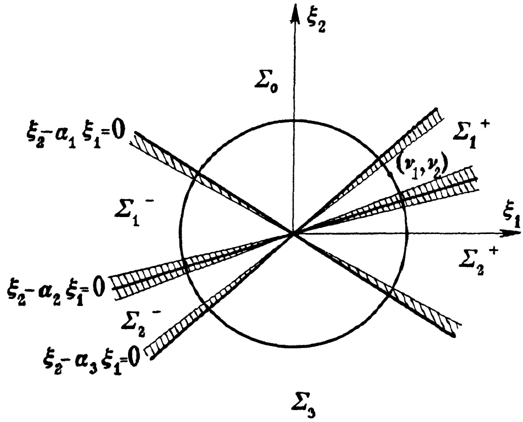

Fig. 1.

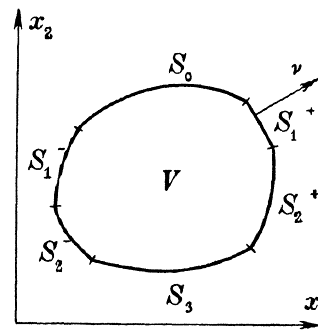

Fig. 2.

This division of the dual space now gives rise to a division of S into parts S_i so that a point of S belongs to S_i if and only if the exterior normal ν of S in this point belongs to Σ_i $\left(\nu=(\nu_1, \nu_2);\ |\nu|=(\nu_1^2+\nu_2^2)^{\frac{1}{2}}=1\right)$. The notations S_1^-, S_1^+, S_2^- and S_2^+ correspond in an obvious way to Σ_1^-, Σ_1^+, Σ_2^- and Σ_2^+ (see fig. 2).

Certain gas-dynamical considerations (see e.g. [1]) make it probable that, roughly speaking, if φ is a given function in V and φ_{ki} $(0 \leqq k < i)$ are given functions on S_i $(i=1, 2, 3)$, the differential equation

$$Lu = \varphi \tag{3}$$

will have a unique solution u such that $d^k u/d\nu^k=\varphi_{ki}$ on S_i when $k<i$ $(i=1, 2, 3)$. We shall prove the uniqueness part of this statement under the following restrictions on the boundary S of V,

$$\text{(a)} \qquad \inf_{S_1 \cup S_2} |\lambda_k| > 0 \qquad (k = 1, 2, 3)$$

where $\lambda_k=\nu_2-\alpha_k\nu_1$ and

(b) $S_1^+ \cup S_2^+$ has a positive distance to $S_1^- \cup S_2^-$.

The first of these conditions means that on S_1 and S_2, the normal must avoid the shaded areas in fig. 1 in the neighbourhoods of the lines $\xi_2 - \alpha_k \xi_1 = 0$ $(k = 1, 2, 3)$. Observe that this implies that a passage from one part S_i of S to another is accompanied by a jump in the normal derivative. The condition (b) means that any part of $S_1^- \cup S_2^-$ is separated from any part of $S_1^+ \cup S_2^+$ by a part of S_0 or S_3.

A solution u of (3) is to be understood here as a real function whose derivatives of order $\leqq 3$ are continuous in V.

THEOREM 1. *Let V be a region whose boundary S is piecewise smooth and satisfies* (a) *and* (b). *Then a solution of the equation*

$$Lu = 0 \tag{4}$$

with the boundary conditions $d^k u/d\nu^k = 0$ for $k < i$ on S_i $(i = 0, 1, 2, 3)$ vanishes identically in V.

In the proof we shall introduce a new differential operator

$$M = (D_2 - \beta_1 D_1)(D_2 - \beta_2 D_1) \tag{5}$$

where β_1 and β_2 are continuously differentiable functions satisfying

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3$$

at each point of V. They will be subject to additional conditions later. First we shall prescribe their values on the different parts of the boundary S in a suitable way, and then choose β_1 and β_2 as continuously differentiable continuations of these values to the whole of V.

The notations

$$l_k = D_2 - \alpha_k D_1 \qquad (k = 1, 2, 3)$$

and

$$L_k = \prod_{i \neq k} (D_2 - \alpha_i D_1) \qquad (i, k = 1, 2, 3) \tag{6}$$

will be of constant use.

We first prove the following

LEMMA. *Suppose that L only contains the principal part*

$$(D_2 - \alpha_1 D_1)(D_2 - \alpha_2 D_1)(D_2 - \alpha_3 D_1)$$

and that the β_i of (5) *are constants so that $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3$. Then we have the identity*

$$2\,Lu\,Mu = \sum_{k=1}^{3} l_k A_k (L_k u)^2 \tag{7}$$

where A_k are positive constants.

PROOF. The relation

$$Mu = \sum_{k=1}^{3} A_k L_k u \tag{8}$$

is evidently valid if the constants A_k satisfy the system

$$A_1 + A_2 + A_3 = 1,$$
$$A_1(\alpha_2+\alpha_3) + A_2(\alpha_3+\alpha_1) + A_3(\alpha_1+\alpha_2) = \beta_1 + \beta_2,$$
$$A_1\alpha_2\alpha_3 + A_2\alpha_3\alpha_1 + A_3\alpha_1\alpha_2 = \beta_1\beta_2,$$

which has the unique solution

$$\begin{cases} A_1 = (\beta_1-\alpha_1)(\beta_2-\alpha_1)/(\alpha_2-\alpha_1)(\alpha_3-\alpha_1), \\ A_2 = (\beta_1-\alpha_2)(\beta_2-\alpha_2)/(\alpha_1-\alpha_2)(\alpha_3-\alpha_2), \\ A_3 = (\beta_1-\alpha_3)(\beta_2-\alpha_3)/(\alpha_1-\alpha_3)(\alpha_2-\alpha_3). \end{cases} \tag{9}$$

Here A_k are seen to be positive if $\alpha_1<\beta_1<\alpha_2<\beta_2<\alpha_3$. Now multiplying (8) by $2Lu$, we get

$$2Lu\, Mu = \sum_{k=1}^{3} 2A_k\, L_k u\, Lu = \sum_{k=1}^{3} 2A_k\, L_k u\, l_k L_k u = \sum_{k=1}^{3} l_k A_k (L_k u)^2$$

which proves the lemma.

To prove Theorem 1 we observe that even if Lu contains derivatives of lower order and β_1 and β_2 are not constants but continuously differentiable functions of x on V, the difference between the right and left sides of (7) does not contain derivatives of order three, i.e. we have

$$2Lu\, Mu = \sum_{k=1} l_k A_k (L_k u)^2 + R(u,u) \tag{10}$$

where $R(u,u)$ is a quadratic form in $u_{ik}=D_iD_ku$, $u_i=D_iu$ and u with continuous coefficients.

Following Hörmander [4] we multiply (10) by a weight function $e^{-\gamma x_2}$, where the constant $\gamma>0$ will be chosen later. After easy computations we get

$$2e^{-\gamma x_2} Lu\, Mu \tag{11}$$
$$= \sum_{k=1}^{3} l_k \left[e^{-\gamma x_2} A_k (L_k u)^2\right] + e^{-\gamma x_2}\left[\gamma \sum_{k=1}^{3} A_k (L_k u)^2 + R(u,u)\right].$$

One immediately verifies the identity

$$0 = l_2\left[e^{-\gamma x_2}(u_1{}^2+u_2{}^2+u^2)\right] - e^{-\gamma x_2} l_2 (u_1{}^2+u_2{}^2+u^2) + \tag{12}$$
$$+ \gamma e^{-\gamma x_2}(u_1{}^2+u_2{}^2+u^2).$$

Multiplying (12) by a non-negative constant δ and adding (11) we get

$$(13) \qquad 2e^{-\gamma x_2} Lu\, Mu$$
$$= \sum_{k=1}^{3} l_k [e^{-\gamma x_2} A_k (L_k u)^2] + l_2 [e^{-\gamma x_2} \delta (u_1{}^2 + u_2{}^2 + u^2)] + e^{-\gamma x_2} [\gamma V(u,u) + Q(u,u)]$$

where

$$(14) \qquad V(u,u) = \sum_{k=1}^{3} A_k (L_k u)^2 + \delta (u_1{}^2 + u_2{}^2 + u^2)$$

and $Q(u,u)$ is a quadratic form in u_{ik}, u_i and u with continuous coefficients.

Integrating (13) over V and using Greens formula we get

$$(15) \quad \int_V 2e^{-\gamma x_2} Lu\, Mu\, dV = \int_S e^{-\gamma x_2} S(u,u)\, dS + \int_V e^{-\gamma x_2} [\gamma V(u,u) + Q(u,u)]\, dV$$

where

$$(16) \qquad S(u,u) = \sum_{k=1}^{3} \lambda_k A_k (L_k u)^2 + \lambda_2 \delta (u_1{}^2 + u_2{}^2 + u^2).$$

For a solution of (4) the left side of (15) vanishes and we shall prove that $S(u,u)$ and $V(u,u)$ can be made positive definite on S and in V respectively by a suitable choice of the functions β_1 and β_2 and the constant δ. In this way, if γ is sufficiently large, we have written the right side of (15) as a sum of two non-negative terms, which must therefore both vanish. The vanishing of the second term combined with the positive definiteness of the integrand implies that u vanishes in V, which is what we wanted to prove.

We start with $S(u,u)$ and consider first the various parts of S separately.

By virtue of the boundary conditions, $S(u,u)$ vanishes identically on S_3 and thus we can choose β_1, β_2 and δ arbitrarily on S_3. As $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3$ the A_k are positive and according to the definition of S_0 the λ_k are positive on S_0. Therefore, as δ is chosen positive, we have that $S(u,u)$ is non-negative on S_0. On S_1 and S_2, $S(u,u)$ will be proved to be positive definite in the derivatives which do not vanish according to the boundary conditions. Thus we express $S(u,u)$ in terms of u and its derivatives in S and orthogonal to S. After the transformation

$$D_s = \nu_1 D_2 - \nu_2 D_1$$
$$D_\nu = \nu_1 D_1 + \nu_2 D_2$$

we obtain from (16) after easy computations

$$S(u,u) = \lambda_1\lambda_2\lambda_3\mu_1\mu_2\left\{\sum_{k=1}^{3} A_k'(L_k'u)^2+\lambda_2\delta(u_s^2+u_v^2+u^2)+W(u,u)\right\}$$

where $W(u,u)$ is a quadratic form which is linear in u_{ss}, u_{sv} and u_{vv}, $\mu_k=\nu_2-\beta_k\nu_1$, and L_k' and A_k' are the expressions corresponding to (6) and (9) with α_i and β_i replaced by

$$\alpha_i' = (\nu_1+\alpha_i\nu_2)/(\alpha_i\nu_1-\nu_2)$$

and

$$\beta_i' = (\nu_1+\beta_i\nu_2)/(\beta_i\nu_1-\nu_2)\,,$$

respectively.

Here $\alpha_i\nu_1-\nu_2=-\lambda_i$ is never zero on S_1 and S_2 because of the condition (a), and β_1 and β_2 are to be chosen in a way which makes $\beta_i\nu_i-\nu_2=-\mu_i$ different from zero on S_1 and S_2.

On S_2 we have $u=u_s=u_v=u_{ss}=u_{sv}=0$ and we get

$$S(u,u) = \lambda_1\lambda_2\lambda_3\mu_1\mu_2\, u_{vv}^2.$$

To make the coefficient of u_{vv}^2 positive on S_2^+ we have only to choose β_1 sufficiently close to α_2 and β_2 arbitrarily in the interval (α_2, α_3). To see this we notice that for an arbitrary point of S_2^+ we have $\lambda_1>0$, $\lambda_2<0$, $\lambda_3<0$ and $\mu_2<0$. Thus it suffices to choose β_1 in a way which makes $\mu_1<0$ for all points on S_2^+. Now choosing β_1 sufficiently close to α_2 we obtain that the part of the line $\xi_2-\beta_1\xi_1=0$ which corresponds to positive values of ξ_1 belongs to the shaded area of Σ_2^+ close to the line $\xi_2-\alpha_2\xi_1=0$ in fig. 1. That means that for all allowed ν on S_2^+ we have $\mu_1<0$. Similarly, to make the coefficient of u_{vv}^2 positive on S_2^- we have only to choose β_2 sufficiently close to α_2 and β_1 arbitrarily in the interval (α_1, α_2). The proof is analogous to that for S_2^+.

On S_1 we have $u=u_s=u_{ss}=0$. We shall prove that after having fixed β_1 arbitarily in the interval (α_1,α_2) we have only to choose β_1 sufficiently close to α_3 in order to make $S(u,u)$ positive definite on S_1^+ and similarly that after having fixed β_2 arbitrarily in the interval (α_2, α_3) we have only to choose β_2 sufficiently close to α_1 in order to make $S(u,u)$ positive definite on S_1^-. It is sufficient to prove this for S_1^+, the proof for S_1^- beeing analogous.

We first restrict ourselves to such β_2 which satisfy

$$\lambda_1\lambda_2\lambda_3\mu_1\mu_2 > 0$$

Similarly to the discussion on S_2^+ above, we realize that this is obtained if we choose β_2 sufficiently close to α_3. In a fixed point of S_1^+ we have

$$\beta_2' < \alpha_3' < \alpha_1' < \beta_1' < \alpha_2'$$

if β_2 is sufficiently close to α_3, and we start by proving that in this fixed point $S(u,u)$ can be made positive definite. Now we notice that choosing β_2 close to α_3 is equivalent to choosing β_2' close to α_3'. The main part of $S(u,u)$ is

$$\sum_{k=1}^{3} A_k'(L_k'u)^2$$

$$= u_{\nu\nu}^2 - 2(\beta_1+\beta_2)u_{s\nu}u_{\nu\nu} + [A_1'(\alpha_2'+\alpha_3') + A_2'(\alpha_3'+\alpha_1') + A_3'(\alpha_1'+\alpha_2')]u_{s\nu}^2$$

$$= [u_{\nu\nu} - (\beta_1'+\beta_2')u_{s\nu}]^2 + D(\beta_1',\beta_2')u_{s\nu}^2$$

where

$$D(\beta_1',\beta_2') = A_1'(\alpha_2'+\alpha_3') + A_2'(\alpha_3'+\alpha_1') + A_3'(\alpha_1'+\alpha_2') - (\beta_1'+\beta_2')^2.$$

For $\alpha_1' < \beta_1' < \alpha_2'$ we get

$$D(\beta_1', \alpha_3') = (\beta_1'-\alpha_1')(\alpha_2'-\beta_1') > 0.$$

Since $D(\beta_1', \beta_2')$ is a continuous function of β_2' in the neighbourhood of α_3', it follows that $D(\beta_1',\beta_2')$ is positive in a sufficiently small neighbourhood of α_3'. For a fixed β_1 we have thus found that for each point of S_1^+ there exists a neighbourhood of α_3 in which

$$\sum_{k=1}^{3} A_k'(L_k'u)^2 \tag{17}$$

is positive definite for all β_2. By continuity it follows that we can choose β_2 as a constant so that (17) is positive definite in $u_{\nu\nu}$ and $u_{s\nu}$ on the whole of S_1^+. Choosing δ sufficiently large we obtain that

$$\sum_{k=1}^{3} A_k'(L_k'u)^2 + \lambda_2\delta u_\nu^2 + W(u,u)$$

is positive definite in $u_{\nu\nu}$, $u_{s\nu}$ and u_ν on S_1^+, and thus $S(u,u)$ is positive definite on S_1^+.

We have now actually proved that we can choose β_1 and β_2 as constants on $S_1^+ \cup S_2^+$ and $S_1^- \cup S_2^-$, respectively. Take for instance $S_1^+ \cup S_2^+$, the reasoning for $S_1^- \cup S_2^-$ beeing analogous. Then we first fix β_1 sufficiently close to α_2 to get $S(u,u)$ positive definite on S_2^+ and after that we choose β_2 so that $S(u,u)$ becomes positive definite on S_1^+.

Since $S_1^+ \cup S_2^+$ has a positive distance to $S_1^- \cup S_2^-$ it is evident that there exists continuously differentiable functions β_1 and β_2 in V taking the chosen constant values on $S_1^+ \cup S_2^+$ and $S_1^- \cup S_2^-$ and which in V attain only values between the values attained on $S_1^+ \cup S_2^+$ and $S_1^- \cup S_2^-$.

This choice of β_1 and β_2 guarantees the existence of an $\varepsilon > 0$ so that

$$\alpha_1+\varepsilon \leqq \beta_1 \leqq \alpha_2-\varepsilon$$

and

$$\alpha_2+\varepsilon \leqq \beta_2 \leqq \alpha_3-\varepsilon.$$

This now implies that $\inf_V A_k > 0$ $(k=1, 2, 3)$, that is, the quadratic form $V(u,u)$ defined in (14) is positive definite in V. This completes the proof of Theorem 1.

2. Finally we prove that it is possible to obtain an estimate for the solution of the equation (3) in terms of $\varphi = Lu$ and the boundary values.

THEOREM 2. *For a solution of* (3) *in a region* V *whose boundary* S *is submitted to the same restrictions as in Theorem* 1 *we have the following estimate*

$$(18)\quad \int_V \Big(\sum_{i,k=1}^{2} u_{ik}^2 + \sum_{i=1}^{2} u_i^2 + u^2 \Big) dV \leqq C \Big\{ \int_V \varphi^2 dV + \int_{S_1} (u_{ss}^2 + u_s^2 + u^2)\, dS + \int_{S_2} (u_{ss}^2 + u_{s\nu}^2 + u_s^2 + u_\nu^2 + u^2)\, dS + \int_{S_3} (u_{ss}^2 + u_{s\nu}^2 + u_{\nu\nu}^2 + u_s^2 + u_\nu^2 + u^2)\, dS \Big\}$$

where C *is a constant independent of the function* u.

PROOF. Having chosen the functions β_1 and β_2 and the constants γ and δ as in the proof of Theorem 1 we get from (15) (observe that C does not necessarily denote the same constant during the course of the proof)

$$\int_V \Big(\sum_{i,k=1}^{2} u_{ik}^2 + \sum_{i=1}^{2} u_i^2 + u^2 \Big) dV$$

$$\leqq C \Big\{ \int_V Lu\, Mu\, dV - \int_S S(u,u)\, dS \Big\}$$

$$\leqq C \Big\{ \Big[\int_V (Lu)^2 dV \int_V (Mu)^2 dV \Big]^{\frac{1}{2}} - \int_S S(u,u)\, dS \Big\}$$

$$\leqq C \Big\{ \Big[\int_V \Big(\sum_{i,k=1}^{2} u_{ik}^2 + \sum_{i=1}^{2} u_i^2 + u^2 \Big) dV \int_V \varphi^2\, dV \Big]^{\frac{1}{2}} - \int_S S(u,u)\, dS \Big\}$$

$$= \Big[\int_V \Big(\sum_{i,k=1}^{2} u_{ik}^2 + \sum_{i=1}^{2} u_i^2 + u^2 \Big) dV \int_V C^2 \varphi^2\, dV \Big]^{\frac{1}{2}} - C \int_S S(u,u)\, dS$$

$$\leqq \tfrac{1}{2} \int_V \Big(\sum_{i,k=1}^{2} u_{ik}^2 + \sum_{i=1}^{2} u_i^2 + u^2 \Big) dV + \tfrac{1}{2} C^2 \int_V \varphi^2\, dV - C \int_S S(u,u)\, dS\,,$$

that is,

$$\int_V \Big(\sum_{i,k=1}^{2} u_{ik}^2 + \sum_{i=1}^{2} u_i^2 + u^2 \Big) dV \leqq C \Big\{ \int_V \varphi^2\, dV - \int_S S(u,u)\, dS \Big\}.$$

The inequality (18) is now obtained by estimating

$$-\int_S S(u,u)\, dS$$

on the various parts of S, and this is easily done, using that $S(u,u)$ is positive definite when the boundary values vanish.

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